Do all problems. Present adequate work to justify your answers.

Notation: \( B(0; r) = \{ z \in \mathbb{C} : |z| < r \} \), \( \mathbb{D} = B(0; 1) \), \( \text{ann}(a; \alpha, \beta) = \{ z \in \mathbb{C} : \alpha < |z-a| < \beta \} \)

1. Let \( G \) be the region in the first quadrant bounded by the line segment \([0, 1]\) and the arc of the circle which passes through 0 and 1 and which is tangent to the line \( \text{Re} z = \text{Im} z \) at \( z = 0 \). Construct a one-to-one, conformal map of \( G \) onto \( \mathbb{D} \).

2. Consider the rational function \( f(z) = \frac{z^2 - 2z}{z(1-z)(2-z)^2} \).
   
   a) Classify all of the singularities of \( f \), including the singularity at \( \infty \).
   
   b) Classify all of the singularities of \( f(z^2) \), including the singularity at \( \infty \).
   
   c) Find the Laurent expansion of \( f \) on the annulus \( \text{ann}(0; 1, 2) \).

3. State and prove Louiville’s Theorem.

4. Let \( \mathbb{D}' = \mathbb{D} \setminus \{0\} \) and let \( \mathcal{A}(\mathbb{D}') \) denote the set of analytic functions on \( \mathbb{D}' \). Let \( \{ f_n \} \subset \mathcal{A}(\mathbb{D}') \) and \( f \in \mathcal{A}(\mathbb{D}') \) such that \( f_n \) converges to \( f \) in the topology of local uniform convergence on compacta. Let \( \sum_{k=-\infty}^{\infty} a_k^{(n)} z^k \) be the Laurent series expansion of \( f_n \) on \( \mathbb{D}' \) and \( \sum_{k=-\infty}^{\infty} a_k z^k \) be the Laurent series expansion of \( f \) on \( \mathbb{D}' \). Prove for each \( k \) that the sequence \( \{ a_k^{(n)} \} \) converges to \( a_k \) as \( n \to \infty \).

5. Let \( f \) be analytic on \( B(0; 10) \) such that for \( z \in \partial B(0; 1) \) that \( \text{Im} f(z) = \text{Im} z \). Find a representation for \( f \) if \( f(0) = 1 \).

6. Show for \( \alpha > 1 \) that \( \alpha z^3 e^z = 1 \) has exactly three roots in \( B(0; 2) \).

7. For \( f \) analytic on \( \mathbb{C} \) we say that \( \zeta \) is an attractive fixed point of \( f \) if \( \zeta \) is a fixed point of \( f \) and if there exists a \( \delta > 0 \) such that \( |f(z) - \zeta| < |z - \zeta| \) for \( 0 < |z - \zeta| < \delta \). Let \( f(z) = z^2 - (2 - \frac{1}{2}i)z \). Find the attractive fixed points of \( f \).

8. Let \( G \) be a region in \( \mathbb{C} \) and let \( f \) be analytic on \( G \). Suppose there exists \( \overline{B(a; r)} \subset G \) such that for \( z \in \partial B(a; r) \), \( |f(z)| = 1 \). If \( \inf_{z \in G} |f(z)| > 0 \), show that \( f \) is constant.

9. Prove that the function \( f(z) = \frac{1}{z^2} \) cannot be uniformly approximated by polynomials on the annulus \( \text{ann}(0; 1, 2) \).

10. Let \( G \) be a region in \( \mathbb{C} \) and let \( \mathcal{A}(G) \) denote the set of analytic functions on \( G \). For any subset \( \mathcal{F} \) of \( \mathcal{A}(G) \) let \( \mathcal{F}' = \{ f' : f \in \mathcal{F} \} \). If \( \mathcal{F} \) is a normal subset of \( \mathcal{A}(G) \), prove that \( \mathcal{F}' \) is also normal.