Preliminary Examination 2000
Complex Analysis

Do all problems.

Notation.

\[ \mathbb{C} = \{ z : z \text{ is a complex number} \} \quad \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \]

1. Show that if \( u \) is a real-valued harmonic function in a domain \( \Omega \subset \mathbb{C} \) such that \( u^2 \) is harmonic in \( \Omega \), then \( u \) is constant.

2. For \(|z| < 1\) let \( f(z) = \frac{1}{1-z} \exp \left[ - \frac{1}{1-z} \right] \), and for \( 0 \leq \theta < 2\pi \) let \( \ell_\theta = \{ z : z = re^{i\theta}, 0 \leq r < 1 \} \).
   Show that \( f \) is bounded on each set \( \ell_\theta \). Is \( f \) bounded on \( \mathbb{D} \)? Explain.

3. Let \( \Omega \subset \mathbb{C} \) be the intersection of the two disks of radius 2 whose centers are at \( z = 1 \) and \( z = -1 \). Find an explicit conformal mapping of \( \Omega \) onto the upper-half plane.

4. Does there exist a function \( f \) that is analytic in a neighborhood of \( z = 0 \), for which
   (a) \( f(1/n) = f(-1/n) = 1/n^2 \) for all sufficiently large integers \( n \)?
   (b) \( f(1/n) = f(-1/n) = 1/n^3 \) for all sufficiently large integers \( n \)?
   In each case, either give an example or prove that no such function exists.

5. (a) Let \( f \) be analytic on \( \mathbb{D} \) with \( \lim_{|z| \to 1^-} f(z) = 0 \).
   Prove \( f \equiv 0 \).
   (b) Let \( g \) be analytic on \( \mathbb{D} \).
   Prove that the statement \( \lim_{|z| \to 1^-} g(z) = \infty \) is impossible.

6. Let \( f : \mathbb{D} \to \mathbb{D} \) be analytic. Suppose there exists \( z_0 \in \mathbb{D} \) with \( f(z_0) = z_0 \) and \( f'(z_0) = 1 \).
   Prove that \( f(z) \equiv z \).

7. Find all Laurent expansions of \( \frac{1}{(z-2)(z-3)} \) in powers of \( z \) and state where they converge.

8. Use the Theorem of Residues and an appropriate contour to evaluate
   \[
   \int_{-\infty}^{\infty} \frac{\sqrt{x + i}}{1 + x^2} \, dx ,
   \]
   where on \( \{ \text{Im } z > 0 \} \), we choose the branch of \( \sqrt{z + i} \) whose value at 0 is \( e^{\pi i/4} \). Describe your method carefully, and include verification of all relevant limit statements.

9. Show that there exists an unbounded analytic function \( f \) on \( \mathbb{D} \) such that
   \[
   \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < +\infty ,
   \]
   where \( dA \) is area measure on \( \mathbb{D} \).

10. Show that every function that is meromorphic on the extended complex plane is rational.