Hysteresis Parameter Identification with Limited Experimental Data

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***Abstract*— The Preisach operator and its variants have been successfully used in the modeling of hysteresis observed in ferromagnetic, magnetostrictive and piezoelectric materials. In an application involving these smart materials, one has to determine a density function for the Preisach operator using the input-output behavior of the material at hand. In this paper, we describe a method for numerically determining an approximation of the density function when there is not enough experimental data to uniquely solve for the density function. We present theoretical justification for our method by establishing links to regularization methods for ill-posed problems. We also present numerical results where we estimate an approximate density function from data published in the literature for a magnetostrictive actuator and two electro-active polymers.**

***Index Terms*— Hysteresis, Preisach Operator, Density function Identification, Constrained least squares, Electro-active polymers, Magnetostrictive actuators, Smart materials.**

I. INTRODUCTION

The phenomena of nonlinear hysteresis has been well documented in magnetism and electricity. Hysteresis comes from a Greek term meaning "to lag behind," and describes a relationship between inputs and outputs of a certain system. A system with scalar inputs and outputs is said to exhibit rate-independent *hysteresis* if: (a) the outputs of the system do not depend on the rate at which the input is applied and (b) the outputs of the system at a certain time, depend on the past history of the input function. Though dynamical systems exhibit the "memory" requirement (b), they do not exhibit the "rate independence" requirement (a).

The Preisach operator is a mathematical tool that has been used to model the phenomena of hysteresis for many years [1], [2]. Part of what is needed to describe the Preisach operator for a particular system is a density function defined for certain parameters. In the past, several researchers have addressed the problem of identifying the Preisach density function. Mayergoyz [2] first described a method to identify the density function in the proof of his representation theorem. However, this method has limited applicability in practice when the output signal is corrupted by noise, as it involves a differentiation of the output signal.

Other approaches are divided along three main lines. In the first method, one assumes the density function to be of a particular type (for example, factored-Lorentzian or a Lognormal-Gauss distributions) and then sets about identifying the parameters that define these distributions (see Della Torre [3] for a discussion using the Gaussian distribution function for magnetically hard materials). The modeling and prediction errors that result from this approach can be significant as there is no real justification for assuming one particular distribution function over another [4]. In the second approach, one uses a family of density functions as a set of basis vectors, and the required density function is assumed to be expressible as a linear combination of a finite subset. The selection of this finite subset is the main problem in this approach. It is shown in Galinaitis, Joseph and Rogers [5] that the best approximation in a chosen subset can yield very poor predictions. The third method involves discretizing the Preisach plane (described in Section 2), and identifying a discrete approximation to the actual density function via a linear least-squares method. Hoffman and Meyer [6] were perhaps the first to use this method. Banks, Kurdila and Webb [7] address the problem of convergence of the approximate measures identified through experiments to the true density function. Galinaitis and Rogers [8] used of this method to identify an approximate Preisach density function for piezoelectric actuators. In Venkataraman, Tan and Krishnaprasad, a constrained linear least squares method is used to explicitly constrain the approximate density function to be positive [9], and an application to magnetostrictive actuators can be found. In our opinion, the third approach is the best suited to obtain the best possible approximation to the actual density function as no assumptions are made about the actual density function. Furthermore, the work of Banks, Kurdila and Webb [7] provide the justification for this method, as we are assured that finer discretizations of the input signal will yield better approximations of the density function.

In this paper, we consider the identification of the Preisach density function when there is not sufficient experimental data. Mayergoyz's representation theorem [2] yields a sufficiency condition on the input-output signal, so that a complete identification of the measure can be achieved. However, in practice, this would involve a very large amount of data that has to be processed to obtain the density function. There is a need for a method that uses limited information to obtain an approximation of the density function. What is needed is a method that utilizes all the available "information" in the experimental data to obtain the best approximation of the actual density function. In this paper, we use the well-known singular value decomposition along with a linear least squares method to efficiently identify the best approximation to the density function. We also present theoretical justification for our method by establishing links to regularization methods for ill-posed problems.

We cast the identification problem as a constrained mini-

Manuscript submitted to the IEEE Trans. Magnetics.

mization problem (the details are in Section 3):

$$\min \quad \frac{1}{2}\|AX - Y\|^2 \quad \text{subject to} \quad X \geq 0,$$

where A and Y are computed using the input and output signals respectively, and X is unknown. Our methodology involves the identification of the nullspace of the matrix $A^T A$ and obtaining a solution in its orthogonal complement. In Section 2, we briefly describe the Preisach operator and describe the problem in Section 3. In Section 4 we describe the results of our numerical experiments for a magnetostrictive actuator and two commonly used electro-active polymers.

II. THE PREISACH OPERATOR

The Preisach operator has been used to model hysteresis in many applications. In the following section, we will define the Preisach operator, and discuss its properties.

Consider a relay $R_{\beta,\alpha}$ which at any given time is at one of two states: +1 or -1. A descriptive definition of the relay can be given using Figure 1. The relay is parametrized by scalars α and β. Consider an input function $u(t);\ t \in [0,T]$ for the relay, with $v(0)$ either $+1$ or -1. If the input monotonically increases, the ascending branch $abcde$ is followed by the output $v(t)$, while if the input monotonically decreases, the descending branch $edfba$ is followed. Mathematically, one defines this elementary hysteresis operator or *hysteron* as follows (our definition is a modification of the one in Krasnoselskii [10], taking care of the hysterons with $\alpha = \beta$):

$$v(t) = \begin{cases} -1 & \text{if} \quad u(t) < \beta \\ +1 & \text{if} \quad u(t) \geq \alpha \\ -1 & \text{if} \quad \beta \leq u(t) < \alpha, \text{ and, } \exists \quad t_1 : u(t_1) < \beta \\ & \quad \text{and} \quad \forall \tau \in (t_1, t),\ u(\tau) \notin [\alpha, \infty) \\ +1 & \text{if} \quad \beta \leq u(t) < \alpha, \text{ and, } \exists \quad t_1 : u(t_1) \geq \alpha, \\ & \quad \text{and} \quad \forall \tau \in (t_1, t),\ u(\tau) \notin (-\infty, \beta) \end{cases} \tag{1}$$

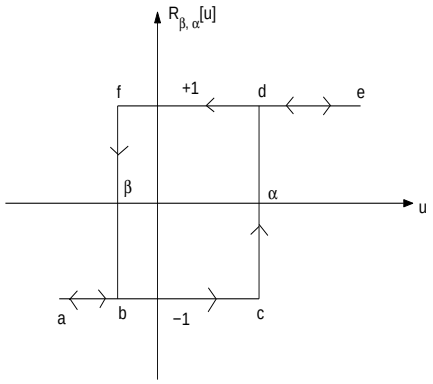

Fig. 1. Input-output relationship for the relay.

