Directions: Do all of the following eight problems. Show all your work and justify your answers. Each problem is worth 10 points.

Notation: \( \mathbb{C} \) — the complex plane; \( \mathbb{Z} \) — the set of integers; \( \mathbb{D} := \{ z : |z| < 1 \} \) — the unit disk; \( \Re(z) \) and \( \Im(z) \) denote the real part of \( z \) and the imaginary part of \( z \), respectively.

1. Let \( f(z) = e^z \).
   (a) Use the Cauchy-Riemann Equations to prove that \( f(z) \) is analytic on \( \mathbb{C} \).
   (b) Prove that \( f(z) \) is conformal at every point \( z \in \mathbb{C} \).
   (c) Prove that \( f(z) \) is one-to-one on the domain \( D \), where

   \[
   D := \{ z = x + iy : -\infty < x < \infty, x < y < x + 2\pi \}.
   \]

2. (a) State Liouville’s Theorem.
    (b) Show that there is no non-constant bounded analytic function on \( \mathbb{C} \setminus \mathbb{Z} \).
    (c) Give an example of a function \( f(z) \) which is analytic on \( \mathbb{C} \setminus \mathbb{Z} \) but is not entire.

3. Let

   \[
   f(z) = \cot z + \cos \left( \frac{1}{1-z} \right) - \frac{1}{z}.
   \]

   Locate and classify all the singularities of \( f(z) \) (including any singularity at \( z = \infty \)) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential). Calculate the residues of \( f(z) \) at its poles.

4. Let

   \[
   f(z) = \frac{cz^2 - cz + 1}{z^2(z-1)},
   \]

   where \( c \in \mathbb{C} \) is constant.
   (a) Find the principal part of the Laurent expansion of \( f(z) \) convergent in the domain

   \[
   D := \{ z : 0 < |z| < 1 \}.
   \]
   (b) Find all values of \( c \) for which \( f(z) \) has a primitive in \( D \).

5. Let

   \[
   f(z) = \begin{cases} 
   \sin z & \text{if } \Im(z) \geq 0 \\
   1/\sin z & \text{if } \Im(z) < 0.
   \end{cases}
   \]

   Prove that there is a sequence of polynomials \( p_n(z) \), \( n = 1, 2, 3, \ldots \) such that \( p_n(z) \) converges to \( f(z) \) point-wise on \( \mathbb{C} \).

6. Use the Residue Theorem to evaluate the integral

   \[
   \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - 2x + 10} \, dx.
   \]

7. Let \( g(z) \) be analytic on the disk \( \{ z : |z| < 2 \} \). Suppose that \( g(z) \neq 0 \) for all \( z \) such that \( |z| = 1 \) and \( \Re \left( \frac{\sin(z^2)}{g(z)} \right) > 0 \) for all \( z \) such that \( |z| = 1 \). Find the number of zeros (counting multiplicity) of \( g(z) \) in the unit disk \( \mathbb{D} \).

8. Let \( \mathcal{A}(\mathbb{D}) \) be the set of analytic functions on the unit disk. Let \( F \) be the set of all functions \( f \in \mathcal{A}(\mathbb{D}) \) such that \( f(0) = 1 \) and \( |\arg(f(z))| < \pi/4 \) for all \( z \in \mathbb{D} \). Use Schwarz’s lemma to find

   \[
   \max_{f \in F} |f(1/2)|.
   \]