

# On the global stability of cholera models

Jianjun Paul Tian<sup>a</sup> and Jin Wang<sup>b</sup>

<sup>a</sup> Department of Mathematics  
College of William and Mary  
Williamsburg, VA 23187, USA  
Email: jptian@math.wm.edu

<sup>b</sup> Department of Mathematics and Statistics  
Old Dominion University  
Norfolk, VA 23529, USA  
Email: j3wang@odu.edu

## Abstract

We conduct global stability analysis for the endemic equilibria of several deterministic cholera models. These models, incorporating both human population and pathogen concentration, constitute four-dimensional nonlinear autonomous systems where the classical Poincaré-Bendixson theory is not applicable. We employ three different techniques, including the monotone dynamical systems, geometric approach, and Lyapunov functions, to investigate the endemic global stability for several biologically important cases. The analysis and results presented in this paper make building blocks towards a comprehensive study of the general mathematical cholera model.

*Keywords:* cholera model, global stability, dynamical systems

## 1 Introduction

Cholera, characterized by severe diarrhea and rapid dehydration, is a water-borne infectious disease caused by the bacterium *Vibrio cholerae*. Despite many clinical and theoretical studies [1, 14, 18, 31, 34, 35, 47] and tremendous administrative efforts [52], cholera remains a significant threat to public health in the developing countries [50–52].

Understanding of the fundamental mechanism in the disease transmission is crucial for effective prevention and intervention strategies against cholera outbreak. To achieve this goal, many mathematical models have been proposed to investigate the complex epidemic and endemic behavior of cholera. The difficulty in studying cholera dynamics stems from the coupling between its multiple transmission pathways which involve both direct human-to-human and indirect environment-to-human modes and which lead to combined human-environment epidemiological models. Representative works on mathematical cholera modeling include those by Capasso and Paveri-Fontana [4], Pourabbas *et al.* [36], Codeço [6], Ghosh *et al.* [11], Hartley *et al.* [12], and Mukandavire *et al.* [32].

Most recently, Wang and Liao [48] proposed a generalized cholera model which unifies previous mathematical models by introducing a general formulation for the incidence and

pathogen concentration. The model consists of the following differential equations:

$$\frac{dS}{dt} = bN - Sf(I, B) - bS, \quad (1.1)$$

$$\frac{dI}{dt} = Sf(I, B) - (\gamma + b)I, \quad (1.2)$$

$$\frac{dR}{dt} = \gamma I - bR, \quad (1.3)$$

$$\frac{dB}{dt} = h(I, B). \quad (1.4)$$

Here  $S$ ,  $I$  and  $R$  denote the susceptible, the infected, and the recovered populations, respectively, and  $B$  denotes the concentration of the vibrios in the contaminated water. The total population  $N = S + I + R$  is assumed to be a constant. The parameter  $b$  represents the natural human birth/death rate, and  $\gamma$  represents the rate of recovery from cholera. The function  $f(I, B)$  represents the incidence which determines the rate of new infection, and the function  $h(I, B)$  describes the rate of change for the pathogen in the environment.

Under biologically reasonable conditions for  $f(I, B)$  and  $h(I, B)$ , it is shown in [48] that there exists a forward transcritical bifurcation at  $R_0 = 1$  for this model. Specifically, the following theorem summarizes the dynamics known for the system (1.1)-(1.4).

**Theorem 1.1** [48] *The basic reproduction number of the model (1.1)-(1.4) is*

$$R_0 = \frac{N}{\gamma + b} \left[ \frac{\partial f}{\partial I}(0, 0) - \frac{\partial f}{\partial B}(0, 0) \left( \frac{\partial h}{\partial B}(0, 0) \right)^{-1} \frac{\partial h}{\partial I}(0, 0) \right]. \quad (1.5)$$

*When  $R_0 < 1$ , there is a unique disease-free equilibrium (DFE) which is both locally and globally asymptotically stable; when  $R_0 > 1$ , the DFE becomes unstable, and there is a unique positive endemic equilibrium which is locally asymptotically stable.*

The global stability of the endemic equilibrium, however, remains unresolved for the system (1.1)-(1.4). In fact, to our knowledge, none of the previous works on cholera modeling have addressed the question of this kind. The study of the endemic global stability is not only mathematically important, but also essential in predicting the long-term behavior of the disease. The challenge, however, in the global analysis of cholera models is that due to the incorporation of the environmental component  $B$ , the models usually constitute three-dimensional nonlinear autonomous systems for which the classical Poincaré-Bendixson theory [13] is no longer valid. Hence, other analytical tools must be employed, and possibly new methods need to be created, to overcome this challenge.

As a step towards completely answering this difficult question, we apply three established methods, i.e., those based on the monotone dynamical systems [22, 24, 44–46], geometric approach [9, 25, 27], and Lyapunov functions [17, 19], to conduct global stability analysis for several cholera models in this paper. The theory of monotone dynamical systems and geometric approach is relative new compared to the Poincaré-Bendixson framework. Meanwhile, although the method of Lyapunov functions has been widely applied to various dynamical systems, the essential part of our analysis is based on the less well known results of Volterra-Lyapunov stable matrices [40–42]. The models investigated in this paper represent several

important, and nontrivial, choices of the incidence  $f(I, B)$  in the most general model (1.1)-(1.4). These include the cases when  $f$  is bilinear in  $I$  and  $B$  due to the standard mass action law, when  $f$  is only depending on  $B$  in a nonlinear manner so that human-to-human transmission is not present, and when  $f$  has a linear dependence on  $I$  and a nonlinear dependence on  $B$ . We have found that it is convenient and illustrative (and, in some case, necessary) to employ different approaches to deal with these different situations. The analysis and results presented in this paper can be viewed as building blocks towards a comprehensive study for the global dynamics of the general model (1.1)-(1.4).

We organize this paper as follows. In Section 2, we apply the theory of monotone dynamical systems to analyze models with nonlinear environment-dependent-only incidence, where the disease transmission is characterized solely by the environment-to-human pathway. A typical example is given by Codeço's model [6]. In Section 3, we apply the geometric approach to investigate models with incidence depending linearly on human and nonlinearly on environment, which in general do not satisfy the requirement of monotone systems. A representative example is the model of Mukandavire *et al.* [32]. In Section 4, we consider models with bilinear incidence but with a general nonlinear representation of the pathogen growth rate. Such models are neither monotone nor can be easily analyzed by the geometric approach. Fortunately, the method of Lyapunov functions combined with the Volterra-Lyapunov matrix properties lead to the proof of the endemic global stability. Finally, we close the paper by conclusions and discussion.

## 2 Incidence with environment-to-human transmission only

We first consider the following model

$$\frac{dS}{dt} = bN - Sf(B) - bS, \quad (2.1)$$

$$\frac{dI}{dt} = Sf(B) - (\gamma + b)I, \quad (2.2)$$

$$\frac{dB}{dt} = eI - mB, \quad (2.3)$$

with the incidence function  $f(I, B) = f(B)$  depending only on the environmental factor  $B$ , and  $h(I, B) = eI - mB$  being linear. For convenience of discussion, we have dropped the equation for  $R$ , i.e., (1.3), since  $S(t) + I(t) + R(t) = N$  and  $R$  is not independent. Here the parameter  $e$  represents the rate of contribution (e.g., shedding) to *V. cholerae* and  $m$  represents the net death rate of vibrios in the environment. The incidence function satisfies (see [48])

$$f(0) = 0, \quad f' \geq 0, \quad \text{and} \quad f'' \leq 0. \quad (2.4)$$

Note that

$$B' = eI - mB \leq eN - mB.$$

It is easy to obtain that  $B(t) \leq \frac{eN}{m}$ . Thus we will study system (2.1)-(2.3) in the feasible region

$$\Delta = \left\{ (S, I, B) \mid S \geq 0, I \geq 0, 0 \leq S + I \leq N, 0 \leq B \leq \frac{eN}{m} \right\}. \quad (2.5)$$

It can be easily verified that  $\Delta$  is positively invariant.

