Do eight of the following nine problems. Clearly indicate which eight are to be graded. Calculators are not allowed.

1. Let $F(x)$ have a fixed point $s$, and assume that there exists an integer $q \geq 2$ such that $F^{(k)}(s) = 0$ for $1 \leq k \leq q - 1$, but $F^{(q)}(s) \neq 0$. Prove that the sequence $[x_n]$, defined by $x_{n+1} = F(x_n)$ converges to the fixed point $s$ with order of convergence $q$. Assume $F(x)$ to be smooth enough.

2. Consider the iterative procedure:

$$\tilde{y}_{j+1} = \tilde{y}_j + \frac{h}{24} \left[ 9 \tilde{f}(\tilde{y}_{j+1}) + 19 \tilde{f}(\tilde{y}_j) - 5 \tilde{f}(\tilde{y}_{j-1}) \right],$$

where $\tilde{y}_{j+1}$, $\tilde{y}_j$, and $\tilde{y}_{j-1} \in \mathbb{R}^n$, $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\tilde{y}_j$ and $\tilde{y}_{j-1}$ are given. Assume that $\|\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{w})\|_\infty \leq \frac{3}{9} \|\tilde{z} - \tilde{w}\|_\infty$ for all $\tilde{z}$ and $\tilde{w} \in \mathbb{R}^n$. Prove that if $h$ is sufficiently small, then $\|\tilde{f}(\tilde{y}_{j+1}) - \tilde{f}(\tilde{y}_{j+1})\|_\infty \to 0$ as $k \to \infty$, where $\tilde{y}_{j+1}$ satisfies

$$\tilde{y}_{j+1} = \tilde{y}_j + \frac{h}{24} \left[ 9 \tilde{f}(\tilde{y}_{j+1}) + 19 \tilde{f}(\tilde{y}_j) - 5 \tilde{f}(\tilde{y}_{j-1}) \right].$$

3. Let $\| \cdot \|$ be any induced matrix norm. Prove that if $E$ is an $n \times n$ matrix for which $\|E\|$ is sufficiently small, then

$$\|(I - E)^{-1} - (I + E)\| \leq 3\|E\|^2.$$ Determine how small $\|E\|$ should be.

4. Let the $n \times n$ matrix $A$ be diagonalized by similarity transformation $D = P^{-1}AP$. Consider any $n \times n$ matrix $B$ and let $C = P^{-1}BP$.

a) Show that $A + B$ and $D + C$ have the same eigenvalues.  
b) Show that the eigenvalues of $A + B$ lie in the union of the disks

$$\{ \lambda \in \mathbb{C} : |\lambda - \lambda_i| \leq \kappa_\infty(P)\|B\|_\infty \} \quad (1 \leq i \leq n)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$, and $\kappa_\infty(P)$ is the condition number of $P$.

5. Let the $3 \times 3$ matrix $A$ have eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 8$ with corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. Consider the iterative method

$$\vec{x}_{k+1} = \frac{1}{3}(I - A)(A - 5I)^{-1}\vec{x}_k,$$

where $\vec{x}_0 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$. Prove that $\|\vec{x}_k - \vec{z}\| \to 0$ as $k \to \infty$, where $\vec{z}$ is one of the eigenvectors $\vec{v}_1$, $\vec{v}_2$ or $\vec{v}_3$.  

1
6. Suppose that \( f(x) \) satisfies a Lipschitz condition \(|f(x) - f(y)| \leq L|x - y|\) for all \( x, y \in [0, 1] \). Let \( \Psi(x) \) be a piecewise constant approximation to \( f(x) \) such that
\[
\Psi(x) = \frac{f(x_i) + f(x_{i+1})}{2}, \quad \text{for } x_i \leq x < x_{i+1}, \text{ for } i = 0, 1, \ldots, N - 1
\]
with \( x_i = ih \) and \( h = 1/N \). Prove that
\[
\max_{0 \leq x \leq 1} |\Psi(x) - f(x)| \leq c h
\]
for some constant \( c \).

7. Suppose we wish to approximate an odd function by a polynomial of degree \( \leq n \) (\( n \) odd) using the norm \( \|f\| = \left( \int_{-\alpha}^{\alpha} |f(x)|^2 w(x) dx \right)^{1/2} \), where \( w(x) \) is a even positive weight function. Prove that the best approximation is also odd.

8. a) Find the constants \( A \) and \( B \) such that the formula
\[
\int_0^{2\pi} f(x) dx \approx A f(0) + B f(\pi),
\]
is exact for any function of the form \( f(x) = a + b \cos x \).

b) Prove that the resulting formula is exact for any function of the form
\[
f(x) = \sum_{k=0}^{n} \left[ a_k \cos((2k + 1)x) + b_k \sin(kx) \right].
\]

9. Consider the initial-value problem \( \frac{dy}{dt} = 1 + 2t + 3 \cos(y(t)), \ 0 \leq t \leq 1 \), with \( y(0) = 1 \). Suppose that the solution satisfies \( \max_{0 \leq t \leq 1} |y''(t)| = M < \infty \). Consider the approximation
\[
y_{k+1} = y_k + h(1 + 2t_k + 3 \cos(y_k))
\]
for \( k = 0, 1, 2, \ldots, N - 1 \), \( y_0 = y(0) = 1 \), and \( h = 1/N \) and \( t_k = kh \). Prove that
\[
\|y(1) - y_N\| \leq \frac{Mh}{6} (e^3 - 1).
\]