Algebra Preliminary Examination
May 2012

Work any eight problems. Clearly indicate which eight are to be graded.

1. Let $G$ be a group in which every element has order at most 2.
   (a) Show that $G$ is Abelian.
   (b) Show that the order of $G$, if finite, is a power of 2.

2. Let $p$ be a prime. Show that there is no simple group of order $8p$.

3. Let $G$ be a finite group. Show that the number of conjugacy classes in $G$ is greater than or equal to $[G : G']$, with equality if and only if $G$ is abelian. As usual, $G'$ denotes the commutator subgroup of $G$.

4. Let $G$ be a group of order $8 \cdot 7^m$ for some $m \in \mathbb{N}$.
   (a) Show that the number of 7-Sylow subgroups of $G$ is 1 or 8.
   (b) Let $\phi : G \longrightarrow \Sigma_8$ be a group homomorphism into the symmetric group on 8 letters. Show that the image of $\phi$ has strictly less than 60 elements.

5. List all prime ideals in the ring $\mathbb{Z}[x]/(30, x^2 + 1)$. List each ideal exactly once and indicate which ones are maximal.

6. Suppose that $S$ is a unique factorization domain. Let $R$ be a subring of $S$ with the following property: If $s \in S$ and $r \in R$ such that $s$ divides $r$, then $s$ is an element of $R$. Show that $R$ is a unique factorization domain.

7. Let $R$ be a commutative ring and let $\mathfrak{N}(R)$ be the set of nilpotent elements of $R$, that is
   \[
   \mathfrak{N}(R) = \{ x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{Z}^+ \} \]
   (a) Show that $\mathfrak{N}(R)$ is an ideal of $R$.
   (b) Show that 0 is the only nilpotent element of $R/\mathfrak{N}(R)$.

8. Let $R$ be a commutative ring. Assume that for any two principal ideals $(a)$ and $(b)$ in $R$ one has $(a) \subseteq (b)$ or $(b) \subseteq (a)$. Show that for any two ideals $I$ and $J$ in $R$ one has $I \subseteq J$ or $J \subseteq I$.

9. Let $K$ be a field, and let $f(x)$ and $g(x)$ be non-constant polynomials in $K[x]$ with $\gcd(f, g) = 1$. Show: For every $h(x) \in K[x]$ such that
deg(h) < deg(f) + deg(g), there exists unique \( p(x), q(x) \in K[x] \) with \( \deg(p) < \deg(f) \), \( \deg(q) < \deg(g) \) and

\[
\frac{h(x)}{f(x)g(x)} = \frac{p(x)}{f(x)} + \frac{q(x)}{g(x)}.
\]

Here \( \deg(f) \) denotes the degree of the polynomial \( f \).

10. What is the Galois group of \( p(x) = x^3 - x + 4 \), considered over the ground fields
(a) \( \mathbb{Z}/3\mathbb{Z} \),
(b) \( \mathbb{R} \),
(c) \( \mathbb{Q} \)?
Justify your answers.

11. Let \( F \) be a field, and let \( \alpha, \beta \) be algebraic over \( F \). Denote their minimal polynomials by \( \text{minpol}_\alpha, \text{minpol}_\beta \), respectively. Show that \( \text{minpol}_\alpha \) is irreducible over \( F(\beta) \) if and only if \( \text{minpol}_\beta \) is irreducible over \( F(\alpha) \).

12. Prove that the polynomial \( p(x) = x^4 - 3x^2 - 3 \) is irreducible over \( \mathbb{Q} \) and compute its Galois group.