The result below directly follows Theorem 1.1.

**Theorem 2.1** *The basic reproduction number of the model (2.1)-(2.3) is*

$$R_0 = \frac{N}{\gamma + b} f'(0) \frac{e}{m}. \quad (2.6)$$

If  $R_0 < 1$ , there is only one nonnegative equilibrium point  $X_0 = (N, 0, 0)$ , which is the disease free equilibrium, and it is globally asymptotically stable. If  $R_0 > 1$ , there are two nonnegative equilibria,  $X_0$  and the endemic equilibrium  $X^* = (S^*, I^*, B^*)$ , where  $X_0$  is unstable and  $X^*$  is locally asymptotically stable.

In order to show the global stability of the endemic equilibrium  $X^*$ , we will use a method based on monotone dynamical systems, as developed in [24]. Below we briefly review this method.

Given a  $C^1$  function  $x \mapsto F(x) \in \mathbb{R}^n$  for  $x$  in a bounded convex open set  $D \subset \mathbb{R}^n$ . Define the differential equation

$$\frac{dx}{dt} = F(x). \quad (2.7)$$

Denote by  $x(t, x_0)$  the solution of (2.7) such that  $x(0, x_0) = x_0$ . A subset  $K$  is said to be *absorbing* in  $D$  if  $x(t, K_1) \subset K$  for any compact subset  $K_1 \subset D$  and sufficiently large  $t$ . To study the global stability of an equilibrium solution,  $\bar{x}$ , we assume

**(H1)** There exists a compact absorbing set  $K \subset D$ .

**(H2)** The system (2.7) has a unique equilibrium point  $\bar{x}$  in  $D$ .

The system (2.7) is said to be *uniformly persistent* if there exists a constant  $c > 0$  such that each component of any solution  $x(t)$  with  $x(0) = x_0 \in D$  satisfies

$$\liminf_{t \rightarrow \infty} x_1(t) > c, \quad \liminf_{t \rightarrow \infty} x_2(t) > c, \quad \dots, \quad \liminf_{t \rightarrow \infty} x_n(t) > c. \quad (2.8)$$

The boundedness of  $D$  and uniform persistence imply that the system has a compact absorbing subset of  $D$  [3].

The system (2.7) is called *competitive* if there exists a diagonal matrix  $H$  with entries  $\pm 1$  such that each off-diagonal entry of  $H \frac{\partial F}{\partial x} H$  is nonpositive in  $D$ , where  $\frac{\partial F}{\partial x}$  is the variational matrix of (2.7). It is known that three-dimensional competitive systems have the Poincaré-Bendixson property:

**Theorem 2.2** [44] *For a competitive system defined in a three-dimensional convex open domain, if a nonempty compact  $\omega$ -limit set contains no equilibria, then it is a closed orbit.*

We recall here basic definitions related to orbital stability of a periodic orbit [7]. Suppose (2.7) has a periodic solution  $x = p(t)$  with the least period  $\omega > 0$  and orbit  $\gamma = \{p(t) \mid 0 \leq t \leq \omega\}$ . This orbit is *orbitally stable* if for each  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that any solution  $x(t)$ , for which the distance of  $x(0)$  from  $\gamma$  is less than  $\delta$ , remains at a distance less

than  $\varepsilon$  from  $\gamma$  for all  $t \geq 0$ . It is *asymptotically orbitally stable* if the distance of  $x(t)$  from  $\gamma$  also tends to zero as  $t \rightarrow \infty$ . The orbit  $\gamma$  is asymptotically orbitally stable with *asymptotic phase* if it is asymptotically orbitally stable and there is an  $\eta > 0$  such that, any solution  $x(t)$ , for which the distance of  $x(0)$  from  $\gamma$  is less than  $\eta$ , satisfies  $|x(t) - p(t - \tau)| \rightarrow 0$  as  $t \rightarrow \infty$  for some  $\tau$  which may depend on  $x(0)$ . We now state a criterion given in [33] for the asymptotic orbital stability of a periodic orbit of (2.7).

**Theorem 2.3** *A sufficient condition for a periodic orbit  $\gamma = \{p(t) | 0 \leq t \leq \omega\}$  of (2.7) to be asymptotically orbitally stable with asymptotic phase is that the linear system*

$$\frac{dz}{dt} = \left( \frac{\partial F^{[2]}}{\partial x}(p(t)) \right) z(t) \quad (2.9)$$

*is asymptotically stable, where  $\frac{\partial F^{[2]}}{\partial x}$  is the second compound matrix of the Jacobian  $\frac{\partial F}{\partial x}$ .*

Recall for a  $n \times n$  matrix  $A$  and integer  $1 \leq k \leq n$ , the  $k$ th *additive compound matrix* of  $A$ , denoted by  $A^{[k]}$ , is defined by

$$A^{[k]} = D_+(I + hA)^{(k)}|_{h=0}, \quad (2.10)$$

where  $(I + hA)^{(k)}$  is the  $k$ th exterior power of  $(I + hA)$ , and  $D_+$  is the corresponding right-hand derivative [33].

Then we state a theorem implicitly given in [24].

**Theorem 2.4** *Assume that*

- (1) *conditions (H1) and (H2) hold;*
- (2)  *$\bar{x}$  is locally asymptotically stable;*
- (3) *the system (2.7) satisfies the Poincaré-Bendixson Property;*
- (4) *each periodic orbit of (2.7) in  $D$  is orbitally asymptotically stable.*

*Then the unique equilibrium  $\bar{x}$  is globally asymptotically stable in  $D$ .*

We now examine the system (2.1)-(2.3). We have two propositions which imply that the system is persistent.

**Proposition 2.5** *The disease-free equilibrium point  $E_0$  is the only  $\omega$ -limit point of the system (2.1)-(2.3) on the boundary,  $\partial\Delta$ , of  $\Delta$ .*

**Proof** The boundary of  $\Delta$  has 5 faces. In each face, the vector field of the system (2.1)-(2.3) is transversal to it. For example, in the face of  $\{(S, I, B) | S + I = N, 0 \leq B \leq \frac{eN}{m}\}$ , the vectors point to inside of  $\Delta$ . Meanwhile,  $\partial\Delta$  has 9 edges. Except for the edge on the S-axis, for other edges the vector fields are transversal to them. On the S-axis, the system reduces to  $\frac{dS}{dt} = bN - bS$ , since  $f(0) = 0$ . We have  $S(t) \rightarrow N$  as  $t \rightarrow +\infty$ . Thus,  $X_0$  is the only  $\omega$ -limit point of the system (2.1)-(2.3) on the boundary  $\partial\Delta$ . ■

**Proposition 2.6** When  $R_0 > 1$ ,  $X_0$  cannot be the  $\omega$ -limit point of any orbit starting in the interior,  $\Delta^\circ$ , of  $\Delta$ .