To construct the Preisach operator from the elementary hysterons when the input is $u(\cdot)$, denote $v_{\beta,\alpha}(\cdot) = R_{\beta,\alpha}[u](\cdot)$. The Preisach operator's input is $u(\cdot)$, and the output is given by [2]:

$$\bar{y}(t) = \iint_{\alpha \geq \beta} \mu(\beta, \alpha) R_{\beta,\alpha}[u](t)\, d\beta d\alpha \tag{2}$$

where $\mu(\cdot,\cdot) \in L^2(K)$ where K is a compact region in the (β, α) plane with $\alpha \geq \beta$ and support$(\mu) = K$. Denote the set of relays with value $+1$ at time t as $S_+(t)$, and the relays with value -1 at time t as $S_-(t)$. The curve separating S_+ and S_- (henceforth referred to as a memory curve) corresponds to the state of a dynamical system in the following sense [1]:

- If we knew the memory curve at time t and the input function over the interval $[t, T]$, then the memory curve at time T can be computed uniquely;
- Equation (2) can be considered as a map from the space of memory curves to the set of output values; in other words, the memory curve at a time t yields a unique output value at that time.

Define $r = \frac{\alpha - \beta}{2}$ and $s = \frac{\alpha + \beta}{2}$, the memory curve can be described by a function $s = \phi(r)$. The set of Preisach memory curves is defined as follows [1]:

Definition 2.1: The set $\mathcal{M}_0 \coloneqq \{\phi \mid \phi : \mathbb{R}_+ \to \mathbb{R},\ |\phi(\mathrm{r}) - \phi(\bar{\mathrm{r}})| \leq |\mathrm{r} - \bar{\mathrm{r}}|$ for all, $\mathrm{r}, \bar{\mathrm{r}} \geq 0,\ \mathrm{R}_{\mathrm{supp}}(\phi) < +\infty\}$. is called the set of Preisach memory curves. In the above, $R_{supp}(\phi)$ is the largest value of r such that $\phi(r) \neq 0$.

It can be shown that for a piecewise monotone function $u(t), \quad t \in [0,T]$, and an initial memory curve in $\mathcal{M}_0$, the memory curve at any time $t \in [0,T]$, denoted by $\phi_u(t)$, also belongs in $\mathcal{M}_0$ [1]. The memory curve $\phi_u(t)$ divides the set K into two connected components $S_+(t)$ and $S_-(t)$, which will be exploited in the next section (for a detailed description, please refer to Mayergoyz [2]).

Thus for a given μ, we can define a Preisach operator to be a map $\Gamma_\mu : C_{pm}[0,T] \to C_{pm}[0,T]$, where $C_{pm}[0,T]$ denotes the space of piecewise monotone continuous functions on $[0,T]$. For compatibility with experimental evidence, we restrict $\mu(\cdot,\cdot)$ to be a non-negative function. During the identification experiments, the fixed input $u \in C_{pm}[0,T]$ usually only affects a portion of the set K in the Preisach plane. Without loss of generality, we can restrict attention to this portion. *In the following, the set K_u is the subset of K affected by the input u.* Define the set of density functions:

$$\mathcal{K}_u \coloneqq \{\mu \in \mathrm{L}^2(\mathrm{K_u}) \,|\, \mu \geq 0\}. \tag{3}$$

Due to the assumption on K_u, the output at time 0 given by $\Gamma[u](0)$ is the same for all density functions in $\mathcal{K}_u$.

An important class of Preisach operators from the applications point of view, are the *piecewise, strictly increasing* (PSI) Preisach operators.

Definition 2.2: A Preisach operator is said to be piecewise strictly increasing (PSI) if $(\Gamma_\mu[u](T) - \Gamma_\mu[u](0))(u(T) - u(0)) > 0$ for a monotone input $u \in C[0,T]$ with $u(0) \neq u(T)$.

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Under the mild condition that the density function is continuous, and greater than zero along the diagonal $\alpha = \beta$, it is easy to show that the Preisach operator is PSI. Clearly, the definition of PSI is for a fixed density function and varying monotone increasing input functions u. However, in the identification

problem the input function is fixed while the density function is the independent variable. For a non-constant $u \in C_{pm}[0,T]$, let $0 = T_0 < T_1 < \cdots < T_N = T$ be the standard partition (see Brokate and Sprekels [1]). As u is non-constant, $u(T_{i-1}) \neq u(T_i)$ for $i = 1 \cdots N$. Denote $\Delta_i = [T_{i-1}, T_i]$ for $i = 1 \cdots N$. Define the set of density functions:

$$\begin{aligned} \mathcal{K}_{u,PSI} \;\coloneqq\; & \{0\} \cup \big\{ \mu \in L^2(K) \,|\, \mu \geq 0; \text{ and} \\ & (\Gamma[u](t_1) - \Gamma[u](t_2))(u(t_1) - u(t_2)) > 0 \\ & \text{for } t_1, t_2 \in \Delta_i, \;\; t_1 \neq t_2; \;\; i = 1, \cdots, N. \big\} \end{aligned} \tag{4}$$

The idea is that the density function for a PSI Preisach operator (that can be identified) should come from this set. The sets $\mathcal{K}_u$ and $\mathcal{K}_{u,PSI}$ are clearly non-empty with $\mathcal{K}_{u,PSI} \subset \mathcal{K}_u$. To further study their properties, recall that a convex set C is a subset in a vector space, such that if $x_1, x_2 \in C$, then $\theta\, x_1 + (1-\theta)\, x_2 \in C$ for $0 \leq \theta \leq 1$ (Luenberger [11]). A *cone with vertex at the origin* is a set C in a vector space with the property that if $x \in C$, then $\theta\, x \in C$ for all $\theta \geq 0$. A convex cone is defined as a set that is both convex and a cone. Closed convex sets are important from the point of view of numerical methods for constrained optimization problems.

Lemma 2.1: The set $\mathcal{K}_u$ is a closed convex cone with vertex at the origin, while the set $\mathcal{K}_{u,PSI}$ is a convex cone with vertex at the origin that is not closed.

