Numerical Analysis Preliminary Examination 2002
Department of Mathematics and Statistics

Note: Do eight of the following nine problems. Clearly indicate which eight are to be graded.

1. Let $A$ be a real, symmetric, strictly diagonally dominant $n \times n$ matrix. Suppose $A = D + N$, where $D$ is a diagonal matrix and $N_{ii} = 0$ for each $i$.
   (a) Show that $A\vec{x} = \vec{b}$ if and only if $\vec{x} = D^{-1}(b - N\vec{x})$.
   (b) Show that there exists a $\rho < 1$ such that for $f(\vec{x}) = D^{-1}(b - N\vec{x})$,
       $$||f(\vec{x}) - f(\vec{y})||_\infty \leq \rho ||\vec{x} - \vec{y}||_\infty.$$
   (c) Show that the sequence $\vec{x}^{(k+1)} = D^{-1}(b - N\vec{x}^{(k)})$ converges to $\vec{x}$.

2. Let the $n \times n$ matrix $A$ have elements $a_{ij} = \int_0^1 e^{ix} e^{jx} dx$ for $1 \leq i, j \leq n$. Prove that $A$ has a Cholesky Factorization $A = L^T L$.

3. Let $A$ be a nonsingular $n \times n$ real matrix and $||A^{-1}B|| = r < 1$.
   (a) Show that $A + B$ is nonsingular and $||A^{-1}B|| \leq \frac{||A^{-1}||}{1 - r}$.
   (b) Show that $||(A + B)^{-1} - A^{-1}|| \leq \frac{||B|| \cdot ||A^{-1}||^2}{1 - r}$.

4. Let $x_i^*$ for $i = 1, 2, \ldots, n$ be positive numbers on a computer. With a unit round-off error $\delta$, $x_i^* = x_i (1 + \epsilon_i)$ with $|\epsilon_i| \leq \delta$, where $x_i$ for $i = 1, 2, \ldots, n$ are the exact numbers.
   (a) Consider the product $P_n = \prod_{i=0}^{n} x_i$ and its floating point approximation $P_n^* = \prod_{i=0}^{n} x_i^*$.
       Show that if $P_n^* = P_n (1 + \epsilon)$, then $\epsilon$ satisfies
       $$\epsilon \leq e^{\delta(2n+1)} - 1.$$
   (b) Consider the scalar product $S_2 = \vec{a}^T \vec{b}$ where $\vec{a} = (x_1, x_2)^T$ and $\vec{b} = (x_3, x_4)^T$. Let $S_2^*$ be the floating point approximation of $S_2$. Prove that
       $$\frac{S_2^*}{S_2} \leq e^{4\delta}.$$

5. Assume that $f \in C^3[a, b]$ and $x_0, x_0 + h, x_0 + 2h \in [a, b]$. Prove that there exist constants $c_1$ and $c_2$ such that
   $$\left| f'(x_0) - \frac{1}{h} \left[ -\frac{3}{2} f(x_0) + c_1 f(x_0 + h) + c_2 f(x_0 + 2h) \right] \right| \leq c h^2 \max_{a \leq x \leq b} |f'''(x)|$$
   where $c > 0$ is a constant independent of $h$. 
6. Let \( f(x) = \frac{1}{x} \) and \( P_2(x) \) be the Lagrange quadratic polynomial that interpolates \( f(x) \) at \( x_0 = 2, \ x_1 = 2.5 \) and \( x_2 = 4 \). Recall the error formula
\[
f(x) - P_2(x) = \frac{1}{6} (x-x_0)(x-x_1)(x-x_2)f'''(\xi(x)), \quad x_0 < x < x_2.
\]
(a) Using the error formula, obtain a sharp error bound for \(|f(3) - P_2(3)|\).
(b) Find a function \( \xi(x) \) explicitly for this problem.

7. Consider a quadrature formula of the type
\[
\int_0^\infty e^{-x} f(x) \, dx = af(0) + bf(c) + E(f)
\]
where \( E(f) \) is the error in the formula.
(a) Find \( a, b \) and \( c \) such that the formula is exact for polynomials of the highest degree possible. (Note that \( \int_0^\infty e^{-x} x^n \, dx = n! \)).
(b) Let \( P(x) \) be the Hermite polynomial interpolating \( f \) at the (simple) point \( x = 0 \) and double point \( x = 2 \); i.e. \( P(0) = f(0), P(2) = f(2) \) and \( P'(2) = f'(2) \). Determine \( \int_0^\infty e^{-x} P(x) \, dx \) and compare with the result in part (a).
(c) For the values of \( a, b \) and \( c \) found in part (a), obtain the error \( E(f) \) in the form \( E(f) = Cf'''(\xi) \) for some \( \xi > 0 \) where \( C \) is a constant.

8. Consider the initial-value problem \( y'(t) = f(t, y) \) for \( 0 \leq t \leq 1 \) with \( y(0) = a \). Consider the one-step method
\[
y_{k+1} = y_k + h \, \Phi(t_k, y_k, h)
\]
with \( y_0 = a, \ h = 1/N, \) and \( t_k = kh \) for \( k = 0, 1, \ldots, N \). Assume that there is a constant \( L \) such that
\[
|\Phi(t, y, h) - \Phi(t, z, h)| \leq L \, |y - z|
\]
for all \( t, y, z \in \mathbb{R} \). Furthermore, assume that the solution \( y(t) \) satisfies
\[
|y(t+h) - y(t) - h\Phi(t, y(t), h)| \leq c \, h^{p+1}
\]
for all \( t, h \in [0, 1] \). Prove that
\[
|y_N - y(1)| \leq c \, \frac{h^p}{L} \, (e^L - 1).
\]

9. Let \( f : [a, b] \to \mathbb{R} \) be a \( C^1 \) function satisfying \( f'(x) \neq 0 \) for \( x \in [a, b] \). Let \( \{p_n\}_{n=0}^\infty \) be the Newton iteration sequence for solving \( f(x) = 0 \) i.e., \( p_n \) satisfies
\[
p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}.
\]
Assume that \( p_n \in (a, b) \) for \( n \geq 0 \) and \( \lim_{n \to \infty} p_n = r \).
(a) Show that \( f(r) = 0 \).
(b) Prove that
\[
|p_n - r| \leq \max_{x \in [a, b]} \frac{|f(p_n)|}{|f'(x)|}
\]