Instructions:

Do each of the following problems. Show all relevant steps which lead to your solutions.

Notation:

\( \mathbb{C} \) denotes the complex plane.
For \( z \in \mathbb{C} \), \( \Re(z) \) and \( \Im(z) \) denote the real and imaginary parts of \( z \), respectively.
\( \mathbb{D} \) denotes the open unit disk in \( \mathbb{C} \), i.e., \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \).
\( \mathbb{U} \) denotes the upper half-plane in \( \mathbb{C} \), i.e., \( \mathbb{U} = \{ z \in \mathbb{C} : \Im(z) > 0 \} \).
For a region \( G \subset \mathbb{C} \), let \( \mathcal{A}(G) = \{ f : f \text{ is analytic on } G \} \).

1. (a) Find all solutions of the equation \( e^{z^2} = 1 \).
   
   (b) Factor the polynomial \( p(z) = -32z^4 - 32z^3 - 12z^2 - 2z^4 \) given that \( z = -2 \) is a root.

2. Find and classify all isolated singular points of each of the following functions, including any isolated singular points which occur at the point of infinity:
   
   a. \( e^{-z} \cos \frac{1}{z} \)
   
   b. \( \cot \frac{z}{z^2} \)

3. Prove that the function \( f(z) = \sum_{n=1}^{\infty} n^{-z} \) converges for \( \Re(z) > 1 \) and represent its derivative in series form.

4. Use residue calculus to evaluate the integral \( \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - 2x + 10} \, dx \). Verify each step.

5. Let \( A = \{ z : \frac{1}{2} < |z| < 1 \} \) and \( f \in \mathcal{A}(A) \). Suppose there exists a sequence of polynomials \( \{ p_n \} \) such that \( \{ p_n \} \) converges to \( f \) in the topology of uniform convergence on compact subsets of \( A \). Show that \( f \) can be extended to a function which is analytic on \( \mathbb{D} \).

6. Let \( f \) be analytic on a region containing \( \overline{\mathbb{D}} \) such that \( |f(z)| \leq 1 \) for \( z \in \overline{\mathbb{D}} \). Show for all \( a, b \in \mathbb{C} \) that \( |af(0) + bf'(0)| \leq |a| + |b| \).

7. Determine how many roots \( p(z) = z^4 + z + 1 \) has in the first quadrant.

8. Find a conformal mapping from the sector \( S = \{ z : |z| < 4, 0 < \arg z < \pi/3 \} \) slit along the radial segment \( [0, e^{i\pi/6}] \) onto \( \mathbb{U} \) such that \( f(e^{i\pi/6}) = 0, \ f(2e^{i\pi/6}) = i. \)

9. Let \( f(z) = \frac{1}{1 + z^2 + z^4 + z^6 + z^8 + z^{10}} \). Find the radius of convergence of the Taylor series of \( f \) about \( z = 1 \).