Proof: The assertions about $\mathcal{K}$ follow directly from the definitions of $\mathcal{K}_u$ and of closed convex cone with vertex at the origin $\mu = 0$. Using the definitions, it is easy to see that $\mathcal{K}_{u,PSI}$ is a convex cone with vertex at the origin. However, it is not a closed set for the following reason. Let $\mu^* \in \mathcal{K}_u$ with $\mu > 0$ on K except on the right-triangle formed by the points $u(T_{N-1})$ and $u(T_N)$ with the line $\alpha = \beta$ in the Preisach plane. Then we can construct a sequence $\mu_n > 0$ on K, with $\mu_n \in \mathcal{K}_{u,PSI}$ that converges to μ^* in norm. Now $\mu^* \notin \mathcal{K}_{u,PSI}$, as in the time interval Δ_N, the output $\Gamma_{\mu^*}[u]$ is a constant. Hence $\mathcal{K}_{u,PSI}$ is not closed. ■

The importance of the sets $\mathcal{K}_u$ and $\mathcal{K}_{u,PSI}$ for identification problems will be seen in Theorem 3.1 in the next section.

It has been shown that a PSI Preisach operator has several useful mathematical properties [1]: (a) Lipschitz continuity, (b) regularity, (c) invertibility. In addition it has the so-called "congruency", and "wiping-out" properties (please refer to Mayergoyz for an excellent description [2]). The wiping out property is observed in magnetic materials and "smart" actuators such as those based on piezoelectricity and magneto-striction. This makes the Preisach operator a valuable mathematical tool for researchers in smart structures. However, it is possible that the congruency property is violated in these materials, which can only be verified by exhaustive experimental work.

We take the view (which is implicit in the rest of literature in this area) that we will assume the congruency property to hold and perform identification experiments. Once the best approximation to the density function is obtained, we will perform *prediction* experiments in which the density function will be tested against inputs not used in the identification. This comparison will yield an answer to the suitability of the Preisach model for the given material. An important point to note is that the Preisach operator is rate-independent, and therefore the test signals need only have different local maxima and minima from the signals used in identification - the frequency of the test signals do not matter! In related work by Venkataraman, Tan and Krishnaprasad [9], the identified Preisach model for a magnetostrictive actuator was used in a controller based on an approximate inverse of the Preisach operator. The signals used in the test of the inverse were significantly different in both magnitude and frequency in this test.

III. THE PROBLEM

When using the Preisach operator to model a physical system, it is necessary to find the density function $\mu(\beta, \alpha)$, that models the physical phenomenon. We have to determine μ from experiments, by observing outputs $y(\cdot)$ that correspond to inputs $u(\cdot)$. What we would like is an approximation of the density function μ to put into the Preisach operator which fits the current input and output data.

The proof of Mayergoyz's representation theorem [2] yields a way to exactly compute the density function μ, but we need continuous inputs and no sensor noise. The second condition cannot be assumed from experimental data and so we need another method to determine μ. Due to the experimental limitations, we have to discretize the input values. This leads to a discretization of the Preisach plane. This procedure is described in Shirley and Venkataraman [12]. In this paper, we show that discretization is desirable from a theoretical point of view also. The key is the singular value decomposition of the operator Φ_u described in Equation (5) below and Picard's theorem (Theorem 3.3).

Recall that for all μ in $\mathcal{K}_u$, the initial output $w = \Gamma[u](0)$ is fixed. For a given $u(\cdot) \in C_{pm}[0,T]$, define the operator:

$$\begin{array}{rccl} \Phi_u : & \mathcal{K}_u & \rightarrow & L^2[0,T] \\ & \mu(\beta,\alpha) & \mapsto & y = \Phi_u\, \mu(\cdot) = \Gamma_\mu[u](\cdot) - w \end{array} \tag{5}$$

The operator Φ_u is a *linear* operator between $L^2(K_u)$ and $L^2[0,T]$ which is not true for the Preisach operator Γ! The key is that Φ_u maps the function $\mu = 0$ to the function $y = 0$. The following is an important theorem that concerns the identification problem for a PSI Preisach operator.

Theorem 3.1: Let u be a non-constant function in $C_{pm}[0,T]$ and $\Phi_u : \mathcal{K}_u \rightarrow L^2[0,T]$ be as defined in (5. Then,

1) $\{\mu \in \mathcal{K}_u \,|\, \Phi_u\, \mu(t) = 0\} = \{0\}$;
2) $\Phi_u : \mathcal{K}_u \rightarrow L^2[0,T]$ is injective;
3) $\overline{\text{Range}\,[\Phi_u]} \neq L^2[0,T]$;
4) Suppose that $y \in \text{Range}[\Phi_u]$ and Γ is PSI. Then there exists a unique $\mu \in \mathcal{K}_{u,PSI}$ such that $\Phi_u\, \mu = y$.

Proof:

1) The claim follows from the definition of $\mathcal{K}_u$ and the assumption that $u \in C_{pm}[0,T]$ is non-constant.
2) Suppose $\|\mu_1 - \mu_2\|_{\mathcal{K}_u} \neq 0$. Then there exists a set $C \subset K_u$ in the (β, α) plane, with non-zero measure, such that $\mu_1 \neq \mu_2$ on C. As K_u is the region influenced by the input u, and $u \in C_{pm}[0,T]$, there exists a set of non-zero measure in $[0,T]$ such that $y_1 \neq y_2$ on this set. In other words, $\|y_1 - y_2\|_{L^2[0,T]} \neq 0$.

3) The fact that the range of Φ_u is a strict subset of $L^2[0,T]$ follows from the observation that the standard partition of y is the same as that of u for non-negative measures $\mu \in \mathcal{K}_u$.
4) The last conclusion follows from the injectivity of Φ_u when restricted to the set $\mathcal{K}_{u,PSI}$ (from item 2 above), and the definition of $\mathcal{K}_u$.

■

Remarks

We conclude from the above theorem and Lemma 2.1 that:

- Even though $\mu \in \mathcal{K}_{u,PSI}$ for a PSI Preisach operator, we have to carry out the identification over the set $\mathcal{K}_u$, as $\mathcal{K}_{u,PSI}$ is not a closed convex set.
- The set $\mathcal{K}_u$ is a closed convex cone, and as Φ_u is a linear injective operator on this set, it is "well-suited" from a numerical point of view.
- However, as the closure of the range of Φ_u is not all of $L^2[0,T]$, we need to construct a regularization scheme to counter-act noise (see the remarks after (Picard's) Theorem 3.3 below).

