Preliminary Exam: Algebra  
May 1999  

Work any 3 problems from each part.

**Part I. - Groups**

1. Suppose there is only one group (up to isomorphism) of order $n$. Show that $(n, \phi(n)) = 1$ where $\phi(n) =$ Euler phi function.


3. If a finite group is nilpotent, show that each of its Sylow subgroups is normal.

4. Let $G$ be a group, $Aut\ G$ the group of automorphisms of $G$.
   For $a \in G$, define $f_a : G \to G$ by $f_a(x) = axa^{-1}$ for all $x \in G$, and $Inn\ G = \{f_a : a \in G\}$.
   a. Show that $f_a \in Aut\ G$.
   b. Show that $Inn\ G \triangleleft Aut\ G$ and $Inn\ G \cong G/C(G)$ where $C(G)$ is the center of $G$.

**Part II - Rings and Modules**

1. Let $R$ be a commutative ring with the property that every prime ideal is maximal. Let $R[x]$ be the polynomial ring over $R$.
   Show that if $P$ is a prime ideal of $R$, then $P[x]$ is a prime ideal of $R[x]$. Also, show that no prime ideal of $R[x]$ is properly contained in $P[x]$.
2. A commutative domain $R$ is called a Dedekind ring if all ideals $I$ of $R$ are projective as $R$-modules.

Assume $R$ is a Dedekind ring. Let $A$ and $B$ be $R$-modules such that $A = B \oplus R$ as $R$-modules. Let $M$ be a submodule of $A$.
Show that there is an exact sequence
$$0 \to M \cap B \to M \to J \to 0$$
where $J$ is an ideal of $R$.

3. Let $R$ be a commutative ring with identity. If $I$ and $J$ are ideals, define
$$(I : J) = \{r \in R : rJ \subseteq I\}.$$

Show that $(I : J)$ is an ideal.

4. Prove that if $R$ is a simple ring with identity, then $M_n(R)$ is also a simple ring.

Part III - Fields and linear algebra

1. Suppose $K \subseteq L$ is a Galois field extension with $[L : K] = 20$.
Show that there exists an irreducible polynomial $f(x) \in K[x]$ of degree 5 such that $L$ contains a root of $f$.

2. Find the rational and Jordan canonical forms of the complex matrix
$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-6 & 1 & 0 & 1 \\
-3 & 0 & 1 & 0
\end{pmatrix}.$$