**Proof** We define a function

$$L = eI + (\gamma + b)B \geq 0.$$

We consider a small neighborhood of  $X_0$  in  $\Delta^\circ$  such that  $B > 0$  is sufficiently small and  $S > 0$  is sufficiently close to  $N$ . In this neighborhood the orbital derivative of  $L$  is

$$\begin{aligned} L' &= eI' + (\gamma + b)B' = eSf(B) - m(\gamma + b)B \\ &= m(\gamma + b) \left[ \frac{eSf(B)}{m(\gamma + b)} - B \right] \geq m(\gamma + b) \left[ \frac{e}{m} \frac{S}{\gamma + b} f'(0)B - B \right] \\ &= Bm(\gamma + b) \left[ \frac{e}{m} f'(0) \frac{N}{\gamma + b} \frac{S}{N} - 1 \right] > 0, \end{aligned}$$

where we have used the facts  $R_0 = \frac{e}{m} \frac{N}{\gamma + b} f'(0)$  and  $f(B) = f(0) + f'(0)B + f''(0)B^2 + \dots \geq f'(0)B$ , and  $f(0) = 0, f' \geq 0, f'' \leq 0$ ,  $B$  is positive but small. Therefore,  $X_0 = (N, 0, 0)$  cannot be the  $\omega$ -limit point of any orbit starting in  $\Delta^\circ$ . ■

The following theorem is immediately obtained based on Propositions 2.5 and 2.6.

**Theorem 2.7** The system (2.1)-(2.3) is uniformly persistent.

The variational matrix of the system (2.1)-(2.3) is given by

$$J = \begin{pmatrix} -f(B) - b & 0 & -Sf'(B) \\ f(B) & -(\gamma + b) & Sf'(B) \\ 0 & e & -m \end{pmatrix}.$$

If we set  $H = \text{diag}\{1, -1, 1\}$ , then  $HJH$  has nonpositive off-diagonal entries. Hence, it is a three-dimensional competitive system which possesses the Poincaré-Bendixson property [44]. We have the following theorem.

**Theorem 2.8** Any compact  $\omega$ -limit set of the system (2.1)-(2.3) in the interior of  $\Delta$  is either a closed orbit or the endemic equilibrium  $X^*$ .

**Proof** Suppose  $\gamma$  is an  $\omega$ -limit set of the system (2.1)-(2.3) in the interior of  $\Delta$ . If  $\gamma$  does not contain  $X^*$ , then it contains no equilibria since  $X^*$  is the only interior equilibrium point. Theorem 2.2 implies that  $\gamma$  is a closed orbit. If, instead,  $\gamma$  contains  $X^*$ , then any orbit that gets arbitrarily close to  $X^*$  will converge to  $X^*$  since  $X^*$  is locally asymptotically stable. Thus  $\gamma = X^*$ . ■

A closed orbit corresponds to a periodic solution. If it exists for the system (2.1)-(2.3), it will be stable. Specifically, we have the following result.

**Theorem 2.9** The trajectory of any nonconstant periodic solution of the system (2.1)-(2.3), if it exists, is asymptotically orbitally stable with asymptotic phase.

**Proof** The second compound matrix of the system (2.1)-(2.3) is given by

$$J^{[2]} = \begin{pmatrix} -2b - \gamma - f(B) & Sf'(B) & Sf'(B) \\ e & -b - m - f(B) & 0 \\ 0 & f(B) & -b - m - \gamma \end{pmatrix}.$$

Then the second compound system defined along the periodic solution  $(S(t), I(t), B(t))$  of the system (2.1)-(2.3) is given by

$$X'(t) = -(2b + \gamma + f(B))X + Sf'(B)(Y + Z) \quad (2.11)$$

$$Y'(t) = eX - (b + m + f(B))Y \quad (2.12)$$

$$Z'(t) = f(B)Y - (b + m + \gamma)Z \quad (2.13)$$

Based on Theorem 2.3, if we can prove the system (2.11)-(2.13) is asymptotically stable, then the periodic solution is asymptotically orbitally stable with asymptotic phase.

We define a Lyapunov function

$$V(X, Y, Z, S, I, B) = \sup \left\{ |X|, \frac{I}{B}(|Y| + |Z|) \right\}.$$

Since the system (2.1)-(2.3) is persistent, any periodic solution  $(S(t), I(t), B(t))$  is at a positive distance from the boundary  $\partial\Delta$ . So  $\frac{I}{B}$  is well-defined, and there is a constant  $c > 0$  such that  $\frac{I}{B} > c$ . Hence, for some positive constant  $c_0$ , we have

$$V(X, Y, Z, S, I, B) \geq c_0 \sup\{|X|, |Y|, |Z|\},$$

for any  $(X, Y, Z) \in \mathbb{R}^3$  and any periodic solution  $(S(t), I(t), B(t))$  of the system (2.1)-(2.3).

Let us estimate the right-derivative of  $V$  along a solution  $(X(t), Y(t), Z(t))$  of the system (2.11)-(2.13) and  $(S(t), I(t), B(t))$  of the system (2.1)-(2.3).

$$\begin{aligned} D_+|X(t)| &\leq -(2b + \gamma + f(B))|X| + \frac{Sf'(B)B}{I} \frac{I}{B}(|Y| + |Z|), \\ D_+|Y(t)| &\leq e|X| - (b + m + f(B))|Y|, \\ D_+|Z(t)| &\leq f(B)|Y| - (b + m + \gamma)|Z|, \end{aligned}$$

and

$$\begin{aligned} &D_+ \frac{I}{B}(|Y| + |Z|) \\ &= \left(\frac{I'}{I} - \frac{B'}{B}\right) \frac{I}{B}(|Y| + |Z|) + \frac{I}{B} D_+(|Y| + |Z|) \\ &\leq \left(\frac{I'}{I} - \frac{B'}{B}\right) \frac{I}{B}(|Y| + |Z|) + \frac{I}{B}(e|X| - (b + m)(|Y| + |Z|)) \\ &= e \frac{I}{B}|X| + \left(\frac{I'}{I} - \frac{B'}{B} - (b + m)\right) \frac{I}{B}(|Y| + |Z|). \end{aligned}$$

We then need to estimate the following two functions

$$\begin{aligned} g_1(t) &= -2b - \gamma - f(B) + \frac{Sf'(B)B}{I}, \\ g_2(t) &= e \frac{I}{B} + \frac{I'}{I} - \frac{B'}{B} - (b + m). \end{aligned}$$

From the system (2.1)-(2.3), we have  $\frac{B'}{B} = e\frac{I}{B} - m$ . Then

$$g_2(t) = \frac{I'}{I} - b.$$

Similarly,  $\frac{I'}{I} = \frac{Sf(B)}{I} - (\gamma + b)$ . Then  $\frac{S}{I} = (\frac{I'}{I} + (\gamma + b))\frac{1}{f(B)}$ . Since  $f(0) = 0$ ,  $f'' \leq 0$  (which implies  $f'$  is decreasing), we obtain  $\frac{f(B)}{B} = \frac{f(B)-f(0)}{B-0} = f'(\eta) > f'(B)$ , where  $0 < \eta < B$ . Thus  $f(B) > f'(B)B$ , and

$$\begin{aligned} g_1(t) &= -2b - \gamma - f(B) + f'(B)B\left(\frac{I'}{I} + (\gamma + b)\right)\frac{1}{f(B)} \\ &= -2b - \gamma - f(B) + \frac{f'(B)B}{f(B)}\left(\frac{I'}{I} + (\gamma + b)\right) \\ &\leq -2b - \gamma - f(B) + \frac{I'}{I} + (\gamma + b) \\ &= -b - f(B) + \frac{I'}{I} \leq \frac{I'}{I} - b. \end{aligned}$$

Therefore,

$$D_+V(t) \leq \max\{g_1(t), g_2(t)\}V(t), \quad (2.14)$$

and

$$\max\{g_1(t), g_2(t)\} \leq \frac{I'}{I} - b. \quad (2.15)$$

Denote the period of the periodic solution  $(S(t), I(t), B(t))$  by  $\tau$ . We have

$$\int_0^\tau \max\{g_1(t), g_2(t)\} dt \leq \int_0^\tau \left(\frac{I'}{I} - b\right) dt = \ln I(t)\Big|_0^\tau - b\tau = -b\tau < 0. \quad (2.16)$$

Thus, the system (2.11)-(2.13) is asymptotically stable. Then, the periodic solution  $(S(t), I(t), B(t))$  is asymptotically orbitally stable with asymptotic phase. ■

Summing up these results together, we have

**Theorem 2.10** *The endemic equilibrium  $X^*$  of the system (2.1)-(2.3) is globally asymptotically stable in  $\Delta$ .*

The proof follows Theorem 2.4 by combining Theorems 2.1, 2.7 and 2.9.