□

Next, we recall some standard facts about linear operators that can be utilized to solve for μ, in the operator equation $\Phi_u\,\mu = y$. It is also useful to think of Φ_u as acting not just on $\mathcal{K}_u$, but on all of $L^2(K_u)$.

Lemma 3.1: The operator $\Phi_u : L^2(K_u) \to L^2[0,T]$ is a compact linear operator.

Proof: For a given input function $u(\cdot) \in C_{pm}[0,T]$, the memory curve $\phi_u(t);\ t \in [0,T]$ defined in the previous section uniquely determines the values of the kernel function $R_{\beta,\alpha}[u](t);\ t \in [0,T]$ in Equation (2). For fixed t, this kernel function satisfies:

$$R_{\beta,\alpha}[u](t) = \begin{cases} +1; & \text{if } (\beta,\alpha) \in S_+(t); \\ -1; & \text{if } (\beta,\alpha) \in S_-(t). \end{cases} \tag{6}$$

For fixed (β,α), the kernel function takes $+1$ or -1 on finite intervals for $u(\cdot) \in C_{pm}[0,T]$. These facts imply that the kernel function $R_{\cdot,\cdot}[u](\cdot)$ belongs in $L^2(K_u \times [0,T])$. Therefore, Φ_u is a Hilbert-Schmidt and hence a compact linear operator. ■

The adjoint of operator Φ_u is defined to be : $\Phi_u^* : L^2[0,T] \to L^2(K_u)$ with

$$\begin{aligned} &< \Phi_u^* y, \mu >_{L^2(K_u)} \quad = \quad < y, \Phi_u \mu >_{L^2[0,T]}; \\ &\text{where } \ y \in L^2[0,T] \quad \text{and} \quad \mu \in L^2(K_u), \end{aligned} \tag{7}$$

where $< \cdot, \cdot >$ denotes the inner product of functions. Specifically:

$$\begin{aligned} < \phi, \mu >_{L^2(K_u)} &= \iint_{K_u} \phi(\beta,\alpha)\,\mu(\beta,\alpha)\,d\beta\,d\alpha \\ < y_1, y_2 >_{L^2[0,T]} &= \int_0^T y_1(t)\,y_2(t)\,dt. \end{aligned}$$

One defines norms on the spaces $\mathcal{K}$ and $L^2[0,T]$ by setting:

$$\|\mu\|_{L^2(K_u)} = \left[< \mu, \mu >_{L^2(K_u)}\right]^{\frac{1}{2}} \tag{8}$$

$$\|y\|_{L^2[0,T]} = \left[< y, y >_{L^2[0,T]}\right]^{\frac{1}{2}}. \tag{9}$$

A basic fact about the compact linear operator Φ_u is that its adjoint Φ_u^*, and the composition $\Phi_u^*\,\Phi_u$ are also compact linear operators (Theorem 4.10, Kress [13]).

Definition 3.1: (Singular Values) The non-negative square roots of the eigenvalues of the non-negative self-adjoint compact linear operator $\Phi_u^*\,\Phi_u : L^2(K_u) \to L^2(K_u)$ are called the singular values of Φ_u. The singular values of Φ_u will be denoted by $\{\sigma_n\}_{n=1}^{\infty}$.

Note that the singular values could be repeated. The important fact regarding singular values relevant to this paper is that as $n \to \infty$, $\sigma_n \to 0$. The following theorem combines the statements of Theorem 8.3-1 in Kreyszig [14], and Theorem 15.16 in Kress [13].

Theorem 3.2: The set of singular values of the operator $\Phi_u : L^2(K_u) \to L^2[0,T]$ is countable (perhaps finite or even empty), and the only possible point of accumulation is $\sigma = 0$. Furthermore, there exists orthonormal sequences $\{\phi_n\}$ in $L^2(K_u)$ and $\{g_n\}$ in $L^2[0,T]$ such that

$$\Phi_u\,\mu_n = \sigma_n\,y_n; \quad \Phi_u^*\,y_n = \sigma_n\,\mu_n, \tag{10}$$

for all $n \in \mathbb{N}$. For each $\mu \in \mathcal{K}_u$ we have the singular value decomposition

$$\mu = \sum_{n=1}^{\infty} \ < \mu, \mu_n >_{\mathcal{K}_u} \ \mu_n, \tag{11}$$

and

$$\Phi_u\,\mu = \sum_{n=1}^{\infty} \ \sigma_n \ < \mu, \mu_n >_{\mathcal{K}_u} \ g_n. \tag{12}$$

Proof: The proof follows from those of Theorem 8.3-1 in Kreyszig [14], and Theorem 15.16 in Kress [13], combined with item 1 of Theorem 3.1. ■

Using Theorem 3.2, the solution to the operator Equation $\Phi_u\,\mu = y$ is given by Picard's theorem [13] in the case when μ is unconstrained:

Theorem 3.3: (Picard) Let $\Phi_u : L^2(K_u) \to L^2[0,T]$ have the singular system (σ_n, μ_n, g_n). The equation

$$\Phi_u\,\mu = y$$

is solvable if and only if y belongs to $[\text{Null}(\Phi_u^*)]^{\perp}$ (which is the same as the closure of Range$[\Phi_u]$ in $L^2[0,T]$) and satisfies

$$\sum_{n=1}^{\infty} \ \frac{1}{\sigma_n^2} \ | < y, y_n > |^2 < \infty$$

In this case, a solution is given by:

$$\mu = \sum_{n=1}^{\infty} \ \frac{1}{\sigma_n} \ < y, y_n > \ \mu_n.$$

As observed by Kress [13], the above theorem clearly demonstrates the ill-posed nature of the equation $\Phi_u\mu = y$. If the right hand side is perturbed to $y^{\epsilon} = y + \epsilon\,y_n$, we obtain the solution

$$\mu^{\epsilon} = \sum_{n=1}^{\infty} \ \frac{1}{\sigma_n} \ < y^{\epsilon}, y_n >_{L^2[0,T]} \ \mu_n = \mu + \frac{\epsilon}{\sigma_n}\,\mu_n.$$

Thus the ratio $\frac{\|\mu^\epsilon - \mu\|}{\|y^\epsilon - y\|} = \frac{1}{\sigma_n} \stackrel{n\to\infty}{\longrightarrow} \infty$ by Theorem 3.2. The above argument implies that the inverse operator Φ_u^{-1} is unbounded, and thus there is a need for a *bounded approximation* to the unbounded inverse. This brings to the topic of regularization that is discussed in the next subsection. Another important reason for regularization is that Φ_u for a given $u \in C_{pm}[0,T]$ is *not injective* in $L^2[0,T]$ as observed in Theorem 3.1. In other words, Picard's condition that the closure of Range$[\Phi_u]$ be equal to $L^2[0,T]$, for the solvability of $\Phi_u = y$, where $y \in L^2[0,T]$ is not satisfied when $u \in C_{pm}[0,T]$. Therefore, when y is perturbed by noise to y^ϵ, it is possible that there is no non-negative solution μ^ϵ to the equation $\Phi_u\, \mu^\epsilon = y^\epsilon$.

