Numerical Analysis Preliminary Examination, Aug 2009
Department of Mathematics and Statistics

Do 9 of the following 10 problems. Clearly indicate which 9 are to be graded. Calculators are not allowed.

1. Consider the function \( g : \mathbb{R}^2 \to \mathbb{R} \). Assume that there exists a \( z \in [a, b] \) such that \( g(z, z) = z \) and \( g \) satisfies \( |g(x, y) - g(\hat{x}, \hat{y})| \leq \lambda \max\{|x - \hat{x}|, |y - \hat{y}|\} \) for all \( x, y, \hat{x}, \hat{y} \in [a, b] \), where \( 0 < \lambda < 1 \). Consider the iterative method \( x_{j+1} = g(x_j, x_{j-1}) \) for \( j = 2, 3, \ldots \), where \( x_0, x_1 \in [a, b] \). Show that \( |x_j - z| \to 0 \) as \( j \to \infty \) and \( z \) is the only point in \( [a, b] \) such that \( g(z, z) = z \).

2. Prove that if \( A \) is invertible and \( \|B - A\| < \|A^{-1}\|^{-1} \), then \( B \) is invertible and

\[
B^{-1} = A^{-1} \sum_{k=0}^{\infty} (I - BA^{-1})^k.
\]

3. Prove that, if the matrix \( A \) is strictly diagonally dominant and \( Q \) is the lower triangular part of \( A \), including the diagonal, then the Gauss-Seidel method

\[
Qx^{(k)} = (Q - A)x^{(k-1)} + b, \quad k \geq 1,
\]

converges to the solution of \( Ax = b \), for any starting vector \( x^{(0)} \).

4. Prove that if \( A \) is symmetric and positive definite, then the problem of solving \( Ax = b \) is equivalent to the problem of minimizing the quadratic form

\[
q(x) = x^T Ax - 2x^T b.
\]

5. a) Show that the trace of a matrix \( A \) equals the sum of its eigenvalues. (Schur’s Theorem may be useful).
   b) Prove that if the eigenvalues of \( A \) satisfy \( |\lambda_1| > |\lambda_i| \) for \( i = 2, 3, \ldots, n \), then

\[
\lambda_1 = \lim_{m \to \infty} \frac{\text{tr}(A^{m+1})}{\text{tr}(A^m)}
\]

6. A natural cubic spline \( S \) on \([0, 2]\) is defined by

\[
S_0(x) = 1 + 2x - x^3 \quad \text{on } 0 \leq x < 1,
\]
\[
S_1(x) = 2 + b(x - 1) + c(x - 1)^2 + d(x - 1)^3 \quad \text{on } 1 \leq x \leq 2.
\]

Find \( b, c \) and \( d \).
7. Let \( e(h) = \int_0^1 f(x)dx - h \sum_{i=1}^{N} f(ih - \frac{h}{2}) \), where \( h = \frac{1}{N} \)
   a) For \( f \in C^1[0, 1] \), prove that for all \( N \) there exists a constant \( c_1 > 0 \) such that
   \[
   |e(h)| \leq c_1 h.
   \]

   b) For \( f \in C^2[0, 1] \), prove that for all \( N \) there exists a constant \( c_2 > 0 \) such that
   \[
   |e(h)| \leq c_2 h^2.
   \]

8. Suppose that \( x_i \) and \( A_i \), for \( i = 0, 1, 2 \), are selected so that the quadrature formula
   \[
   \int_{-1}^{1} x^2 f(x)dx \approx \sum_{i=0}^{2} A_i f(x_i),
   \]
   is exact for any polynomial of degree 5. Find the third degree polynomial \( q_3(x) \)
   such that \( q_3(x_i) = 0 \), for \( i = 0, 1, 2 \).

9. Consider the initial value problem
   \[
   \begin{align*}
   x'(t) &= f(t, x) \\
   x(t_0) &= x_0
   \end{align*}
   \]
   Show that, if \( w_1, w_2, \alpha \) and \( \beta \) satisfy
   \[
   \begin{align*}
   w_1 + w_2 &= 1 \\
   w_2 \alpha &= \frac{1}{2} \\
   w_2 \beta &= \frac{1}{2}
   \end{align*}
   \]
   then
   \[
   \begin{align*}
   x(t + h) &= x(t) + (w_1 F_1 + w_2 F_2) + O(h^3) \\
   F_1 &= h f(t, x) \\
   F_2 &= h f(t + \alpha h, x + \beta F_1)
   \end{align*}
   \]

10. Show how the Shooting method can be used to solve the two-point boundary value problem of the following type, in which the constants \( \alpha, \beta \) and \( c_{ij} \) and the functions \( u(t), v(t) \) and \( w(t) \) are all given:
    \[
    \begin{align*}
    x'' &= u(t) + v(t)x + w(t)x' \\
    c_{11}x(a) + c_{12}x'(a) &= \alpha \\
    c_{21}x(b) + c_{22}x'(b) &= \beta
    \end{align*}
    \]
    Assume that the solution exists and is unique.