Let $M$ be an oriented surface in $\mathbb{R}^3$, let $\xi$ be the unit vector field normal to $M$:

$$A_p = -d\xi_p : T_p M \to T_{\xi(p)} S^2 \cong T_p M$$

is the shape operator of $M$.

The trace of $A_p$ is twice the mean curvature $H(p)$ at $p \in M$. 
Definition 1

$M$ is an $H$-surface means that it has constant mean curvature $H$.

Definition 2

$M$ is an $H$-surface $\iff M$ is a critical point for the area functional under compactly supported variations preserving the volume.

- Sphere
- Cylinder
- Delaunay surfaces
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- Sphere
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Introduction to the theory of CMC surfaces.

**Definition**

An H-surface \( M \) is a **minimal surface** \( \iff H \equiv 0 \iff M \) is a critical point for the area functional under compactly supported variations.

- **Catenoid**
- **Helicoid**
Soap films are minimal surfaces.

Soap bubbles are nonzero $H$-surfaces.
Notation and Language

- \( \text{Ch}(Y) = \inf_{K \subset Y \text{ compact}} \frac{\text{Area}(\partial K)}{\text{Volume}(K)} = \text{Cheeger constant of } Y. \)

- \( H(Y) = \inf \{ \max |H_M| : M = \text{immersed closed surface in } Y \} \), where \( \max |H_M| \) denotes \( \max \) of absolute mean curvature function \( H_M \).

- The number \( H(Y) \) is called the \textit{critical mean curvature} of \( Y \).
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Theorem (Meeks-Mira-Pérez-Ros)

- If \( Y \) is a simply connected homogeneous 3-manifold, then:
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Remark

Proof uses \( H(Y) \)-foliations of \( Y \) to show that if \( \Omega(n) \subset Y \) is a sequence of isoperimetric domains in \( Y \) with \( \text{Volume}(\Omega(n)) \to \infty \), then

\[
H_{\partial \Omega(n)} \geq H(Y) \quad \text{and} \quad \lim_{n \to \infty} H_{\partial \Omega(n)} = H(Y).
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Fact:
Simply connected homogeneous 3-manifolds $X$ are either isometric to $S^2(\kappa) \times \mathbb{R}$ or to a metric Lie group.
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Any two spheres in $M$ of the same absolute constant mean curvature differ by an isometry of $M$. Moreover:

1. If $X$ is not diffeomorphic to $\mathbb{R}^3$, then, for every $H \in \mathbb{R}$, there exists a sphere of constant mean curvature $H$ in $M$. 

Bill Meeks at the University of Massachusetts  

The theory of surfaces of constant mean curvature
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2. If $X$ is diffeomorphic to $\mathbb{R}^3$, then the values $H \in \mathbb{R}$ for which there exists a sphere of constant mean curvature $H$ in $M$ are exactly those with $|H| > \text{Ch}(X)/2$. 

Bill Meeks at the University of Massachusetts
The theory of surfaces of constant mean curvature
Theorem

Let $S$ be an $H$-sphere in $M$ and let $\tilde{S} \subset X$ be a lift.

1. If $X$ is a product $S^2 \times \mathbb{R}$, where $S^2$ is a sphere of constant curvature, and $\tilde{S} = S^2 \times \{t_0\}$, for some $t_0 \in \mathbb{R}$, then $S$ is totally geodesic, stable and has nullity 1 for its Jacobi operator.

2. Otherwise, $S$ has index 1 and nullity 3 for its Jacobi operator and the immersion of $S$ into $X$ extends as the boundary of an isometric immersion $F: B \to M$ of a Riemannian 3-ball $B$ which is mean convex.

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The theory of surfaces of constant mean curvature
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Previous results on the Hopf Uniqueness Problem are the following:

Theorem (Hopf, 1950)

$H$-spheres in $\mathbb{R}^3$ are round.

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If $M$ has a 4-dimensional isometry group, then $H$-spheres in $M$ are surfaces of revolution and they are characterized by their mean curvatures.

