Variational Geometry

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Geometric Variational Problems

- Geometry: Metric (measurement) and curvature (shape).
- Ricci flow: Specific variations.
- Minimal surfaces: Critical points of the area functional.
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• Einstein Structures: Critical points of a natural geometric functional.
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Differentiable Manifolds

- Differentiable manifold $M$: locally Euclidean.
- Riemannian metric $g$: measure length/distance/volume.
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Curvature

• Levi-Civita connection: Allow differentiation.

• Curvature: (determined by derivatives of the metric) Measure non-flatness.

• Intrinsic: Riemannian curvature, Ricci curvature, scalar curvature.

• Extrinsic: Mean curvature, second fundamental form.
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A tangent plane
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Space Forms

Constant curvature models: Euclidean space (flat), round sphere (constant positive), hyperbolic space (constant negative).
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Poincare model
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Poincare model
Ricci Flow

\[
\frac{\partial}{\partial t} g = -2Rc
\]

Figure: Ricci flow on a neck

Ricci Flow

$(M, g(t))$ is a Ricci flow solution if

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Ricci flow on a neck

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Overview
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- Fundamental questions:
  - Convergence.
  - Formulation of Singularities.

Celebrated applications:
- G. Perelman's proof of the Poincare's conjecture.
- The proof of the differentiable sphere theorem by S. Brendle and R. Schoen.

Technicality: Parabolic PDE, maximum principle.
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- Technicality: Parabolic PDE, maximum principle.
Contributions

- Harnack inequalities crucial in Perelman’s singularity analysis.
- Obtain analogous estimates in generalized settings:
  - Ricci flow on warped Products (2015, JGA)
  - (with Mihai Bailesteanu) Ricci-Harmonic map flow (2017, PEMS)
  - (with Xiaodong Cao, Hongxin Guo) Generalized abstract flow (2015, MZ)
- (with X. Cao) Behavior of curvature towards the singular time (2015, MRL)
Einstein Structures

- $(M, g)$ is an Einstein structure if, for a constant $\lambda$, $Rc = \lambda g$.
- Critical points of the Hilbert functional.
- Generalized Structures: Gradient Ricci soliton, Harmonic curvature, Harmonic Weyl tensor.
- Quest for the best metric.
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Feb 20th, 2018 9 / 15
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Round sphere

Non-Round sphere
Einstein Structures

Fundamental questions:
• Existence.
• Uniqueness/moduli space.

Open question:
Conjecture: A non-flat simply connected Einstein four-manifold with non-negative sectional curvature must be either $S^4$, $\mathbb{CP}^2$, $S^2 \times S^2$.

Technicality: Non-linear PDE, elliptic methods.
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Contributions

- (with X. Cao) Rigidity of a gradient Ricci soliton (2016, GT)
- Rigidity of closed manifolds with harmonic Weyl curvature (2017, AiM)
- (with X. Cao) Progress towards E4M conjecture (2016, Preprint)
Free Boundary Minimal Surfaces

\[ \Sigma \subset B_3, \quad \partial \Sigma \subset \partial B_3, \quad \Sigma \text{ is a FBMS if } H \equiv 0 \text{ and } \Sigma \text{ meets } \partial B_3 \text{ perpendicularly.} \]

FBMS are critical points of the area functional with the free boundary condition (extension of Plateau's problem).

Figure: Critical Catenoid

Images courtesy of Peter McGrath
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\(^2\)Images courtesy of Peter McGrath
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**Conjecture**

* A free boundary minimal annulus must be the critical catenoid.*
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  Technicallity: Elliptic PDE, PDE, GMT, and complex methods.
Contributions

• Stability (quantitatively measured by the Morse index) is crucial to answer uniqueness questions.

• Develop a natural method to compute the Morse index (2016, CAG).

• (with Graham Smith, Ari Stern, and Detang Zhou) Study the growth of Morse indices of higher dimensional catenoids (2017, Preprint).

• Characterize the critical catenoid by a natural condition on its Gauss map (2017, Preprint).
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Thank You