1. Let $A$ be a real symmetric matrix.
   
   (a) Prove that $A$ has a Cholesky factorization if and only if $A$ is positive definite.

   (b) Assuming $A$ has a Cholesky factorization, find the number of operations (to leading order) required to compute that factorization.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x - \frac{3}{4}e^{-x}$. Prove that $f(x) = 0$ has a unique solution $x^*$, and that the fixed-point iteration $x_{n+1} = \frac{3}{4}e^{-x_n}$ converges to $x^*$ from any initial $x_0 \in \mathbb{R}$.

3. Let $f$ be any non-constant $C^\infty$ function on $\mathbb{R}$, and let $f'_h(x_0)$ be the centered difference approximation to $f'(x_0)$ computed with stepsize $h$,
   
   $$f'_h(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}.$$

   (a) Suppose $f'_h(x_0)$ is computed in exact arithmetic. Prove that $f'_h(x_0) \to f'(x_0)$ as $h \to 0$.

   (b) Suppose $f'_h(x_0)$ is computed in idealized floating point arithmetic with specified $\epsilon_m$.
      
      i. Prove that $f'_h(x_0)$ does not converge as $h \to 0$.
      
      ii. Find the step $h^*$ that minimizes the absolute error $|f'(x_0) - f'_h(x_0)|$.

4. Consider the midpoint method for the approximate solution of a scalar initial value problem $y' = f(x, y)$, $f \in C^{(1,1)}$, with initial conditions $y(x_0) = y_0$ and stepsize $h$. A step of the midpoint method is given by

   $$\tilde{y}_n = y_n + \frac{1}{2}hf(x_n, y_n)$$

   $$\tilde{x}_n = x_n + \frac{h}{2}$$

   $$x_{n+1} = x_n + h$$

   $$y_{n+1} = y + hf(\tilde{x}_n, \tilde{y}_n).$$

   (a) State carefully definitions of the following:
      
      i. Absolute stability
      
      ii. $A$-stability
      
      iii. Local truncation error

   (b) Find the region of absolute stability for the midpoint method. Is the midpoint method $A$-stable?

   (c) Prove that the local truncation error of the midpoint method is $O(h^3)$.  

5. Consider approximation of the integral \( I(f) = \int_{-1}^{1} f(x) \, dx \).

(a) Find a three-point quadrature rule \( Q \) that computes \( I(f) \) exactly for all \( f \in P^5 \), where \( P^5 \) is the space of all polynomials of degree at most 5.

(b) Derive a bound on the error \( I(f) - Q(f) \) in terms of \( \|f^{(k)}\|_{\infty} \). State as needed any assumptions about \( k \), the degree of differentiability of \( f \).

6. Let \( A \) be any \( M \times N \) complex matrix. The notation \( A^H \) denotes the conjugate transpose of \( A \).

(a) Describe the properties (size, shape, structure, or other notable attributes) of the factors of the singular value decomposition (SVD) of \( A \). When appropriate, distinguish between the full and reduced SVDs.

(b) Use the SVD of \( A \) to compute the condition number (with respect to the Euclidean norm) of \( A^H A \).

(c) Prove that for every \( \epsilon > 0 \), the matrix \( \epsilon I + A^H A \) is Hermitian positive definite.

7. Let the inner product \((\cdot,\cdot)\) be defined by
\[
(u,v) = \int_{-1}^{1} u(x) v(x) \, dx,
\]
and let \( \|\cdot\| \) be the norm induced by that inner product: \( \|v\| = \sqrt{(v,v)} \). Let \( P^N \) be the space of polynomials of degree at most \( N \).

(a) Find the first-degree polynomial \( u_1 \in P^1 \) that best approximates in the \( \|\cdot\| \) norm the function \( f(x) = x^5 \).

(b) Let \( f \) be continuous on \([-1,1]\), and let \( u_N \) be the \( N \)-th degree polynomial that best approximates \( f \) in the \( \|\cdot\| \) norm. Prove that \( u_N \to f \) uniformly as \( N \to \infty \).