A typical example with nonlinear environment-dependent-only incidence is Codeço's model [6]:

$$\frac{dS}{dt} = n(H - S) - a\frac{B}{K + B}S, \quad (2.17)$$

$$\frac{dI}{dt} = a\frac{B}{K + B}S - rI, \quad (2.18)$$

$$\frac{dB}{dt} = eI - (mb - nb)B, \quad (2.19)$$



where, in their original notations [6],  $H$  stands for the total human population,  $n$  denotes the natural human birth/death rate,  $r$  corresponds to  $\gamma + b$  in equation (2.2), and  $mb - nb > 0$  represents the net death rate of vibrios. The nonlinear incidence is  $f(B) = a \frac{B}{K+B}$  with  $a$  being the contact rate with contaminated water and  $K$  the half saturation rate (i.e.,  $ID_{50}$ , the infectious dose in water sufficient to produce disease in 50% of those exposed). The local stability of the endemic equilibrium for this model was originally analyzed in [6], and can also be obtained from Theorem 1.1 as a special case, whereas the global endemic stability is established by Theorem 2.10.

### 3 Incidence depending linearly on human and nonlinearly on environment

Next, we consider models with incidence depending linearly on  $I$  and nonlinearly on  $B$ . A representative example in this category is the model of Mukandavire *et al.* [32] which takes the form

$$\frac{dS}{dt} = \mu N - \beta_1 \frac{SB}{K+B} - \beta_2 SI - \mu S, \quad (3.1)$$

$$\frac{dI}{dt} = \beta_1 \frac{SB}{K+B} + \beta_2 SI - (\gamma + \mu)I, \quad (3.2)$$

$$\frac{dB}{dt} = \xi I - \delta B. \quad (3.3)$$

Here we have kept the original notations in [32], with  $\mu$  denoting the natural human birth/death rate,  $\xi$  representing the rate of contribution to *V. cholerae*, and  $\delta$  representing the net death rate of vibrios. The parameters  $\beta_1$  and  $\beta_2$  are rates of ingesting vibrios from contaminated water and through human-to-human interaction, respectively. The parameter  $K$  is the same as that defined in Codeço's model (2.17)-(2.19). Note that, similar to the model (2.1)-(2.3), the environmental function  $h(I, B) = \xi I - \delta B$  is linear. We have also dropped the equation for  $R$  from the above system:

$$\frac{dR}{dt} = \gamma I - \mu R. \quad (3.4)$$

Using the same argument as before, it is clear to see the region

$$\Delta = \left\{ (S, I, B) \mid S \geq 0, I \geq 0, 0 \leq S + I \leq N, 0 \leq B \leq \frac{\xi}{\delta} N \right\} \quad (3.5)$$

is a positive invariant domain of the system (3.1)-(3.3).

The result below follows Theorem 1.1 and is similar to Theorem 2.1.

**Theorem 3.1** *The basic reproduction number of the model (3.1)-(3.3) is*

$$R_0 = \frac{N}{\gamma + \mu} \left( \beta_1 \frac{\xi}{K\delta} + \beta_2 \right). \quad (3.6)$$

*If  $R_0 < 1$ , there is only one nonnegative equilibrium point  $X_0 = (N, 0, 0)$ , which is the disease free equilibrium, and it is globally asymptotically stable. If  $R_0 > 1$ , there are two nonnegative equilibria,  $X_0$  and the endemic equilibrium  $X^* = (S^*, I^*, B^*)$ , where  $X_0$  is unstable and  $X^*$  is locally asymptotically stable.*

Similar to the model (2.1)-(2.3), the system (3.1)-(3.3) is uniformly persistent which can be derived from the following two propositions.

**Proposition 3.2** *The disease-free equilibrium point  $X_0$  is the only  $\omega$ -limit point of the system (3.1)-(3.3) on the boundary  $\partial\Delta$  of  $\Delta$ .*

We skip the proof since it is very similar to that of Proposition 2.5.

**Proposition 3.3** *When  $R_0 > 1$ ,  $X_0$  cannot be the  $\omega$ -limit point of any orbit starting in the interior  $\Delta^\circ$  of  $\Delta$ .*

**Proof** Take the initial value  $(S_0, I_0, B_0)$  close to  $X_0 = (N, 0, 0)$ . If  $B' > 0$ , then  $B > 0$  and is increasing, thus moving away from  $X_0$ . If, instead,  $B' \leq 0$ , then  $B \geq \frac{\xi}{\delta}I$ . Assuming  $B$  is small, we have

$$\begin{aligned} \frac{dI}{dt} &= \beta_1 \frac{SB}{K} \frac{1}{1+B/K} + \beta_2 SI - (\gamma + \mu)I \doteq \frac{\beta_1}{K} SB + \beta_2 SI - (\gamma + \mu)I \\ &\geq \frac{\beta_1 \xi}{K \delta} SI + \beta_2 SI - (\gamma + \mu)I = (\gamma + \mu) \left[ \frac{N}{\gamma + \mu} \left( \frac{\beta_1 \xi}{K \delta} + \beta_2 \right) \frac{S}{N} - 1 \right] I \\ &= (\gamma + \mu) \left[ R_0 \frac{S}{N} - 1 \right] I > 0, \end{aligned}$$

as long as  $S$  is close to  $N$ . Therefore, the trajectory always moves away from  $X_0$ . ■

Combining these two propositions and Theorem 3.1, we obtain

**Corollary 3.4** *The system (3.1)-(3.3) is uniformly persistent, and satisfies the assumptions (H1) and (H2).*

It can be easily verified, however, that the model (3.1)-(3.3) is not monotone or competitive due to the incidence depending on both  $B$  and  $I$ . Thus the analysis conducted in the previous section cannot be extended to this case. Instead, we employ the geometric approach [9, 25, 27] to analyze the global stability of this model.

To that end, we first recall a Bendixson criterion in  $\mathbb{R}^n$  developed in [23, 27]. Consider the system (2.7). A Bendixson criterion is a condition satisfied by  $F$  which precludes the existence of nonconstant periodic solutions. For any solution  $x(t, x_0)$  in  $D$ , define the second compound equation

$$\frac{dz}{dt} = \frac{\partial F^{[2]}}{\partial x}(x(t, x_0)) z(t). \quad (3.7)$$

If  $D$  is simply connected, the uniformly asymptotical stability of solutions of (3.7) and uniformly exponential decay of solutions with respect to initial values in each compact subset of  $D$  preclude the existence of any invariant simple closed rectifiable curve of the system (2.7) in  $D$ . A very useful criterion is given in [27] to characterize this stability, which is a Bendixson criterion for high dimensional systems. The detail is provided below.

