Solve eight (8) of the twelve (12) problems below. If you provide solutions (full or partial) to more than eight problems, clearly indicate which eight should be graded.

**Group theory**

(1) Give examples of groups of the following type:
   (a) an infinite non-abelian group,
   (b) an infinite torsion group,
   (c) a subgroup that is not normal,
   (d) a subgroup that is normal,
   (e) a group homomorphism that is neither injective nor surjective.

(2) Consider the group \( GL(2, \mathbb{F}_2) \) of invertible \( 2 \times 2 \) matrices with entries from the field of two elements.
   (a) Show that this group has 6 elements.
   (b) Show that it is isomorphic to the symmetric group on 3 letters.

(3) Let \( G \) be the set of all rational functions \( p/q \) in \( \mathbb{R}(t) \) for which \( \deg(p) < \deg(q) \) (with the convention that \( \deg(0) = -\infty \)), and define an operation \( * \) by
   \[ f * g = f + g + fg. \]
   Show that \( (G, *) \) is a torsion-free Abelian group.

(4) Let \( G \) be a group, and assume that \( G/Z(G) \) is cyclic. Show that \( G \) is Abelian.

**Ring theory and modules**

(5) Let \( i = \sqrt{-1} \in \mathbb{C} \), and let \( x \) be an indeterminate.
   (a) Show that the sets \( \mathbb{Z} \times \mathbb{Z}, \mathbb{Z}[i] \), and \( \mathbb{Z}[x]/(x^2) \) considered as additive groups are isomorphic.
   (b) Show that they are pairwise non-isomorphic as rings.

(6) Let \( R \) be a commutative ring and \( M \) be a nonzero finitely generated \( R \)-module.
   (a) Prove that the set of proper submodules of \( M \) has maximal elements.
   (b) Let \( N \) be a maximal submodule of \( M \). Show that \( M/N \) is isomorphic as an \( R \)-module to \( R/m \), where \( m \subset R \) is a maximal ideal of \( R \).

(7) Let \( R \) be any ring and \( a \) be an element of \( R \).
   (a) Prove that if \( a \) has a left inverse, then \( a \) is not a left 0-divisor.
   (b) Prove that the converse holds if \( a \in aRa \).

(8) Prove that a finite ring is a field if and only if it is an integral domain.
Fields and Galois theory

(9) Let $R \subset \mathbb{Z}[x]$ be the subring of polynomials without linear or quadratic terms. Show that the field of fractions of $R$ is $\mathbb{Q}(x)$.

(10) Let $\mathbb{F}$ be a field. Let $\alpha$ and $\beta$ be algebraic over $\mathbb{F}$. Denote their minimal polynomials by $f$ and $g$ respectively.
   
   (a) Show that $g$ is irreducible over $\mathbb{F}(\alpha)$ if and only if $f$ is irreducible over $\mathbb{F}(\beta)$.
   
   (b) Let $K$ be the splitting field of $x^4 - 6$ over $\mathbb{Q}$. Determine $|K : \mathbb{Q}|$.

(11) Let $K$ and $L$ be two different $C_4$-extensions of $\mathbb{Q}$ inside $\mathbb{C}$. Assume that they have the same quadratic subextension. Find $\text{Gal}(KL/\mathbb{Q})$.

(12) Let $K$ be a field. A derivation on $K$ is a map $D: K \to K$ satisfying
   
   $D(a + b) = D(a) + D(b)$ \quad and \quad $D(ab) = D(a)b + aD(b)$
   
   for all $a, b \in K$. Show that
   
   $\text{Tr}(A^{-1}D(A)) = \frac{D(\det A)}{\det A}$
   
   for all invertible $n \times n$ matrices $A$ over $K$. (Here, $D(A)$ means that $D$ has been applied to each entry in $A$.)