A. Collocation and Singular Value Decomposition based Regularization

As Φ_u is a compact linear operator (Lemma 3.1), it does not possess a bounded linear inverse Φ_u^{-1} (Kress [13], Kirsch [15]). The computation of the density function needs a regularization scheme that yields a bounded approximation of the unbounded inverse operator. To be precise:

Definition 3.2: A regularization scheme is a family of linear and bounded operators

$$\Psi_\lambda : L^2[0,T] \to \mathcal{K}_u, \quad \lambda > 0$$

such that

$$\lim_{\lambda\to 0} (\Phi_u \circ \Psi_\lambda) y = P\, y, \quad \text{for all} \quad y \in L^2[0,T],$$

where $P : L^2[0,T] \to \overline{\text{Range}(\Phi_u)}$ is the orthogonal projection operator.

In other words, the operators $\Phi_u \circ \Psi_\lambda$ converge pointwise to the identity operator on $\overline{\text{Range}(\Phi_u)}$.

The following is a standard result for compact linear operators that says that the family $\Phi_u \circ \Psi_\lambda$ does not converge in the norm. The implication is that one can only expect pointwise convergence as in the definition of the regularization scheme.

Theorem 3.4: Let $\Psi_\lambda : L^2[0,T] \to \mathcal{K}_u, \quad \lambda > 0$ be a regularization scheme for Φ_u. Then the operators Ψ_λ cannot be uniformly bounded with respect to λ, and the operators Ψ_λ cannot be norm convergent as $\lambda \to 0$.

Proof: The proof is an elementary consequence of the fact that Φ_u is a compact linear operator and can be found in Kress [13](page 299). ∎

One of the standard regularization strategies is to compute the singular value decomposition of a compact linear operator and then truncate the singular values if they are smaller than a tolerance value (Kirsch [15]). In this paper, we first construct a finite dimesional approximation to Φ_u and then carry out a singular value decomposition. Another standard method of regularization for ill-posed problems is *discretization* [13], [15]. This approach leads to a moment problem that is discussed below. The regularization scheme discussed in this paper involves collocation in time, and singular value decomposition of a finite-dimensional approximation to the operator Φ_u.

Let $0 = t_1 < \cdots < t_n = T$ be a discretization of time, so that we have :

$$\left.\begin{array}{rcl} (\Phi_u\, \mu)(t_j) & = & y(t_j), \\ \int_K R_{\beta,\alpha}[u](t_j)\, \mu(\beta,\alpha)\, d\beta d\alpha & = & y(t_j), \end{array}\right\} \; j = 1, \cdots, n. \tag{13}$$

where $\mu(\beta,\alpha)$ is unknown, while $u(\cdot)$ and $y(t_j)$ are known. The time instants $t_1, \cdots, t_n$ are known as *collocation* points. It has been shown that for such a *moment* problem, the sequence of approximations $\mu_n(\beta,\alpha)$ to the solution $\mu(\beta,\alpha)$, is a finite linear combination of the functions $R_{\beta,\alpha}[u](t_j)$ [15]. Therefore the moment solutions $\mu_n(\beta,\alpha)$ are as smooth as the functions $R_{\beta,\alpha}[u](t_j)$ even if the true solution is smoother. For the Preisach operator, $R_{\beta,\alpha}[u](t_j)$ are discontinuous functions by Equation (6), and thus the numerically obtained approximations of $\mu(\beta,\alpha)$ tend to have discontinuous character (please see Figures 3(a), 6(a), 8(a) and 9(a) at this time).

Furthermore, due to the time discretization, the operator $\Phi_u : \mathcal{K}_u \to \mathbb{R}^n$ is no longer injective (as in Theorem 3.1). Thus the density function μ cannot be exactly identified, but only a step function approximation can be identified. If the minimum and maximum values of the input signal $u(t)$ are u_{min} and u_{max} respectively, then for experimental implementation, one considers the input to take one of the discrete values $u_{min} = u_1 < \cdots < u_N = u_{max}$ (where $u_{i+1} - u_i = \Delta u$ is a constant), at the collocation times t_i. Therefore the memory curves, at the collocation times, have corners that lie on the grid (u_m, u_n) where $u_1 \le u_m \le u_n \le u_N$. Hence the change in the output can only occur in discrete values, and one can consider the density function to be piecewise constant on the rectangles of the grid. Denote the set of step functions defined on the grid by $\mathcal{K}_m$.

As step functions are dense in $\mathcal{K}_u$ (Littlewood's second principle in Royden [16] (page 127), and Vitali's convergence theorem in Folland [17]), given any $\mu \in \mathcal{K}_u$, there exists a sequence $\mu_m \in \mathcal{K}_m$ such that

$$\lim_{m\to\infty} \|\mu - \mu_m\|_{\mathcal{K}_u} = 0.$$

As the kernel function satisfies $|R_{\beta,\alpha}[u](t)| = 1$, we have:

$$\begin{aligned} \|\Phi_u\, \mu_m - \Phi_u\, \mu\|^2 &\le \int_0^T \left(\iint_{\mathcal{K}_u} R_{\beta,\alpha}[u](t)(\mu_m - \mu) d\alpha d\beta \right)^2 dt \\ &\le (\text{meas}(K_u))^2 \int_0^T \|\mu - \mu_m\|^2_{\mathcal{K}_u} dt \\ &\quad \text{(by Cauchy-Schwartz Inequality)} \\ &= (\text{meas}(K_u))^2\, T\, \|\mu - \mu_m\|^2_{\mathcal{K}_u}. \end{aligned}$$

Thus the output functions converge in norm.