Let  $x \mapsto P(x)$  be a  $\binom{n}{2} \times \binom{n}{2}$  matrix-valued  $C^1$  function in  $D$ . Assume  $P^{-1}$  exists and is continuous in a compact subset  $K$  of  $D$ . Set

$$Q = P_F P^{-1} + P \frac{\partial F^{[2]}}{\partial x} P^{-1}, \quad (3.8)$$

where  $P_F$  is the derivative of  $P$  (entry-wise) along the direction of  $F$ . Let  $m(Q)$  be the Lozinskii measure of  $Q$  with respect to a matrix norm [7], i.e.,

$$m(Q) = \lim_{h \rightarrow 0^+} \frac{|I + hQ| - 1}{h}. \quad (3.9)$$

Define a quantity  $\bar{q}_2$  as

$$\bar{q}_2 = \limsup_{t \rightarrow \infty} \sup_{x_0 \in K} \frac{1}{t} \int_0^t m(Q(x(s, x_0))) ds. \quad (3.10)$$

Then the Bendixson criterion is given by

$$\bar{q}_2 < 0. \quad (3.11)$$

Recall that a local  $\epsilon$ -perturbation of  $F$  in a neighborhood  $U$  of  $x_1 \in D$  is a function  $g \in C^1(D \rightarrow \mathbb{R}^n)$  such that the support,  $\text{Supp}(F - g) \subset U$ , and  $|F - g|_{C^1} < \epsilon$ , where

$$|F - g|_{C^1} = \sup \left\{ |F(x) - g(x)| + \left| \frac{\partial F}{\partial x}(x) - \frac{\partial g}{\partial x}(x) \right| : x \in D \right\}. \quad (3.12)$$

A Bendixson criterion is said to be *robust under  $C^1$  local perturbations of  $F$*  if for each local  $\epsilon$ -perturbation  $g$  of  $F$  at  $x_1 \in D$ ,  $g$  also satisfies the Bendixson criterion. A point  $x_0 \in D$  is *wandering* for (2.7) if there exists a neighborhood  $U$  of  $x_0$  and  $T > 0$  such that  $U \cap x(t, U)$  is empty for all  $t > T$ . For example, all equilibria and limit points are non-wandering.

Now we state the closing lemma of Pugh [37–39].

**Lemma 3.5** *Let  $F \in C^1(D \rightarrow \mathbb{R}^n)$ . Suppose that  $x_0$  is a non-wandering point of (2.7) and that  $F(x_0) \neq 0$ . Also assume that the positive semi-orbit of  $x_0$  has compact closure. Then, for each neighborhood  $U$  of  $x_0$  and  $\epsilon > 0$ , there exists a  $C^1$  local  $\epsilon$ -perturbation  $g$  of  $F$  at  $x_0$  such that*

- (1)  $\text{Supp}(F - g) \subset U$ ; and
- (2) the perturbed system  $\frac{dx}{dt} = g(x)$  has a nonconstant periodic solution whose trajectory passes through  $x_0$ .

Using the closing lemma, the following two theorems were established in [25].

**Theorem 3.6** *Suppose that assumptions (H1) and (H2) hold, and assume that (2.7) satisfies a Bendixson criterion which is robust under  $C^1$  local perturbations of  $F$  at all non-equilibrium non-wandering points for (2.7). Then the unique equilibrium  $\bar{x}$  is globally stable in  $D$  provided it is locally asymptotically stable.*

**Theorem 3.7** *Assume that  $D$  is simply connected and the assumptions (H1) and (H2) hold. Then the unique equilibrium  $\bar{x}$  of (2.7) is globally stable in  $D$  if  $\bar{q}_2 < 0$ .*

We now apply these theorems to our model (3.1)-(3.3). Based on Corollary 3.4, we only need to check the Bendixson criterion  $\bar{q}_2 < 0$ .

The Jacobian matrix of the system (3.1)-(3.3) is

$$J = \begin{pmatrix} -\frac{\beta_1 B}{K+B} - \beta_2 I - \mu & -\beta_2 S & -\frac{\beta_1 K S}{(K+B)^2} \\ \frac{\beta_1 B}{K+B} + \beta_2 I & \beta_2 S - (\gamma + \mu) & \frac{\beta_1 K S}{(K+B)^2} \\ 0 & \xi & -\delta \end{pmatrix}.$$

The second compound matrix of the system (3.1)-(3.3) is

$$J^{[2]} = \begin{pmatrix} -\frac{\beta_1 B}{K+B} - \beta_2 I + \beta_2 S - \gamma - 2\mu & \frac{\beta_1 K S}{(K+B)^2} & \frac{\beta_1 K S}{(K+B)^2} \\ \xi & -\frac{\beta_1 B}{K+B} - \beta_2 I - \mu - \delta & -\beta_2 S \\ 0 & \frac{\beta_1 B}{K+B} + \beta_2 I & \beta_2 S - \gamma - \mu - \delta \end{pmatrix}.$$

We set the matrix function  $P$  by

$$P(S, I, B) = \text{diag}\left\{1, \frac{I}{B}, \frac{I}{B}\right\}.$$

Then

$$P_F P^{-1} = \text{diag}\left\{0, \frac{I'}{I} - \frac{B'}{B}, \frac{I'}{I} - \frac{B'}{B}\right\},$$

and

$$P J^{[2]} P^{-1} = \begin{pmatrix} -\frac{\beta_1 B}{K+B} - \beta_2 I + \beta_2 S - \gamma - 2\mu & \frac{\beta_1 K S}{(K+B)^2} \frac{B}{I} & \frac{\beta_1 K S}{(K+B)^2} \frac{B}{I} \\ \xi \frac{I}{B} & -\frac{\beta_1 B}{K+B} - \beta_2 I - \mu - \delta & -\beta_2 S \\ 0 & \frac{\beta_1 B}{K+B} + \beta_2 I & \beta_2 S - \gamma - \mu - \delta \end{pmatrix}.$$

The matrix  $P_F P^{-1} + P J^{[2]} P^{-1}$  defined in (3.8) can then be written in a block form:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

with

$$Q_{11} = -\frac{\beta_1 B}{K+B} - \beta_2 I + \beta_2 S - \gamma - 2\mu, \quad Q_{12} = \left[ \frac{\beta_1 K S}{(K+B)^2} \frac{B}{I}, \frac{\beta_1 K S}{(K+B)^2} \frac{B}{I} \right],$$

$$Q_{21} = \left[ \begin{array}{c} \xi \frac{I}{B} \\ 0 \end{array} \right], \quad Q_{22} = \left[ \begin{array}{cc} -\frac{\beta_1 B}{K+B} - \beta_2 I - \mu - \delta + \frac{I'}{I} - \frac{B'}{B} & -\beta_2 S \\ \frac{\beta_1 B}{K+B} + \beta_2 I & \beta_2 S - \gamma - \mu - \delta + \frac{I'}{I} - \frac{B'}{B} \end{array} \right].$$

Now we define a norm in  $\mathbb{R}^3$  as

$$|(u, v, w)| = \max\{|u|, |v| + |w|\}$$

for any vector  $(u, v, w) \in \mathbb{R}^3$ . Let  $m$  denote the Lozinskii measure with respect to this norm. We can then obtain

$$m(Q) \leq \sup\{g_1, g_2\}, \quad (3.13)$$

with

$$\begin{aligned} g_1 &= m_1(Q_{11}) + |Q_{12}|, \\ g_2 &= |Q_{21}| + m_1(Q_{22}), \end{aligned}$$

where  $|Q_{12}|$  and  $|Q_{21}|$  are matrix norms induced by the  $L_1$  vector norm, and  $m_1$  denotes the Lozinskii measure with respect to the  $L_1$  norm. Specifically,

$$m_1(Q_{22}) = \frac{I'}{I} - \frac{B'}{B} - \mu - \delta + \sup\{2\beta_2 S - \gamma, 0\},$$

and

$$\begin{aligned} g_2 &= \frac{I'}{I} - \frac{B'}{B} - \mu - \delta + \sup\{2\beta_2 S - \gamma, 0\} + \xi \frac{I}{B} \\ &= \frac{I'}{I} - \mu + \sup\{2\beta_2 S - \gamma, 0\} \\ &\leq \frac{I'}{I} - \mu, \end{aligned}$$

provided that

$$N \leq \frac{\gamma}{2\beta_2}.$$

Meanwhile,

$$\begin{aligned} g_1 &= -\frac{\beta_1 B}{K+B} - \beta_2 I + \beta_2 S - \gamma - 2\mu + \frac{\beta_1 K S}{(K+B)^2} \frac{B}{I} \\ &= -\frac{\beta_1 B}{K+B} - \beta_2 I + \beta_2 S - \gamma - 2\mu + \frac{K}{K+B} \left( \frac{I'}{I} - \beta_2 S + \gamma + \mu \right) \\ &\leq -\frac{\beta_1 B}{K+B} - \beta_2 I + \beta_2 S - \gamma - 2\mu + \frac{I'}{I} - \beta_2 S + \gamma + \mu \\ &= \frac{I'}{I} - \mu - \frac{\beta_1 B}{K+B} - \beta_2 I \\ &\leq \frac{I'}{I} - \mu. \end{aligned}$$