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If $X$ is the Lie group $\text{Sol}_3$ with any of its most symmetric left invariant metrics, then $H$-spheres in $X$ have index 1 and nullity 3 and they are characterized by their mean curvatures.
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Definition

Given an oriented immersed surface $f: \Sigma \to X$ with unit normal vector field $N: \Sigma \to TX$, the left invariant Gauss map of $\Sigma$ is the map $G: \Sigma \to S^2 \subset T_eX$ that assigns to each $p \in \Sigma$, the unit tangent vector to $X$ at the identity element $e$ given by left translation:

$$(dl_{f(p)})_e(G(p)) = N_p.$$
Theorem (Representation Theorem, Meeks-Mira-Perez-Ros)

- Suppose $\Sigma$ is a simply connected Riemann surface with conformal parameter $z$, $X$ is a simply connected metric Lie group, $H \in \mathbb{R}$ and $R(q): \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the $H$-potential.

- Let $g: \Sigma \to \overline{\mathbb{C}}$ be a solution of the complex elliptic PDE

$$
g_{zz} = \frac{R_q}{R}(g)g_zg_{\overline{z}} + \left(\frac{R_{\overline{q}}}{R} - \frac{R_q}{R}\right)(g)|g_z|^2,
$$

such that $g_z \neq 0$ everywhere\(^a\), and such that the $H$-potential $R$ of $X$ does not vanish on $g(\Sigma)$ (for instance, this happens if $\Sigma$ is closed).

- Then, there exists an immersed $H$-surface $f: \Sigma \hookrightarrow X$, unique up to left translations, whose Gauss map is $g$.

- Conversely, if $g: \Sigma \to \overline{\mathbb{C}}$ is the Gauss map of an immersed $H$-surface $f: \Sigma \hookrightarrow X$ in a metric Lie group $X$, and the $H$-potential $R$ of $X$ does not vanish on $g(\Sigma)$, then $g$ satisfies the equation (1), and moreover $g_z \neq 0$ holds everywhere.

\(^a\)By $g_z \neq 0$ we mean that $g_z(z_0) \neq 0$ if $g(z_0) \in \mathbb{C}$ and that $\lim_{z \to z_0} (g_z/g^2)(z) \neq 0$ if $g(z_0) = \infty$. 

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The theory of surfaces of constant mean curvature
Theorem (Classification Theorem for $H$-spheres, Meeks-Mira-Pérez-Ros)

Suppose $X$ is a simply connected 3-dimensional metric Lie group.

- $X$ is diffeomorphic to $\mathbb{R}^3 \implies$ the moduli space of $H$-spheres in $X$ is parameterized by the mean curvature values $H$ in $(H(X), \infty)$.

- $X$ is diffeomorphic to $S^3 \implies$ the moduli space of $H$-spheres in $X$ is parameterized by the mean curvature values $H$ in $[0, \infty)$.

- $X$ diffeomorphic to $S^3 \implies$ the areas of all $H$-spheres form a half-open interval $(0, A(X)]$.

- $H$-spheres in $X$ are **Alexandrov embedded** with index 1, nullity 3.
Steps of the proof of the Classification Theorem for $H$-spheres.

Throughout $\Sigma$ denotes a fixed $H_0$-sphere in $X$ of index 1.
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The theory of surfaces of constant mean curvature
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  (A) If $X$ is isomorphic to SU(2), areas of spheres in $\mathcal{M}(X)$ are uniformly bounded.
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The theory of surfaces of constant mean curvature
Steps of the proof continued.

- **Step 5:** Each component of $\mathcal{M}(X)$ is an interval parameterized by the mean curvature values in a subinterval $I_X \subset [0, \infty)$. $I_X = [0, \infty)$ if $X$ is isomorphic to $SU(2)$ and otherwise $I_X = (H(X), \infty)$.  

