Solve eight of the twelve problems below. If you provide solutions (full or partial) to more than eight problems, clearly mark which eight should be graded.

Group theory.

1. Let $G$ be a finite Abelian group. A group character of $G$ is a homomorphism $\chi$ from $G$ to the multiplicative group of non-zero complex numbers $\mathbb{C}^*$. Let $\hat{G}$ denote the set of group characters of $G$.

   (i) Show that $\hat{G}$ is a group with the operation of point-wise multiplication, i.e., $\chi_1 \chi_2$ is the map sending $g$ to $\chi_1(g)\chi_2(g)$.

   (ii) Prove that if $G$ is cyclic, then $G \cong \hat{G}$.

2. Prove that a group of order 255 is Abelian. Then use this to determine how many groups there are of order 255 (up to isomorphism).

3. Let $G$ be a finite group, and let $N$ be a normal subgroup of $G$. Let $p$ be a prime, and let $P$ denote a $p$-Sylow subgroup of $N$. Assume $P$ is normal in $N$. Show that $P$ is normal in $G$.

4. An automorphism $\varphi$ on a group $G$ is called inner, if it has the form $\varphi(g) = aga^{-1}$ for some fixed $a \in G$. Show that all automorphisms on the symmetric group $S_4$ are inner.

Ring theory and modules. All rings are assumed to be commutative, and to have an identity element 1.

5. Let $F$ be a field, and let $F[x, 1/x]$ be the ring of Laurent polynomials over $F$, i.e., polynomials in $x$ and $1/x = x^{-1}$. Show that $F[x, 1/x]$ is a principal ideal domain.

6. Let $R$ be a ring, and let $a$ and $b$ be ideals in $R$ with $a + b = R$ and $a \cap b = \{0\}$. Show that there exists an element $e \in R$ with $a = Re$, $b = R(1 - e)$, and $e^2 = e$.

7. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

8. Let $R$ be a ring. The annihilator of an $R$-module $M$ is the set

   $$\text{Ann}(M) = \{ r \in R \mid rm = 0 \text{ for all } m \in M \}.$$ 

   (i) Show that $\text{Ann}(M)$ is an ideal in $R$.

   (ii) Assume that $\text{Ann}(M) + \text{Ann}(N) = R$ for two modules $M$ and $N$. Show that $M \otimes_R N = 0$.

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Fields and Galois theory.

(9) Let $F$ be a field and let $E = F(x)$ be a purely transcendental extension of $F$. Let $u \in E \setminus F$. Prove that $[F(u) : F] = \infty$.

(10) Let $p(x) = x^4 + ax^2 + 1 \in \mathbb{Q}[x]$ and assume that $p(x)$ is irreducible over $\mathbb{Q}$. Let $\theta$ be a root of $p(x)$. Show that $\mathbb{Q}(\theta)$ is the splitting field of $p(x)$, and that both $\left(\theta + \frac{1}{\theta}\right)^2$ and $\left(\theta - \frac{1}{\theta}\right)^2$ are in $\mathbb{Q}$. Compute the Galois group of $\mathbb{Q}(\theta)/\mathbb{Q}$.

(11) Prove that an algebraically closed field must be infinite.

(12) Let $E/F$ be a Galois extension of fields with $[E : F] = 2$, where $F$ has characteristic $\neq 2$. Prove that there exists an $\alpha \in E \setminus F$ with $\alpha^2 \in F$.