Therefore,

$$m(Q) \leq \frac{I'}{I} - \mu. \quad (3.14)$$

Since  $0 \leq I(t) \leq N$ , there exists  $T > 0$  such that when  $t > T$ ,  $\frac{\ln I(t) - \ln I(0)}{t} < \frac{\mu}{2}$ . As a result,

$$\frac{1}{t} \int_0^t m(Q) dt \leq \frac{1}{t} \int_0^t \left( \frac{I'}{I} - \mu \right) dt = \frac{\ln I(t) - \ln I(0)}{t} - \mu < -\frac{\mu}{2}, \quad (3.15)$$

which implies  $\bar{q}_2 \leq -\frac{\mu}{2} < 0$ . Hence, we have established the following theorem:

**Theorem 3.8** *The endemic equilibrium  $X^*$  of the system (3.1)-(3.3) is globally asymptotically stable in  $\Delta$ .*

## 4 Bilinear incidence and nonlinear environmental function

Now we consider models of the following type:

$$\frac{dS}{dt} = bN - Sf(I, B) - bS, \quad (4.1)$$

$$\frac{dI}{dt} = Sf(I, B) - (\gamma + b)I, \quad (4.2)$$

$$\frac{dB}{dt} = g(I) - \delta B, \quad (4.3)$$

where, again, we have dropped the equation for  $R$ , i.e., (1.3). Now the function  $f$  is bilinear in  $I$  and  $B$ ,

$$f(I, B) = \beta_1 B + \beta_2 I, \quad (4.4)$$

with  $\beta_1 \geq 0$ ,  $\beta_2 \geq 0$ . This represents the classical mass action incidence. The function  $g(I)$ , assumed nonlinear, describes the growth rate of the pathogen in the environment due to the contribution from the infected people (such as shedding *V. cholerae*). We further assume  $g(I)$  satisfies the following two conditions for  $I \geq 0$ :

(A1)  $g(0) = 0$ ;  $g(I) > 0$  if  $I > 0$ .

(A2)  $g'(I) > 0$ ;  $g''(I) < 0$ .

The assumption (A1) is natural; it also ensures the existence of a unique disease-free equilibrium  $X_0 = (N, 0, 0)$ . The assumption (A2) regulates  $g(I)$  as biologically realistic based on a consequence of saturation effects: increased infection leads to higher pathogen growth; however, when the number of the infective is high, the growth of the pathogen concentration will respond more slowly than linearly to the increase in  $I$ .

Based on the assumption (A2), we can easily obtain the following results:

**Lemma 4.1** *The function  $\frac{g(I)}{I}$  is strictly decreasing on  $(0, \infty)$ .*

**Proof** For any  $I > 0$ , we have

$$\frac{g(I)}{I} = \frac{g(I) - g(0)}{I - 0} = g'(\beta)$$

for some  $\beta$  between 0 and  $I$  due to the mean value theorem. Since  $g'' < 0$  on  $(0, \infty)$ , we obtain  $g'(I) < g'(\beta)$ . Thus  $g'(I) < g(I)/I$ , or  $Ig'(I) - g(I) < 0$ . Hence,

$$\left(\frac{g(I)}{I}\right)' = \frac{Ig'(I) - g(I)}{I^2} < 0,$$

which establishes this lemma. ■

**Lemma 4.2** *Let  $I^*$  be a point in  $(0, \infty)$ . Then*

$$\frac{g(I) - g(I^*)}{I - I^*} < \frac{g(I^*)}{I^*} \quad (4.5)$$

for all  $I > 0$  and  $I \neq I^*$ .

**Proof** When  $I < I^*$ , we have  $\frac{g(I)}{I} > \frac{g(I^*)}{I^*}$  due to Lemma 4.1. Thus  $I^*g(I) > Ig(I^*)$ , or

$$I^*g(I) - I^*g(I^*) > Ig(I^*) - I^*g(I^*)$$

Using the fact  $I - I^* < 0$ , we obtain (4.5). Similarly, when  $I > I^*$ , we obtain

$$I^*g(I) - I^*g(I^*) < Ig(I^*) - I^*g(I^*)$$

which yields (4.5) as well. ■

Below we summarize the dynamics already known for the system (4.1)-(4.3), which follows Theorem 1.1.

**Theorem 4.3** *The basic reproduction number of the model (4.1)-(4.3) is*

$$R_0 = \frac{N}{\gamma + b} \left[ \beta_2 + \frac{\beta_1}{\delta} g'(0) \right]. \quad (4.6)$$

When  $R_0 < 1$ , there is a unique DFE,  $X_0 = (N, 0, 0)$ , which is globally asymptotically stable; when  $R_0 > 1$ , the DFE becomes unstable, and there is a unique positive endemic equilibrium,  $X^* = (S^*, I^*, B^*)$ , which is locally asymptotically stable.

At the endemic equilibrium, we have

$$bN - S^*f(I^*, B^*) - bS^* = 0, \quad (4.7)$$

$$S^*f(I^*, B^*) - (\gamma + b)I^* = 0, \quad (4.8)$$

$$g(I^*) - \delta B^* = 0. \quad (4.9)$$

Our goal here is to show that the endemic equilibrium is globally asymptotically stable. With the incidence  $f$  depending on both  $I$  and  $B$ , such models are not monotone or competitive dynamical systems. Meanwhile, since the environmental function  $g(I)$  can be arbitrary, the geometric approach may not be easily applied. It is, however, interesting to note that the classical method of Lyapunov functions combined with the Volterra-Lyapunov matrix properties [40, 41] can lead to the proof of the endemic global stability. The details are provided below.

We will study the system (4.1)-(4.3) in the biologically feasible domain

$$\Delta = \{(S, I, B) \mid S \geq 0, I \geq 0, S + I \leq N, B \geq 0\} \quad (4.10)$$

which is clearly a positively invariant set in  $\mathbb{R}^3$ .

We construct a Lyapunov function in the form of

$$V(S, I, B) = w_1(S - S^*)^2 + w_2(I - I^*)^2 + w_3(B - B^*)^2, \quad (4.11)$$

where  $w_1$ ,  $w_2$  and  $w_3$  are positive constants, the specific values of which are usually difficult to determine and are not of our interest here. We have

$$\begin{aligned} \frac{dV}{dt} &= \nabla V \cdot \frac{dX}{dt} = 2w_1(S - S^*) [bN - Sf(I, B) - bS] \\ &+ 2w_2(I - I^*) [Sf(I, B) - (\gamma + b)I] + 2w_3(B - B^*) [g(I) - \delta B]. \end{aligned} \quad (4.12)$$

Obviously, when  $X = X^*$ ,  $\frac{dV}{dt} = 0$ ; when  $X$  is on the  $S$ -axis (i.e.,  $I = B = 0$ ), the sign of  $\frac{dV}{dt}$  is indefinite. We aim to show that when  $X \neq X^*$  and  $(I, B) \neq (0, 0)$ ,  $\frac{dV}{dt} < 0$  holds everywhere.

