Directions: Do all of the following ten problems. Show all your work and justify your answers. Each problem is worth 10 points.

Notation: \( \mathbb{C} \) — the complex plane; \( D := \{ z : |z| < 1 \} \) — the unit disk; \( \Re(z) \) and \( \Im(z) \) denote the real part of \( z \) and the imaginary part of \( z \), respectively; \( \text{Log} \ z \) denotes the principal branch of the logarithm.

1. Compute all values of the following multi-valued expression: \( (e^i)^i \).

2. Let \( u(x, y) \) be harmonic on a domain \( D \subset \mathbb{C} \) and let \( v(x, y) \) be a harmonic conjugate of \( u(x, y) \) on \( D \).
   (a) Prove that \( u(x, y) v(x, y) \) is harmonic on \( D \).
   (b) Prove that if \( x \ u(x, y) \) is harmonic on \( D \) then \( u(x, y) = ay + b \), where \( a \) and \( b \) are constants.

3. Let \( G \) be a domain in \( \mathbb{C} \), \( a \in G \), and let \( G_a = G \setminus \{a\} \). Suppose that \( f \) is a bounded analytic function on \( G_a \). Prove that an isolated singularity of \( f \) at \( z = a \) is removable.

4. Let \( f \) be an entire function. Suppose that there is a polynomial \( p \) such that for each \( z \in \mathbb{C} \), \( |f(z)| \leq |p(z)| \). Show that \( f \) is also a polynomial.

5. (a) State any version of Runge’s approximation theorem.
   (b) Prove that there is a sequence of polynomials \( p_n \) such that \( p_n(z) \to \sin z \) pointwise if \( \Re z > 0 \), \( p_n(z) \to \cos z \) pointwise if \( \Re(z) < 0 \), and \( p_n(z) \to 0 \) pointwise if \( \Re z = 0 \).

6. Locate and classify for each of the functions all the singularities (including any singularity at \( z = \infty \)) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential):
   (a) \( \frac{1}{e^z - 1} - \frac{1}{z} \)
   (b) \( \frac{1}{\text{Log} z} \)
   (c) \( z^2 \sin(1/z) \)

7. Use the Residue Calculus to evaluate the integral
   \[ \int_0^\infty \frac{x^2 \, dx}{x^4 + x^2 + 1}. \]

8. Let \( A(\mathbb{D}) \) be the set of analytic functions on the unit disk. Let \( F = \{ f \in A(\mathbb{D}) : f(0) = 1, \ f(\mathbb{D}) \subset \mathbb{C} \setminus (-\infty, 0] \} \). Use Schwarz’s lemma to find
   \[ \max_{f \in F} |f'(0)|. \]

9. Find a conformal mapping \( w = f(z) \) from the semi-disk \( \mathbb{D}^+ := \{ z \in \mathbb{D} : \Im(z) > 0 \} \) onto itself with continuous extension to the boundary of \( \mathbb{D}^+ \) such that \( f(-1) = 1, \ f(0) = i, \ f(1) = -1 \).

10. Let \( f \) be a holomorphic function defined in a neighborhood of the closed disk \( \overline{\mathbb{D}} = \{ z : |z| \leq 1 \} \) such that \( f(0) = 1 \) and \( |f(z)| > 1 \) if \( |z| = 1 \). Prove that \( f \) has at least one zero in the unit disk \( \mathbb{D} \).