1. Consider an iteration function \( g(x) \) of the form \( g(x) = x - f(x)f'(x) \). (Note: This is NOT a Newton iteration function.) Assume that \( r \) satisfies \( f(r) = 0 \) and \( f'(r) \neq 0 \). Find the precise conditions on the function \( f \) so that the iterations \( x_{n+1} = g(x_n) \) converge to the fixed-point \( r \) at least cubically if started near \( r \).

2. Consider the problem of finding the least squares polynomial approximation to the function \( f(x) \) on \([0, 1]\), i.e., find \( p(x) = \sum_{j=1}^{n} \alpha_j x^{j-1} \) such that \( \int_0^1 \left( f(x) - \sum_{j=1}^{n} \alpha_j x^{j-1} \right)^2 \, dx \) is minimized for \( \alpha_0, \alpha_1, \ldots, \alpha_n \).

   (a) Find an \( n \times n \) matrix \( A \) and a vector \( \vec{b} \) so that \( A \vec{\alpha} = \vec{b} \) where \( (\vec{\alpha})_i = \alpha_i \).

   (b) Explain why this is not a good approach for computing \( \alpha_1, \alpha_2, \ldots, \alpha_n \).

3. (a) Consider the formula \( \int_0^h f(x) \, dx = h \left\{ A f(0) + B f \left( \frac{h}{3} \right) + C f(h) \right\} \). Find \( A, B, C \) such that this is exact for all polynomials of degree less than or equal to 2.

   (b) Suppose that the Trapezoidal rule applied to \( \int_0^2 f(x) \, dx \) gives the value \( \frac{1}{2} \) while the quadrature rule in part (a) applied to \( \int_0^2 f(x) \, dx \) gives the value \( \frac{1}{4} \). If \( f(0) = 3 \), then show that \( f \left( \frac{2}{3} \right) = 1 \).

4. Let \( s_1(x) = 1 + c(x + 1)^3 \), \( -1 \leq x \leq 0 \), where \( c \) is a (real) parameter. Determine \( s_2(x) \) on \( 0 \leq x \leq 1 \) so that,

\[
s(x) = \begin{cases} 
  s_1(x), & -1 \leq x \leq 0 \\
  s_2(x), & 0 \leq x \leq 1 
\end{cases}
\]

is a natural cubic spline, i.e., \( s''(-1) = s''(1) = 0 \) on \([-1, 1]\) with nodal points at \(-1, 0, 1\). How must \( c \) be chosen if one wants \( s(1) = -1 \) ?

5. Consider the iterative procedure \( y_{j+1}^{(k+1)} = y_j + \frac{h}{2} \left[ f(y_j) + f(y_j^{(k)}) \right] \) for \( k = 0, 1, 2, \ldots \)

where \( y_j \in \mathbb{R} \) is given, \( f \in C(\mathbb{R}) \), and \( y_j^{(0)} = y_j \). Suppose that \( |f(u) - f(v)| \leq L|u - v| \)

for all \( u, v \in \mathbb{R} \) for a constant \( L \). Prove that if \( \frac{hL}{2} < 1 \), then \( y_j^{(k)} \to y_{j+1} \) as \( k \to \infty \)

where \( y_{j+1} \) satisfies \( y_{j+1} = y_j + \frac{h}{2} \left[ f(y_j) + f(y_j+1) \right] \).
6. Consider the boundary-value problem \(y''(x) + y'(x) + xy(x) = \cos(x), 0 < x < 1\), with \(y(0) = 1, y(1) = 2\).

   (a) Show how the solutions of two initial-value problems
   \[ u''(x) + u'(x) + xu(x) = \cos(x), 0 < x < 1, u(0) = 1, u'(0) = 0 \]
   \[ w''(x) + w'(x) + xw(x) = \cos(x), 0 < x < 1, w(0) = 1, w'(0) = 1 \]
   can be combined to find \(y(x)\). In particular, find \(a\) and \(b\) such that \(y(x) = au(x) + bw(x)\).

   (b) Describe a numerical procedure for approximating \(y(x)\) using part (a).

7. Let \(A = D - U - L\) where \(A\) is strictly diagonally dominant, \(D\) is diagonal, \(U\) is upper triangular and \(L\) is lower triangular. Furthermore, assume that \(D, U\) and \(L\) have all nonnegative elements, that is, \(D, U, L \geq 0\). Suppose \(\bar{b} \geq \bar{0}\). Consider the Gauss-Seidel iterative procedure \(\bar{x}^{(m+1)} = (D - L)^{-1}U\bar{x}^{(m)} + (D - L)^{-1}\bar{b}\) for \(m = 0, 1, 2, \ldots\) with \(\bar{x}^{(0)} = \bar{0}\). Assume also that the spectral radius \(\rho((D - L)^{-1}U) = \gamma < 1\). Prove that \(\bar{x}^{(m)} \rightarrow \bar{x}\) where all the elements of \(\bar{x}\) are nonnegative, that is, \(\bar{x} \geq \bar{0}\). (Hint: Show that \(\bar{x}^{(m)} \geq 0\) for each \(m\).)

8. The polynomial \(P_n(x)\) interpolating the function \(f(x)\) at the nodes \(x_k\) for \(k = 0, \ldots, n\) is given by, \(P_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k)\), where \(L_k(x) = \prod_{i=0, i \neq k}^{n} \frac{(x-x_i)}{(x_k-x_i)}\). Let us define
   \[ \psi(x) = \prod_{i=0}^{n} (x-x_i) \text{ and } \lambda_k^{(n)} = \prod_{j=0, j \neq k}^{n} \frac{1}{(x_k-x_j)} \].

   (a) Show that \(\sum_{k=0}^{n} L_k(x) = 1\) and \(P_n(x) = \psi(x) \sum_{k=0}^{n} \frac{f(x_k)}{(x-x)(\psi(x)_k)} \).

   (b) Show that if \(x\) is not a node, then \(P_n(x) = \frac{\sum_{k=0}^{n} f(x_k) \lambda_k^{(n)}(x-x_k)^{-1}}{\sum_{m=0}^{n} \lambda_m^{(n)}(x-x_m)^{-1}}\).

9. Let \(f \in C^1([0, 1] \times [0, 1])\) and \(h = \frac{1}{n}\). Show that \(\left| \int_0^1 \int_0^1 f(x,y) \, dx \, dy - h^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(ih, jh) \right| \leq ch\)

   for some constant \(c\) independent of \(h\). (Recall \(f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(\mu, \xi)(x-a) + \frac{\partial f}{\partial y}(\mu, \xi)(y-b)\) for some \(\mu\) between \(x\) and \(a\) and some \(\xi\) between \(y\) and \(b\).)

10. Suppose \(||I - AB_0|| = c < 1\) and \(B_k = B_{k-1} + B_{k-1}(I - AB_{k-1}), k = 1, 2, \ldots\).

   (a) Show that \(||I - AB_k|| \leq c^{2^k}\).

   (b) Suppose \(A\) is nonsingular, show that \(||A^{-1} - B_k|| \leq ||A^{-1}|| c^{2^k}\).