Once a set of collocation points have been determined, one can discretize the input values so that a uniform grid is established in the region K_u in the Preisach plane. Then the equation $\Phi_u\, \mu = y$ can be written as a linear equation

linear equation

$$Y = AX + \epsilon, \tag{14}$$

as described in Shirley and Venkataraman [12]. Each element of X (except the last) denotes the area under the density function for a particular grid element. The last element denotes

the initial output value (at time 0). To account for noise, we estimate this value also.

We need is a way to solve for X that will best fit the data, but at the same time keep $x_i \geq 0$, since X represents the integral of a density function over a grid element.

We would like to minimize the function

$$f(X) = \frac{1}{2} \|AX - Y\|^2 \tag{15}$$

where

$$A : \mathbb{R}^n \to \mathbb{R}^m, \quad X \in \mathbb{R}^n, \quad Y \in \mathbb{R}^m, \quad \text{and } rank(A) = m$$

with the inequality constraint $g(X) = X \geq 0$.

The Lagrange multiplier theorem yields the existence of a $\lambda \in \mathbb{R}^n$, such that the necessary conditions for X to minimize (15) are [11],

1)
$$\lambda_i = 0 \text{ when } X_i \neq 0 \tag{16}$$

2)
$$\lambda_i \geq 0 \text{ when } X_i = 0 \tag{17}$$

3) The augmented function
$$\begin{aligned} \bar{f}(X) &= f(X) - \lambda^T g(X) \\ &= f(X) - \lambda^T X \end{aligned} \tag{18}$$
satisfies
$$\frac{\partial \bar{f}(X)}{\partial X} = X^T A^T A - Y^T A - \lambda^T = 0 \tag{19}$$

We used the MATLAB routine "quadprog" to solve this problem and the results are described in Section 4. This is also the approach used in Venkataraman, Tan and Krishnaprasad [9]. Once X is obtained we divide the elements by the area of the corresponding grid element to obtain the density function.

B. Limited experimental data

The method described in the last subsection works very well when the experimental input-output data is in a form to yield an $m \times n$ A matrix with rank n. However, if we do not have such an situation (which can easily arise by choosing a finer discretization of the input), we must reformulate the problem. In the following, we consider an even more general problem where the rank of A is less than $\min\{m, n\}$:

$$\text{minimize} \quad f(X) = \frac{1}{2} \|AX - Y\|^2; \;\; rank(A) < \min\{m, n\}, \tag{20}$$

with the conditions:

$$\begin{aligned} &AX = Y + \epsilon \\ &X_i \geq 0, \quad \forall\, i = 1, \ldots, n \end{aligned} \tag{21}$$

Now, we need to minimize (20), and $\frac{1}{2} \|X\|^2$. This problem can be tackled by different methods. One method involves the minimization of a weighted combination:

$$\begin{aligned} &\text{minimize} \quad g(X) = \tfrac{\alpha}{2} \|X^2\| + \tfrac{1}{2} \|AX - Y\|^2, \\ &\text{subject to} \quad X_i \geq 0, \quad \forall\, i = 1, \ldots, n, \quad \text{where} \quad \alpha > 0. \end{aligned}$$

This method when related to the original operator equation is called Tikhonov regularization. We found that the Tikhonov regularization approach is unstable numerically, as the choice of α affects the solution greatly. Another method to solve (21) involves the singular value decomposition of A.

Let $rank(A) = q < \min\{m, n\}$. If we perform a singular value decomposition on $A^T A$, then we get:

$$A^T A = V S V^T, \tag{22}$$

where S is an $n \times n$ diagonal matrix with rank $q < n$, and $V^T V = I_{n \times n}$ (see page 54, Bellman [18]). The n singular values of $A^T A$ are the diagonal elements of S and be ordered as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. By the remarks following Picard's theorem 3.3, we see that "small" singular values should be discarded as they contribute to an amplification of the noise in the solution. We perform the following steps:

1) Pick a tolerance value $\epsilon > 0$, and set those singular values that are less than ϵ to 0. Call the resulting matrix that is obtained from S as S_1.
2) Remove the rows and columns of S_1 that are identically zero, and also remove the corresponding columns of U and V. By this procedure, we obtain a $\bar{q} \times \bar{q}$ diagonal matrix $\hat{S}$, and a $n \times \bar{q}$ matrices $\hat{V}$ that satisfies
$$\hat{V}^T \hat{V} = I_{\bar{q} \times \bar{q}}. \tag{23}$$

The operator norm of the matrix $A^T A - \hat{V} \hat{S} \hat{V}^T$ can be seen to be:

$$\begin{aligned} \|A^T A - \hat{V}\, \hat{S}\, \hat{V}^T\| &= \|A^T A - V\, S_1\, V^T\| \\ &= \|V\, (S - S_1)\, V^T\| \\ &= \|S - S_1\| \\ &< \epsilon. \end{aligned}$$

We seek solutions of the type $X = \hat{V} Z$ for the Problem (15). Using Equations (22) and (23), the cost function (20) can be transformed to $Z^T \hat{S} Z - Y^T A \hat{V} Z$. Thus we form the constrained optimization problem:

$$\text{minimize} \qquad f(Z) = Z^T \hat{S} Z - Y^T A \hat{V} Z, \tag{24}$$

$$\text{subject to} \qquad g(Z) = \hat{V} Z \geq 0. \tag{25}$$

Then once we have a minimizer Z^* to the above problem, then the desired solution is given $X^* = \hat{V} Z^*$. The constrained minimization problem (25) can be solved by using the MATLAB routine "quadprog". Again, once X^* is obtained we divide the elements by the area of the corresponding grid element to obtain the density function.

IV. APPLICATIONS

The following experiments were performed in a MATLAB environment on a PC running a 1.6 GHz Athlon Processor, with a 1 GB RAM.

A. Magnetostrictive actuator

We used experimental data from a commercial magnetostrictive actuator from Venkataraman, Tan and Krishnaprasad [9] in our initial numerical experiments. There, a Preisach operator was used to model the average magnetic field versus magnetization characteristic. Figure 2 shows the average magnetic field versus magnetization obtained in the experiment. In the experiment, the magnetic field input took values between $10\,Oe$ and $410\,Oe$. The output values were measured in Oe. The results of the numerical experiments for various discretization levels for the input magnetic field are tabulated in Table I.