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Conclusions: The space of non-congruent $H$-spheres in $X$ equals $\mathcal{M}(X)$ which is an interval parameterized by the mean curvature values in $[0, \infty)$ if $X$ is isomorphic to $SU(2)$ and otherwise, in the interval $(H(X), \infty)$. Each $H$-sphere in $X$ has index 1 and nullity 3. Each $H$-sphere in $X$ is the boundary of an immersed 3-ball $F: B \to X$ (Alexandrov embedded). If $X$ is isomorphic to $SU(2)$, then the areas of $H$-spheres in $X$ form a half-open interval $(0, A(X))$. 

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The theory of surfaces of constant mean curvature
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The theory of surfaces of constant mean curvature
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The theory of surfaces of constant mean curvature
Step 4(A): Suppose $X$ is isomorphic to $SU(2)$. There exists a uniform bound on the areas of index 1 $H$-spheres in $M(X)$. 

Proof. Arguing by contradiction, $\exists$ a sequence of $H_n$-spheres $\Sigma_n \in M(X)$ with $\text{Area}(\Sigma_n) \geq n$ and $H_n$ uniformly bounded. Some subsequence of compact domains in the $\Sigma_n$ converges to a complete, stable limit $H$-surface $\Sigma_\infty$ with degenerate Gauss map. $\Sigma_\infty$ is invariant under the left action of a 1-parameter subgroup $S^1$ of $X$ generating a tangent right invariant Killing field $K$. $\Sigma_\infty$ can be chosen to be a quasi-periodic cylinder of bounded curvature and linear area growth $\Rightarrow \Sigma_\infty$ is parabolic. Given a point $p \in \Sigma_\infty$, let $K'$ be a right invariant Killing field with $K'(p) \in T_p \Sigma_\infty$ linearly independent from $K(p)$. Jacobi function $\langle K', N \rangle$ changes sign on $\Sigma_\infty$, $N =$ unit normal field. But on a stable parabolic $H$-surface, a bounded Jacobi function cannot change sign, a contradiction.
Steps of the proof continued.

- **Step 4(A):** Suppose $X$ is isomorphic to $SU(2)$. There exists a uniform bound on the areas of index 1 $H$-spheres in $\mathcal{M}(X)$.

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The theory of surfaces of constant mean curvature
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The theory of surfaces of constant mean curvature
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New uniqueness results for CMC surfaces.

Question
Is the round sphere the only complete simply connected surface embedded in $\mathbb{R}^3$ with non-zero constant mean curvature?

NOT simply connected
- Cylinder

NOT embedded
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Round spheres are the only complete simply connected surfaces embedded in $\mathbb{R}^3$ with non-zero constant mean curvature.

1986 - Above result proved by Meeks for properly embedded.

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The theory of surfaces of constant mean curvature
Theorem (Meeks-Tinaglia)

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1986 - Above result proved by \textbf{Meeks} for \textbf{properly embedded}.

2007 - Work of \textbf{Colding-Minicozzi} and \textbf{Meeks-Rosenberg} for $H = 0$ shows that if $M$ is a complete, simply connected 0-surface \textbf{embedded} in $\mathbb{R}^3$, then $M$ is either

\textbf{a plane or a helicoid}. 
Theorem (Meeks-Tinaglia)

Let $M \subset \mathbb{R}^3$ be a complete, connected embedded $H$-surface.

1. $M$ has positive injectivity radius $\implies M$ is properly embedded in $\mathbb{R}^3$. 

2. $M$ has finite topology $\implies M$ has positive injectivity radius.

3. Suppose $H > 0$. Then: $|A_M|$ is bounded $\iff M$ has positive injectivity radius.

When $H = 0$, items 1 and 2 were proved by Meeks-Rosenberg, based on: Colding-Minicozzi: $M$ has finite topology and $H = 0 \implies M$ is proper.

