Answer all 8 questions.
Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $B(a; r) = \{z \in \mathbb{C} : |z - a| < r\}$, $\text{Ann}(a; r_1; r_2) = \{z : r_1 < |z - a| < r_2\}$.

1. (a) State a version of Cauchy’s Integral Formula.
   (b) State a version of Cauchy’s Theorem.
   (c) Use Cauchy’s Integral Formula to derive Cauchy’s Theorem.

2. Suppose $f$ is analytic on a domain $G$. Prove $\overline{z} f(z)$ is analytic if and only if $f(z) = 0$ for all $z \in G$.

3. Find a one-to-one conformal map $f$ of the slit half-plane $\{z : \text{Re} z < 1\} \setminus (0, 1]$ onto the upper half plane $\{z : \text{Im} z > 0\}$ such that on the boundary $f(0) = \infty$, $f(1 + i) = -1$, and $f(1 - i) = 1$.

4. Find the Laurent series centered at $z = 0$ of $\frac{4}{z^3 - 2z^2 - 3z}$ that converges uniformly on $|z| = 2$ and give the largest annulus on which it converges.

5. (a) Suppose $f$ is entire and $a, b \in B(0; r)$. Evaluate $\int_{\gamma} \frac{f(z)}{(z-a)(z-b)} \, dz$, where $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$.
   (b) Use part (a) to give a proof of Liouville’s Theorem.

6. Let $f$ be a nonconstant analytic function whose power series representation $\sum_{n=0}^{\infty} a_n z^n$ converges in $\mathbb{D}$. Suppose $f(z) = f(e^{i\alpha\pi}z)$ for all $z \in \mathbb{D}$ and some $\alpha \in \mathbb{R}$. Prove that $\alpha$ must be rational.

7. (a) Prove that there exists a sequence of polynomials $\{p_n\}$ which converges pointwise to $\frac{1}{z}$ on $\mathbb{C} \setminus (-\infty, 0]$.
   (b) Prove that no sequence of polynomials can converge to $\frac{1}{z}$ uniformly on compact subsets of $\text{Ann}(0; 1; 2)$.

8. Use the Residue Theorem to evaluate $\int_0^\infty \frac{x^{1/2}}{(1 + x)^2} \, dx$. 