Figure 4 and 5 show the approximate densities computed for different discretization levels. As expected from the theory of the moment method (see Subsection III-A), the computed density functions show a discontinuous character. To obtain a smooth density function, one simply convolves the obtained density function with a smooth function of compact support. This procedure is called *mollification* (see Kirsch [15] for a discussion). We do not proceed along these lines as the choice of the convolution kernel greatly influences the smoothed out density function. Furthermore, the smoothed out density function performed poorly in comparison to the original solution, when we tried to match the output of the Preisach operator with the given output waveform.

Figures 4(c) and 5(c) needed the method described in Subsection III-B, as the matrix $A^T A$ (for these two cases) was singular. In other words, these two cases the experimental data available was limited. For a discretization level of 20 Oe for the magnetic field, a comparison between the output of the Preisach operator (AX) and the given output waveform (Y) can be seen in figure 3(b). Similarly, for the discretization level of $10Oe$, the comparison between the output of the Preisach operator (AX) and the given output waveform (Y) can be seen in figure 6(b).

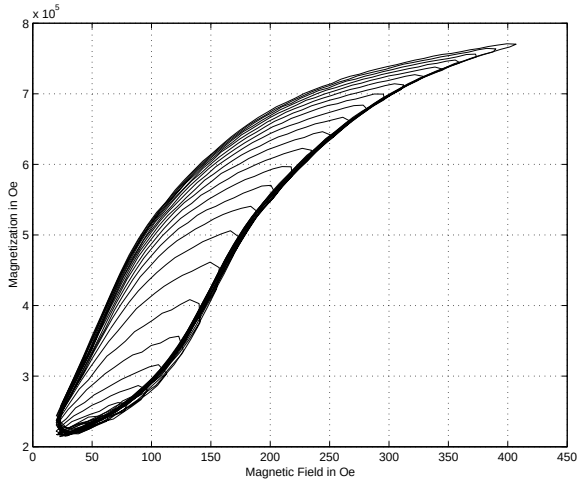
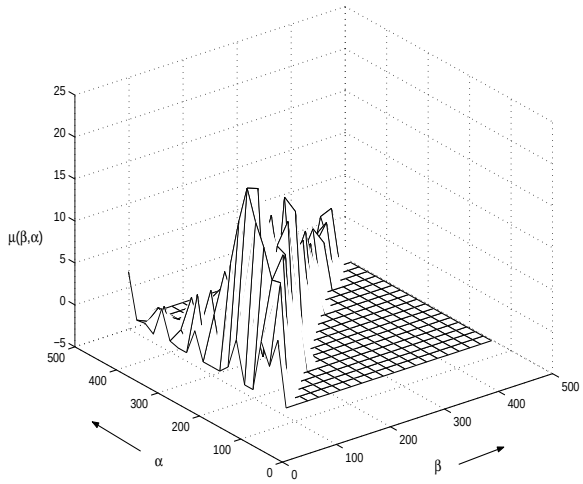
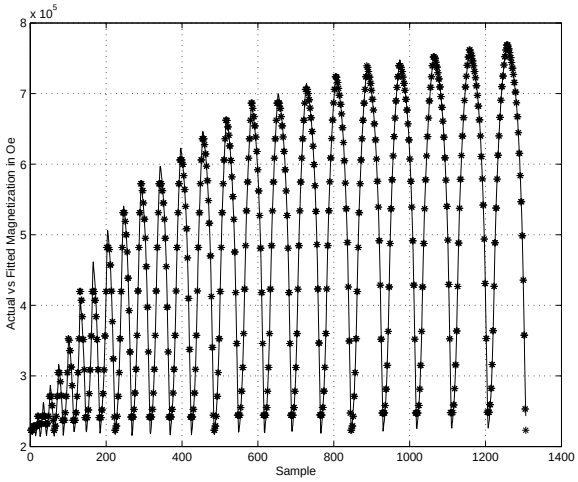


Fig. 2. Experimentally determined Magnetic Field versus Magnetization curves for an ETREMA magnetostrictive actuator.

1) Prediction Experiments: From Table I, we see that the 2 and infinity norm of the error $AX - Y$, is smaller for finer discretizations of the input signal. We also see that



(a) The computed Preisach density function



(b) Comparison between fitted and actual Magnetization versus Magnetic field (* indicates the fitted values)

Fig. 3. Identification experiments for a commercial magnetostrictive actuator [9] with a magnetic field discretization of $20\,Oe$.

the time taken to compute the density functions increases at an exponential rate for finer discretizations. So, the question arises whether one can use a smaller amount of data to speed up the computation of the density for the finer discretization levels. To test this, we conducted numerical experiments where only part of the input signal was used for the identification and then the computed approximate density was used to predict the output for the entire input signal. In this case, we used every other cycle from the input signal to obtain the best approximation to the density function. Again the "quadprog" routine in MATLAB was used to compute the approximation to the density function. The obtained density function can be seen in Figure 7(a). Then the entire input signal was applied to the identified Preisach operator and a comparison between the simulated and the actual experimental Magnetization signal

Magnetic Field Discretization (Oe)	Time taken for Density Computation (s)	$\|\Phi_u\,\mu - y\|_\infty$ (Oe)
6.25	4755.1	3.55×10^4
10	717.69	3.21×10^4
12.5	143.27	4.19×10^4
20	11.70	6.27×10^4
25	1.88	6.70×10^4
40	0.35	7.15×10^4

TABLE I

COMPARISON OF APPROXIMATE DENSITY FUNCTIONS OBTAINED WITH DIFFERENT DISCRETIZATION VALUES FOR THE MAGNETIC FIELD.

Full/Partial Input Data	Input Discretization (Oe)	Time taken (s)	$\|\Phi_u\,\mu - y\|_\infty$ (Oe)
Full	6.25	4755.1	3.55×10^4
	10	717.69	3.21×10^4
Partial	6.25	1228.3	3.16×10^4
	10	108.57	3.09×10^4

TABLE II

COMPARISON OF COMPUTATION TIMES FOR THE NUMERICAL EXPERIMENTS.

were made (see Figure 7(b)). Table II shows the infinity norm of the error between the output of the Preisach operator and the given output waveform. We note the savings in the time of computation and the reduction in the infinity norm of the error. We do not yet fully understand the theoretical reason for the reduction in the error, and is something to be explored in the future. Note that, as the Preisach operator is rate-independent, one does not need to test the prediction of the Preisach operator for varying input frequencies - only the input magnitude needs to be varied. This is precisely what was done in our experiments. To see prediction tests in an open-loop control context, please see Venkataraman, Tan and Krishnaprasad [9].