Item 3 in the above theorem holds for $3$-manifolds which are homogeneously regular; in particular it holds in closed Riemannian $3$-manifolds.
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Theorem (Radius Estimates for $H$-Disks, Meeks-Tinaglia)

$\exists R_0 \geq \pi$ such that every embedded $H$-disk in $\mathbb{R}^3$ has radius $< R_0/H$. 
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Theorem (Curvature Estimates for $H$-Disks, Meeks-Tinaglia)

Fix $\varepsilon, H_0 > 0$ and a complete locally homogenous 3-manifold $X$. $\exists \ C > 0$ s.t. for all embedded $(H \geq H_0)$-disks $D$:

$|A_D|(p) \leq C$ for all $p \in D$ s.t. $\text{dist}_D(p, \partial D) \geq \varepsilon$. 
Theorem (One-sided curvature estimate for $H$-disks, Meeks-Tinaglia)

$\exists C, \varepsilon > 0$ s.t. for any $H$-disk $\Sigma \subset \mathbb{R}^3$ as in the figure below:

$$|A_{\Sigma}| \leq \frac{C}{R} \text{ in } \Sigma \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}.$$ 

This result generalizes the one-sided curvature estimates for minimal disks by Colding-Minicozzi, and uses their work in its proof.
Universal domain for Embedded Calabi-Yau problem?

\[ \mathcal{D}_\infty = \text{the above bounded domain, smooth except at } p_\infty. \]

- Ferrer, Martin and Meeks conjecture: An open surface properly embeds as a complete minimal surface in \( \mathcal{D}_\infty \) \iff every end has infinite genus \iff it admits a complete bounded minimal embedding in \( \mathbb{R}^3 \).
Conjecture (Meeks-Perez-Ros-Tinaglia)

For any complete, connected embedded $H$-surface $\Sigma \subset \mathbb{R}^3$ of finite genus and compact boundary, there exists a constant $K_\Sigma$ s.t. $\forall R \geq 1$,

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**Theorem (Meeks-Perez-Ros)**

Let $\Sigma \subset \mathbb{R}^3$ be a complete, connected embedded $0$-surface of finite genus. Then:

$$\Sigma \text{ is proper } \iff \Sigma \text{ has a countable } \# \text{ of ends.}$$
The family $\mathcal{R}_t$ of Riemann minimal examples

Riemann's Infinite Staircase

- Catenoid Soap Film
- Perturbed Soap Film

Shifted wire

Bill Meeks at the University of Massachusetts
The theory of surfaces of constant mean curvature
I am foliated by circles
The family $\mathcal{R}_t$ of Riemann minimal examples

Riemann's Infinite Staircase

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Shifted wire
Cylindrical parametrization of a Riemann minimal example

Infinite cylinder
Cylindrical parametrization of a Riemann minimal example

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The theory of surfaces of constant mean curvature
Topologically there is only one connected genus-zero surface with two limit ends. Riemann minimal examples have this property.
Properly embedded genus-0 examples - Collin-Meeks-Perez-Ros-Rosenberg

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The theory of surfaces of constant mean curvature

- Catenoid
- Helicoid
- Riemann
- Plane

MODULI SPACE

$\mathbb{R}_t = \text{Riemann Examples}$

CATENOID

HELICOID
Next theorem is motivated by the study of 3-periodic $H$-surfaces that appear as interfaces in material science or as equipotential surfaces in crystals. This result contrasts with the failure of area estimates for compact minimal surfaces of genus $g > 2$ in any flat 3-torus (Traizet).
Next theorem is motivated by the study of 3-periodic $H$-surfaces that appear as interfaces in material science or as equipotential surfaces in crystals. This result contrasts with the failure of area estimates for compact minimal surfaces of genus $g > 2$ in any flat 3-torus (Traizet).

**Theorem (Meeks-Tinaglia)**

Given a flat 3-torus $\mathbb{T}^3$ and $H > 0$, $\exists C_H$ s.t. $\forall g \in \mathbb{N}$, a closed $H$-surface $\Sigma$ embedded in $\mathbb{T}^3$ with genus at most $g$ satisfies $\text{Area}(\Sigma) \leq C_H(g + 1)$.
Definition

Suppose \( f : \Sigma \to N \) is a closed immersed surface positive mean curvature in a Riemannian 3-manifold \( N \).