Substituting equations (4.7)-(4.9) into equation (4.12), we obtain

$$\begin{aligned}
\frac{dV}{dt} &= 2w_1(S - S^*) \left\{ -b(S - S^*) - f(I, B)(S - S^*) - S^*[f(I, B) - f(I^*, B^*)] \right\} \\
&\quad + 2w_2(I - I^*) [Sf(I, B) - S^*f(I^*, B^*) - (\gamma + b)(I - I^*)] + 2w_3(B - B^*) [g(I) - \delta B] \\
&= -2w_1[b + f(I, B)](S - S^*)^2 - 2w_1S^*(S - S^*) [f(I, B) - f(I^*, B^*)] \\
&\quad - 2w_2(\gamma + b)(I - I^*)^2 + 2w_2f(I, B)(I - I^*)(S - S^*) \\
&\quad + 2w_2S^*(I - I^*) [f(I, B) - f(I^*, B^*)] \\
&\quad + 2w_3(B - B^*) [g(I) - \delta B - (g(I^*) - \delta B^*)]. \tag{4.13}
\end{aligned}$$

Now expanding  $f(I, B)$  and using the bilinear assumption (4.4), we obtain

$$f(I, B) = f(I^*, B^*) + \beta_2(I - I^*) + \beta_1(B - B^*). \tag{4.14}$$

Meanwhile, applying the mean value theorem to  $g(I)$ , we obtain

$$g(I) - g(I^*) = g'(\bar{I})(I - I^*) \tag{4.15}$$

for some  $\bar{I}$  between  $I$  and  $I^*$ . Substitution of equations (4.14) and (4.15) into equation (4.13) yields

$$\frac{dV}{dt} = (X - X^*) (WA + A^T W^T) (X - X^*)^T, \tag{4.16}$$

where the matrices  $W$  and  $A$  are given by

$$W = \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix}, \quad A = \begin{bmatrix} -[b + f(I, B)] & -\beta_2 S^* & -\beta_1 S^* \\ f(I, B) & -[\gamma + b - \beta_2 S^*] & \beta_1 S^* \\ 0 & g'(\bar{I}) & -\delta \end{bmatrix}.$$

Assume  $X \neq X^*$  and  $X$  is not on the  $S$  axis. Below we show that there exist  $w_1 > 0$ ,  $w_2 > 0$  and  $w_3 > 0$  such that the matrix  $WA + A^T W^T$  is negative definite.

**Notation 4.4** For convenience, we write a matrix  $B > 0$  ( $< 0$ ) if  $B$  is positive (negative) definite.

**Definition 4.5** We say a nonsingular  $n \times n$  matrix  $B$  is Volterra-Lyapunov stable if there exists a positive diagonal  $n \times n$  matrix  $M$  such that  $MB + B^T M^T < 0$ .

**Notation 4.6** For any  $n \times n$  matrix  $B$ , we let  $\tilde{B}$  denote the  $(n - 1) \times (n - 1)$  matrix obtained from  $B$  by deleting its last row and last column. This notation will be frequently used in what follows.



**Lemma 4.7** [8, 42] *Let  $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$  be a  $2 \times 2$  matrix. Then  $D$  is a Volterra-Lyapunov stable matrix if and only if  $d_{11} < 0$ ,  $d_{22} < 0$ , and  $\det(D) = d_{11}d_{22} - d_{12}d_{21} > 0$ .*

**Lemma 4.8** [40, 41] *Let  $D = [d_{ij}]$  be a nonsingular  $n \times n$  matrix ( $n \geq 2$ ) and  $M = \text{diag}(m_1, \dots, m_n)$  be a positive diagonal  $n \times n$  matrix. Let  $E = D^{-1}$ . Then, if  $d_{nn} > 0$ ,  $\widetilde{M}\widetilde{D} + (\widetilde{M}\widetilde{D})^T > 0$ , and  $\widetilde{M}\widetilde{E} + (\widetilde{M}\widetilde{E})^T > 0$ , it is possible to choose  $m_n > 0$  such that  $MD + D^T M^T > 0$ .*

**Lemma 4.9** *For the matrix  $A$  defined in equation (4.16),  $\widetilde{A}$  is Volterra-Lyapunov stable.*

**Proof**

$$\widetilde{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -[b + f(I, B)] & -\beta_2 S^* \\ f(I, B) & -[\gamma + b - \beta_2 S^*] \end{bmatrix}.$$

Clearly  $a_{11} < 0$ . Next we show  $a_{22} < 0$ , i.e.,

$$\gamma + b - \beta_2 S^* > 0. \quad (4.17)$$

Setting  $I = 0$ ,  $B = B^*$  in equation (4.14), we obtain

$$0 < f(0, B^*) = f(I^*, B^*) - \beta_2 I^*.$$

Thus  $f(I^*, B^*) > \beta_2 I^*$ . Meanwhile, at the endemic equilibrium we have  $S^* f(I^*, B^*) = (\gamma + b)I^*$ . Hence,

$$\gamma + b = \frac{S^* f(I^*, B^*)}{I^*} > \beta_2 S^*.$$

Finally, it is clear to see  $\det(\widetilde{A}) = a_{11}a_{22} - a_{12}a_{21} > 0$  since  $a_{12} < 0$ ,  $a_{21} > 0$ . Therefore,  $\widetilde{A}$  is Volterra-Lyapunov stable based on Lemma 4.7. ■

**Lemma 4.10** *When  $(I, B) \neq (0, 0)$ , the determinant of  $-A$  is positive, where the matrix  $A$  is defined in equation (4.16).*

**Proof** Expanding the matrix  $-A$  by the first column, we obtain

$$\det(-A) = [b + f(I, B)] \left[ \delta(b + \gamma) - \delta\beta_2 S^* - \beta_1 g'(\bar{I}) S^* \right] + f(I, B) \left[ \delta\beta_2 S^* + \beta_1 g'(\bar{I}) S^* \right].$$

The second part of  $\det(-A)$  is clearly positive. Next we show

$$\delta(b + \gamma) - \delta\beta_2 S^* - \beta_1 g'(\bar{I}) S^* > 0. \quad (4.18)$$

Based on Lemma 4.2 and equation (4.15), we have, for all  $I > 0$  and  $I \neq I^*$ ,

$$g'(\bar{I}) = \frac{g(I) - g(I^*)}{I - I^*} < \frac{g(I^*)}{I^*} = \frac{\delta B^*}{I^*}. \quad (4.19)$$

Thus  $I^*g'(\bar{I}) - \delta B^* < 0$ , which yields

$$B_1 \triangleq B^* - \frac{g'(\bar{I})}{\delta} I^* > 0. \quad (4.20)$$

Now, substitute the point  $(I, B) = (0, B_1)$  into equation (4.14) to obtain

$$0 < f(0, B_1) = f(I^*, B^*) - \beta_2 I^* + \beta_1 \frac{g'(\bar{I})}{\delta} I^*. \quad (4.21)$$

Combining the results in (4.21) and (4.8), we obtain (4.18). Hence,  $\det(-A) > 0$ . ■

