1. Let $f \in C^2(\mathbb{R})$. Consider Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n \geq 0$ for solving the nonlinear equation $f(x) = 0$. Let $e_n = x_n - r$ where $r$ is a simple zero of $f$ (i.e. $f(r) = 0 \neq f'(r)$).

(a) Show that $e_{n+1} = \frac{1}{2} f'(x_n) e_n^2$, where $x_n$ is a number between $x_n$ and $r$.

(b) Suppose $f$ is increasing and $f$ is convex (i.e. $f''(x) > 0$) for all $x \in \mathbb{R}$. Show that the Newton iteration will converge to the zero from any starting point.

2. Let $A_\alpha = \alpha D + \alpha L + U$, where $D$ is a $n \times n$ diagonal matrix, $L$ is a $n \times n$ strictly lower triangular matrix, $U$ is a $n \times n$ strictly upper triangular matrix, and $\alpha > 0$ is a positive parameter. Let $A_\alpha \mathbf{x} = \mathbf{b}$. Suppose the Gauss-Seidel method converges for $\alpha = 1$.

(a) Prove that the Gauss-Seidel method converges for any value of $\alpha > 1$.

(b) Let $A_\alpha = \begin{bmatrix} \alpha & \frac{2}{3} \\ \frac{3}{\alpha} & \alpha \end{bmatrix}$. Show that the Gauss-Seidel method converges for $\alpha = 1$ but does not converge for $\alpha = 0.5$.

3. Let $P(x)$ be the continuous piecewise linear interpolant to $f(x) = x^3$ on the interval $[0, 10]$ such that $P(k) = f(k)$ for $k = 0, 1, 2, \ldots, 10$.

(a) Find the exact error $e(x) = |f(x) - P(x)|$ on the interval $[2, 3]$.

(b) Determine the maximum exact error in $[2, 3]$, i.e., find $\max_{2 \leq x \leq 3} e(x)$.

4. Prove that the eigenvalues of matrix $A$ are unaltered if a row of $A$ is multiplied by a number $c \neq 0$ and the corresponding column is multiplied by $\frac{1}{c}$.

5. Let $f$ have derivatives of all orders. For $h > 0$ determine a formula of the form,

$$f''(x) \approx \frac{1}{h^3} [af(x - 2h) + bf(x - h) + cf(x) + df(x + h) + ef(x + 2h)]$$

where the order of the error is $h^2$. Find $a, b, c, d$ and $e$.

6. Consider the initial-value problem $\frac{dy}{dt} = a + by(t) + c \sin(y(t))$, $0 \leq t \leq 1$ where $y(0) = 1$ and $a, b, c > 0$ are constants. Let us suppose that the solution satisfies $\max_{0 \leq t \leq 1} |y''(t)| = M < \infty$. Consider the approximation $y_{k+1} = y_k + (a + by_k + c \sin(y_k))h$ for $k = 0, 1, 2, \ldots, N - 1$, $y_0 = y(0)$, and $h = \frac{1}{N}$. Prove that $|y(1) - y_N| \leq \frac{Mhe^{b+c}}{2(b+c)}$. 

Note: Do nine of the following ten problems. Clearly indicate which nine are to be graded.
7. Let \( g(\vec{x}) = \langle \vec{x}, A\vec{x} \rangle - 2 \langle \vec{x}, \vec{b} \rangle \) where \( A \) is a positive definite and symmetric \( n \times n \) matrix and the inner product is defined as \( \langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} \).

(a) Let \( \vec{d} \) and \( \vec{b} \) be fixed vectors, \( \vec{d} \neq \vec{b} \) and let \( \hat{\vec{t}} \) be a real number variable such that 
\[
\hat{\vec{t}} = \frac{\langle \vec{d}, \vec{b} - A\vec{d} \rangle}{\langle \vec{d}, A\vec{d} \rangle}.
\]
Show that: 
\[
g(\vec{x} + \hat{\vec{t}}\vec{d}) = g(\vec{x}) - \frac{\langle \vec{d}, \vec{b} - A\vec{d} \rangle^2}{\langle \vec{d}, A\vec{d} \rangle}.
\]
(b) Show that if \( \vec{x}^* \) minimizes \( g(\vec{x}) \), then \( \vec{x}^* \) is a solution to the positive definite linear system \( A\vec{x} = \vec{b} \).

8. Consider the following multi-step method:
\[
y_{k+1} + \alpha_0 y_k = h (\beta_2 f(t_{k+1}, y_{k+1}) + \beta_1 f(t_k, y_k) + \beta_0 f(t_{k-1}, y_{k-1}))
\]
for solving the initial-value problem \( y'(t) = f(t, y) \).

(a) Find \( \alpha_0, \beta_0, \beta_1, \beta_2 \) such that the method is third order.
(b) Is the method consistent? If so why? If not why not?
(c) Is the method stable? If so why? If not why not?

9. Let \( f \in C[a, b] \) and let \( P_n(x) = \sum_{k=0}^{n} a_k x^k \) be a polynomial of degree at most \( n \) that minimizes the error \( E = \int_a^b (f(x) - P_n(x))^2 \, dx \).

(a) Prove that \( \sum_{k=0}^{n} a_k \int_a^b x^i x^k \, dx = \int_a^b x^i f(x) \, dx \) for \( i = 0, 1, \ldots, n \).
(b) Also show that the system of equations in part (a) has a unique solution.

10. Determine the total number of operations (addition, subtraction, multiplication, division) for the Gaussian Elimination algorithm described below:

```
input n, (a_{ij})
for k = 1 to n - 1 do
    for i = k + 1 to n do
        z = \frac{a_{ik}}{a_{kk}}
        for j = k to n do
            a_{ij} = a_{ij} - za_{kj}
        end do
    end do
end do
```