1. Let $U \mathbf{x} = \mathbf{b}$ where $U$ is an $n \times n$ nonsingular upper triangular matrix. The vector $\mathbf{x}$ can be computed using the algorithm $x_n = b_n / u_{n,n}$ and $x_k = (b_k - \sum_{j=k+1}^{n} u_{k,j}x_j) / u_{k,k}$ for $k = n - 1, n - 2, \ldots, 1$. Prove that this algorithm requires exactly $(n^2 - n)/2$ subtractions and additions. (Note that $\sum_{i=1}^{M} i = M(M + 1)/2$.)

2. (a) Describe the equations that characterize the best quadratic approximation to $f(x) = x^{1/2}$ in $L^2[0,1]$. That is, find equations for $\alpha, \beta, \gamma$ that minimize

$$\int_{0}^{1} (\alpha + \beta x + \gamma x^2 - f(x))^2 dx.$$ 

Find the best linear approximation to $f(x) = x^{1/2}$ in $L^2[0,1]$. (Find $\alpha$ and $\beta$.)

3. (a) Derive the Trapezoidal rule on interval $[a, b]$ and then the Composite Trapezoidal rule on this interval.

(b) Derive the error in the Trapezoidal rule on interval $[a, b]$ for functions $f \in C^2[a, b]$.

(c) Let $f(x) = x^{8/7}$ and let $T_n(f)$ denote the Composite Trapezoidal rule on $[0,1]$ with spacing $h = 1/n$. Find

$$\lim_{n \to \infty} n^2 \left( \int_{0}^{1} f(x)dx - T_n(f) \right).$$

4. (a) Describe the Inverse Power Method for a matrix that has distinct eigenvalues. Describe the standard error estimates for this method.

(b) Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 4 & 8 & 1 & 1 \\ 0 & 0 & 10 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix}$.

Describe the result of Inverse Power Iteration for the matrix $(A - 2I)^{-1}$. Determine the limiting eigenvalue and eigenvector and estimate the rate of convergence of the iteration method. (First, calculate the eigenvalues of $A$.)
5. Suppose that matrix $A$ is nonsingular, $x$ is the solution of $Ax = b$, $\|A^{-1}\|_2 = 10^3$, and $\|A\|_2 = 10^2$. We wish to solve $Bz = b$ where $B = A - C$ and $\|C\|_2 = 10^{-4}$.

(a) Prove that $B$ is nonsingular.

(b) Find an upper bound on $\|x - z\|_2$ in terms of $\|x\|_2$, that is, find $c > 0$ such that $\|x - z\|_2 < c\|x\|_2$.

6. (a) Prove that if matrix $A = M - N$ is singular and $M$ is nonsingular, then $\|M^{-1}N\| \geq 1$ where $\| \cdot \|$ is any induced matrix norm.

(b) Prove that if matrix $A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ is positive definite, then the Jacobi iteration method converges for a linear system $Ax = b$. (Hint: Consider the eigenvalues of the Jacobi iteration matrix $D^{-1}(D - A)$.)

7. Consider the two-point boundary-value problem $y''(x) - p(x)y'(x) - q(x)y(x) = r(x)$, $0 < x < 1$, with $y(0) = y(1) = 0$. Assume that $q(x) \geq \alpha > 0$ for $0 \leq x \leq 1$. Consider the difference scheme

$$\frac{(y_{j+1} - 2y_j + y_{j-1})}{h^2} - p(x_j)(y_{j+1} - y_{j-1}) - q(x_j)y_j = r(x_j), \quad j = 1, 2, \cdots, N - 1,$$

with $y_0 = y_N = 0$, $x_j = jh$, and $h = 1/N$.

(a) Determine the matrix $A$ so that the above difference equations can be written as the linear system $Ay = h^2r$ with $y = [y_1, y_2, \cdots, y_{N-1}]^T$ and $r = [r(x_1), r(x_2), \cdots, r(x_{N-1})]^T$.

(b) Prove that if $\frac{h}{2} \max_{0 \leq x \leq 1} |p(x)| \leq 1$, then the $(N - 1) \times (N - 1)$ matrix $A$ is strictly diagonally dominant.

8. Consider the two-dimensional quadrature formula

$$\int_{-1}^{1} \int_{-1}^{1} f(x, y)dx dy \approx f(\alpha, \alpha) + f(-\alpha, \alpha) + f(\alpha, -\alpha) + f(-\alpha, -\alpha).$$

Find the value of $\alpha$ such that the formula is exact for every polynomial $f(x, y)$ of degree less than or equal to 3, that is, for $f(x, y) = \sum_{i,j=0}^{3} a_{i,j}x^iy^j$.

9. Consider numerical solution of the initial-value problem $y'(t) = f(t, y(t))$, $0 < t < 1$, $y(0) = y_0 = 0$ using the trapezoidal method $y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})]$, for $k = 0, 1, \cdots, N - 1$, where $N = 1/h$ and $t_k = kh$. Suppose that $\max_{0 \leq t \leq 1} |y''(t)| \leq M$ and that $|f(t, z) - f(t, \tilde{z})| \leq L|z - \tilde{z}|$ for all $z, \tilde{z} \in R$. Assuming that $hL < 1$, prove that $\max_{0 \leq k \leq N} |y_k - y(t_k)| \leq ch^2$ where the constant $c$ does not depend on $h$.

(Note that $\int_{t_k}^{t_{k+1}} g(z)dz - \frac{h}{2}(g(t_{k+1}) + g(t_k)) = -\frac{h^3}{12}g''(\xi_k)$ for some $\xi_k \in (t_k, t_{k+1})$.)

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