Using the transpose of the matrix of cofactors, we write the inverse of  $-A$  by

$$(-A)^{-1} = \frac{1}{\det(-A)} \begin{bmatrix} c_{11}(+) & c_{21}(-) & c_{31}(-) \\ c_{12}(+) & c_{22}(+) & c_{32}(+) \\ c_{13}(+) & c_{23}(+) & c_{33}(+) \end{bmatrix}, \quad (4.22)$$

where  $c_{ij}$  denotes the cofactor of the  $(i, j)$  entry of the matrix  $-A$ , and the  $+$  or  $-$  in the parenthesis indicates the sign of  $c_{ij}$ . Note that  $\det(-A) > 0$  based on Lemma 4.10. Specifically, we have

$$\begin{aligned} c_{11} &= \delta(b + \gamma) - \delta\beta_2 S^* - \beta_1 g'(\bar{I}) S^* > 0, \\ c_{21} &= -(\delta\beta_2 S^* + g'(\bar{I})\beta_1 S^*) < 0, \\ c_{31} &= -\beta_1 \beta_2 (S^*)^2 - \beta_1 S^* [\gamma + b - \beta_2 S^*] < 0, \\ c_{12} &= \delta f(I, B) > 0, \\ c_{22} &= \delta(b + f(I, B)) > 0, \\ c_{32} &= b\beta_1 S^* > 0, \\ c_{13} &= f(I, B)g'(\bar{I}) > 0, \\ c_{23} &= g'(\bar{I})(b + f(I, B)) > 0, \\ c_{33} &= (b + f(I, B))[\gamma + b - \beta_2 S^*] + S^* \beta_2 f(I, B) > 0, \end{aligned}$$

where we have applied (4.18) to obtain  $c_{11} > 0$ , and (4.17) to show  $c_{31} < 0$  and  $c_{33} > 0$ .

**Lemma 4.11** *Let  $D = -A$  and  $E = (-A)^{-1}$ , where  $A$  is defined in equation (4.16). Then there exists a positive  $2 \times 2$  diagonal matrix  $\widetilde{W} = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}$  such that  $\widetilde{W}\widetilde{D} + (\widetilde{W}\widetilde{D})^T > 0$  and  $\widetilde{W}\widetilde{E} + (\widetilde{W}\widetilde{E})^T > 0$ .*

**Proof** Note that  $A^{-1} = -E$ . Using equation (4.22), we obtain

$$\widetilde{A}^{-1} = \frac{1}{\det(-A)} \begin{bmatrix} -c_{11} & -c_{21} \\ -c_{12} & -c_{22} \end{bmatrix}.$$

Based on Lemma 4.7, it is straightforward to verify that  $\widetilde{A}^{-1}$  is Volterra-Lyapunov stable. Hence, there exists a positive  $2 \times 2$  diagonal matrix  $\widetilde{W}$  such that  $\widetilde{W}\widetilde{A}^{-1} + (\widetilde{W}\widetilde{A}^{-1})^T < 0$ . Since  $E = (-A)^{-1}$ , we obtain  $\widetilde{W}\widetilde{E} + (\widetilde{W}\widetilde{E})^T > 0$ , i.e.,

$$\frac{1}{\det(-A)} \begin{bmatrix} 2w_1 c_{11} & w_1 c_{21} + w_2 c_{12} \\ w_1 c_{21} + w_2 c_{12} & 2w_2 c_{22} \end{bmatrix} > 0.$$

Hence, the determinant of the above matrix must be positive, i.e.,

$$4w_1w_2c_{11}c_{22} - (w_1c_{21} + w_2c_{12})^2 > 0.$$

Substituting the expressions for  $c_{ij}$  ( $i, j = 1, 2$ ) and manipulating the algebra, we obtain

$$0 < 4w_1w_2c_{11}c_{22} - (w_1c_{21} + w_2c_{12})^2 = J - 2w_1w_2(2b + f(I, B))g'(\bar{I})\beta_1S^* - (w_1S^*)^2\beta_1g'(\bar{I})[2\beta_2 + \beta_1g'(\bar{I})],$$

where

$$J = 4w_1w_2(b + f(I, B))[\gamma + b - \beta_2S^*] - [w_2f(I, B) - w_1\beta_2S^*]^2.$$

Clearly we must have  $J > 0$ . Now,

$$\widetilde{W}\widetilde{D} + (\widetilde{W}\widetilde{D})^T = \begin{bmatrix} 2w_1[b + f(I, B)] & w_1\beta_2S^* - w_2f(I, B) \\ w_1\beta_2S^* - w_2f(I, B) & 2w_2[\gamma + b - \beta_2S^*] \end{bmatrix}.$$

Note that the (1, 1) and (2, 2) entries of this  $2 \times 2$  matrix are positive, and that its determinant is exactly  $J$ . Hence, it is clear to see  $\widetilde{W}\widetilde{D} + (\widetilde{W}\widetilde{D})^T > 0$ . The proof is then complete. ■

**Theorem 4.12** *The matrix  $A$  defined in equation (4.16) is Volterra-Lyapunov stable.*

**Proof** Based on Lemmas 4.8 and 4.11, there exists a positive  $3 \times 3$  diagonal matrix  $W$  such that  $W(-A) + (-A)^TW^T > 0$ . Thus  $WA + A^TW^T < 0$ . ■

Therefore, we obtain  $\frac{dV}{dt} < 0$  when  $X \neq X^*$  and  $X$  is not on the  $S$ -axis (a set of measure zero). Thus we have established the following theorem:

**Theorem 4.13** *The endemic equilibrium of the model system (4.1)-(4.3) is globally asymptotically stable.*

## 5 Conclusions and Discussion

With the environmental component incorporated and multiple transmission pathways coupled, the cholera models distinguish themselves from regular SIR and SEIR epidemiological models which have been extensively studied and whose global dynamics has been relatively well established (see [5, 15, 16, 21, 22, 24, 26, 28, 30, 43, 49], among others). Using the methods of monotone dynamical systems, geometric approach, and Lyapunov functions, we have investigated in this paper the global asymptotic stability of the endemic equilibria for several deterministic cholera models and obtained new global stability results. These models represent biologically important, and mathematically nontrivial, cases in the study of cholera dynamics. The analysis and results presented in this paper build a solid base for future work on the global dynamics of the most general cholera model.

The three techniques we employed in this paper all have strength and weakness. The method of monotone dynamical systems [10, 24, 44], when applicable, is easier to implement than the geometric approach, since it essentially treats a three-dimensional autonomous system as a two-dimensional one. Unfortunately, most high-dimensional infectious disease

models do not possess the nice properties of monotone systems, which limits the application of this approach. The geometric approach, originally proposed by Li and Muldowney [9, 25, 27], has gained some popularity in recent years (see, e.g., [2, 20]) as it has less constraints on the model systems. Among the three, this method seems to have the best potential to deal with more general model systems. The disadvantage, however, is that the implementation of the geometric approach is not straightforward and involves extra nontrivial technical details, particularly the estimate of the Lozinskiĭ measure. In addition, the method of Lyapunov functions has been known for many decades. The challenge in the application of this method is that there is no systematic way to construct Lyapunov functions (particularly, the determination of the appropriate coefficients is often a matter of luck), so that its success largely depends on trial and error as well as on specific problems. In this paper, by combining this classical approach with the Volterra-Lyapunov matrix analysis [40–42], we have leveraged the difficulty of determining specific coefficient values and, as such, wider application of Lyapunov functions to dynamical systems could be promoted. As can be seen from our analysis in Section 4, the extension of this approach to four- or higher-dimensional systems is possible but becomes much harder, since the proof of Volterra-Lyapunov stable matrices involves considerably more work in higher dimensions.

The work presented in this paper is not limited to cholera models. Indeed, several known infectious diseases [29], such as typhoid fever, amebiasis, and dracunculiasis, involve environmental components and can be possibly modeled in a similar manner as those for cholera. Our analysis and results can thus contribute to a wider range of problems in bi-math studies.

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