\( \Sigma \) is called strongly *Alexandrov embedded* if \( f \) extends to an immersion \( F : W \to N \) of a compact 3-manifold \( W \) with \( \Sigma = \partial W \), where the extended immersion is injective on the interior of \( W \).
Definition

Suppose $f : \Sigma \to N$ is a closed immersed surface positive mean curvature in a Riemannian 3-manifold $N$.

$\Sigma$ is called **strongly Alexandrov embedded** if $f$ extends to an immersion $F : W \to N$ of a compact 3-manifold $W$ with $\Sigma = \partial W$, where the extended immersion is injective on the interior of $W$.

Theorem (Meeks-Tinaglia, 2017)

- Let $N$ be a closed Riemannian 3-manifold.
- Given $H > 0$ and a non-negative integer $g$, then the space of strongly Alexandrov embedded closed surfaces in $N$ of genus at most $g$ and constant mean curvature $H$ is **compact**.
These studies on the geometry of embedded \( H \)-surfaces lead to the following deep results on \textbf{CMC} foliations of 3-manifolds.

**Definition**

A codimension-1 foliation \( F \) of a Riemannian \( n \)-manifold \( X \) is a \textbf{CMC} foliation if it is transversely oriented and the mean curvature function \( H_F : X \to \mathbb{R} \) constant along leaves of \( F \).

**Theorem (CMC Foliation Extension Theorem, Meeks-Perez-Ros)**

Let \( F \) be a weak \textbf{CMC} foliation of a punctured Riemannian 3-ball \( B(p, r) - \{ p \} \).

Then \( F \) extends to a weak \textbf{CMC} foliation of \( B(p, r) \) if and only if the mean curvature function of \( F \) is bounded in some neighborhood of \( p \).

2 key ingredients in the proof.

- Curvature estimates for \textbf{CMC} foliations.
- Local removable singularity theorem for weak \( H \)-laminations.

Bill Meeks at the University of Massachusetts

The theory of surfaces of constant mean curvature
These studies on the geometry of embedded $H$-surfaces lead to the following deep results on CMC foliations of 3-manifolds.

**Definition**

A codimension-1 foliation $\mathcal{F}$ of a Riemannian $n$-manifold $X$ is a CMC foliation if it is transversely oriented and the mean curvature function $H_{\mathcal{F}} : X \to \mathbb{R}$ is constant along leaves of $\mathcal{F}$. 

**Theorem (CMC Foliation Extension Theorem, Meeks-Perez-Ros)**

Let $\mathcal{F}$ be a weak CMC foliation of a punctured Riemannian 3-ball $B(p, r) - \{p\}$. Then $\mathcal{F}$ extends to a weak CMC foliation of $B(p, r) \iff$ the mean curvature function of $\mathcal{F}$ is bounded in some neighborhood of $p$. 

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CMC foliation of $\mathbb{R}^3$ punctured in two points by spheres and planes

Theorem (Meeks-Perez-Ros)

Suppose $\mathcal{F}$ is a CMC foliation of $\mathbb{R}^3 - S$ where $S$ is a closed countable set. Then all leaves of $\mathcal{F}$ are contained in planes and round spheres.
Calabi-Yau type problems for embedded $H$-surfaces

Theorem (Meeks-Tinaglia)
For $H \geq 1$, complete embedded finite topology $H$-surfaces in complete hyperbolic 3-manifolds are proper.

Theorem (Coskunuzer-Meeks-Tinaglia)
- For every $H < 1$, $\exists$ a complete embedded stable $H$-plane that is nonproper in $\mathbb{H}^3$.
- For every $H \in (0, 1/2)$, $\exists$ a complete embedded stable $H$-plane that is nonproper in $\mathbb{H}^2 \times \mathbb{R}$.

Theorem (Tinaglia-Rodriguez)
$\exists$ a complete embedded stable 0-plane that is nonproper in $\mathbb{H}^2 \times \mathbb{R}$.