Instructions:

\( \mathbb{C} \) denotes the complex plane. \( \mathbb{C}_\infty \) denotes the extended complex plane, i.e., \( \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \).

For \( z \in \mathbb{C} \), \( \Re z \) and \( \Im z \) denote the real and imaginary parts of \( z \), respectively.

\( \mathbb{D} \) denotes the open unit disk in \( \mathbb{C} \), i.e., \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \).

\( B(a, r) \) denotes the open disk in \( \mathbb{C} \) centered at \( a \) of radius \( r \), i.e., \( B(a, r) = \{ z \in \mathbb{C} : |z - a| < r \} \).

\( \mathbb{U} \) denotes the upper half-plane in \( \mathbb{C} \), i.e., \( \mathbb{U} = \{ z \in \mathbb{C} : \Im z > 0 \} \).

For a region \( \mathcal{G} \subset \mathbb{C} \), let \( \mathcal{A}(\mathcal{G}) = \{ f : f \text{ is analytic on } \mathcal{G} \} \).

1. Find the Laurent expansion of \( f(z) = \frac{1}{z^2(1-z)^2} \) on the annulus
   a. \( 0 < |z| < 1 \)
   b. \( 1 < |z| < \infty \)

2. Use the residue theorem to evaluate
   a. \( \int_{\gamma} \frac{z(z+1)}{\sin(z+1)} \, dz \) where \( \gamma(t) = 3e^{it}, 0 \leq t \leq 2\pi \)
   b. \( \int_{\gamma} \frac{z(z+1)}{\sin(z+1)} \, dz \) where \( \gamma(t) = 5e^{it}, 0 \leq t \leq 2\pi \)

3. Prove that if \( \mathcal{G} \) is a simply connected region in \( \mathbb{C} \) and if \( f \in \mathcal{A}(\mathcal{G}) \) such that \( f \) has no zeros on \( \mathcal{G} \), then there exists a \( g \in \mathcal{A}(\mathcal{G}) \) such that \( g \) is a branch of logarithm for \( f \).

4. Find the image of the quarter disk, \( \Omega = \{ z \in \mathbb{D} : \Re z > 0, \Im z > 0 \} \) under the map \( w = g(z) = \frac{1}{2i} \left( z - \frac{1}{z} \right) \).
   Prove that \( g \) is one-to-one on \( \Omega \).

5. Let \( f, g \in \mathcal{A}(\mathbb{D}) \). Suppose that \( \frac{f(z)}{g(z)} > 0 \) for \( z \in \partial \mathbb{D} \). Show that \( f, g \) have the same number of zeros in \( \mathbb{D} \).

6. Suppose that \( f \in \mathcal{A}(\mathcal{G}) \), where \( \mathcal{G} \) is a region which contains 0. Suppose that \( \left| f(\frac{1}{n}) \right| \leq e^{-n} \) for all positive integers \( n \). Prove that \( f \equiv 0 \) on \( \mathcal{G} \).

7. Give an explicit example of a function \( f \in \mathcal{A}(\mathbb{D}) \) which is one-to-one on \( \mathbb{D} \) such that the range \( f(\mathbb{D}) \) is dense in \( \mathbb{C} \).

8. Consider the function \( f(z) = \sqrt{1 - z^2} \), where the branch of square root is chosen so that \( f(1) > 0 \). Determine the radius of convergence of the MacLauren series for \( f \).

9. Let \( f \in \mathcal{A}(\mathbb{C}) \). For any point \( \zeta \in \mathbb{C} \), let \( \sum_{n=0}^{\infty} a_n(\zeta)(z - \zeta)^n \) denote the Taylor’s series representation for \( f \) centered at \( \zeta \). Suppose for each such \( \zeta \) that \( a_0(\zeta) = 0 \). Prove that \( f \) is a polynomial.

10. The operators \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \bar{z}} \) are defined as follows:
    \[
    \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
    \]
    Let \( f(z) \) be a nonzero analytic function in a domain \( D \subset \mathbb{C} \). Find \( \frac{\partial}{\partial \bar{z}} f(z) \). Then prove the following:
    \[
    \frac{\partial}{\partial \bar{z}} |f(z)| = \frac{1}{2} |f(z)| \frac{f'(z)}{f(z)},
    \]