B. Electro-active Polymers

In this subsection, we apply our identification method to a couple of electro-active polymers. Electro-active polymer research has been very active over the past few years due to the synthesis of novel materials that have properties very similar to biological muscle. However, these materials like their crystalline counterparts show a hysteretic relationship between input variables (typically the electric field) and output variables (typically electric displacement). As far as we are aware, the Preisach operator has not been used to model hysteresis in electro-active polymers though it is a natural choice. *The data in this section was obtained by enlarging Figure 4 in Petchsuk and Chung [19] and Figure 6 in Lu, Schirokauer and Scheinbeim [20]; manually putting a grid on the figures; and approximating the input and output values*. As we are only interested in demonstrating the utility of our method to cases where the published data is limited, it is possible that there might be slight inaccuracies in the data.

First, we applied our method to the data published by Petchsuk and Chung [19] on an electro-active polymer (VDF/TrFE/HFP $(55.17/42.35/2.46)$ terpolymer 121) at $42^\circ\,C$. This polymer exhibits significant hysteresis in its Electric Displacement vs Electric Field characteristic. The discretization of the electric field was chosen to be $12.5\,MV/m$. The identified Preisach density function can be seen in Figure 8(a). The comparison between the fitted data and the experimental data obtained from Petchsuk and Chung [19] can be seen in Figure 8(b). Finally, we show another application to a VDF/HFP (5 %) electro-active polymer in Figures 9(a) and 9(b), the data for which was obtained from X. Lu, A. Schirokauer and J. Scheinbeim [20]. The discretization of the electric field was again chosen to be $12.5\,MV/m$.

V. CONCLUSIONS

In this paper, we have described a method to compute (an approximation of) the Preisach density function when there is insufficient experimental data. This is a situation that is frequently encountered in practice. Previous methods assumed the availability of sufficient data for the identification to be performed, or considered a very coarse grid on the Preisach plane leading to a poor approximation of the actual density function. We described a method that utilizes all the information present in the input-output data to obtain the best approximation possible. We also presented the theoretical basis for our method and explained why the computed densities where discontinuous in nature. We also presented numerical results using experimental input-output data, where we compare the results of our identification with published data.

ACKNOWLEDGMENT

We gratefully acknowledge Prof. Chung, Prof. Scheinbeim and the Materials Research Society, for allowing us to use the data in [19] and [20] in this manuscript. We also acknowledge the anonymous reviewers for helping us to greatly improve on the original manuscript.

REFERENCES

[1] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*. Springer, 1996.
[2] I. Mayergoyz, *Mathematical Models of Hysteresis*. Springer, 1991.
[3] E. D. Torre, *Magnetic Hysteresis*. IEEE Press, New York., 1999.
[4] O. Henze and W. Rucker, "Identification procedures of preisach model," *IEEE Transactions on Magnetics*, vol. 38, no. 2, pp. 833–836, March 2002.
[5] W. S. Galinaitis, D. Joseph, and R. C. Rogers, "Parameter identification for preisach models of hysteresis," in *Proceedings of DETC 2001 ASME Design Engineering Technical Conferences*, Mar 2001, pp. 267–277.
[6] K. H. Hoffmann and G. H. Meyer, "A least squares method for finding the preisach hysteresis operator from measurements," *Numer. Math.*, vol. 55, pp. 695–710, 1989.
[7] H. T. Banks, A. J. Kurdila, and Webb, "Identification of hysteretic control influence operators representing smart actuators - part i: Formulation," Center for research in scientific computation, North Carolina State University, Tech. Rep. CRSC-TR96-14, 1996.
[8] W. Galinaitis and R. C. Rogers, "Control of a hysteretic actuator using inverse hysteresis compensation," in *Mathematics and Control in Smart Structures, Smart Structures and Materials 1998*, V. V. Varadhan, Ed., vol. 3323, Mar. 1998, pp. 267–277.
[9] R. Venkataraman, X. Tan, and P. Krishnaprasad, "Control of hysteresis: Theory and experimental results," in *Smart Structures and Materials 2001: Modeling, Signal Processing, and Control in Smart Structures*, V. Rao, Ed., vol. 4326, Mar. 2001.

[10] M. Krasnoselskii, I. Mayergoyz, D. Rachinskii, and A. Pokrovskii, "Differential equations with hysteresis nonlinearities of vector relay systems type," *Russian Acad. Sci. Dokl. Math*, vol. 48, no. 1, pp. 105 – 109, 1993.
[11] D. G. Luenberger, *Optimization by Vector Space Methods*. Wiley, 1968.
[12] M. Shirley and R. Venkataraman, "On the identification of Preisach measures," in *Modeling, Signal Processing and Control in Smart Structures and Materials*, R. Smith, Ed., vol. 5049, July 2003, pp. 326–336.
[13] R. Kress, *Linear Integral Equations*, 2nd ed. Springer Verlag, 1999.
[14] E. Kreyszig, *Introductory Functional Analysis with Applications*. John Wiley & Sons, 1978.
[15] A. Kirsch, *A Mathematical Introduction to the Theory of Inverse Problems*. Springer Verlag, 1996.
[16] H. L. Royden, *Real Analysis*. Macmillan Publishing Company, New York, NY 10022., 1989.
[17] G. Folland, *Real Analysis: Modern Techniques and their Applications*. John Wiley and Sons, Inc., 1984.
[18] R. Bellman, *Introduction to Matrix Analysis*. SIAM, Philadelphia, PA 19104., 1995.
[19] A. Petchsuk and T. Chung, "Synthesis and electric property of VDF/TrFE/HFP TerPolymers," in *Electroactive Polymers (EAP)*, Y. B. Q. M. Zhang, Takeo Furukawa and J. Scheinbeim, Eds., vol. 600, Nov-Dec 1999, pp. 53–60.
[20] X. Lu, A. Schirokauer, and J. Scheinbeim, "Giant electrostrictive response in Poly(vinylidene-fluoride hexafluoropropylene) copolymer," in *Electroactive Polymers (EAP)*, Y. B. Q. M. Zhang, Takeo Furukawa and J. Scheinbeim, Eds., vol. 600, Nov-Dec 1999, pp. 61–69.

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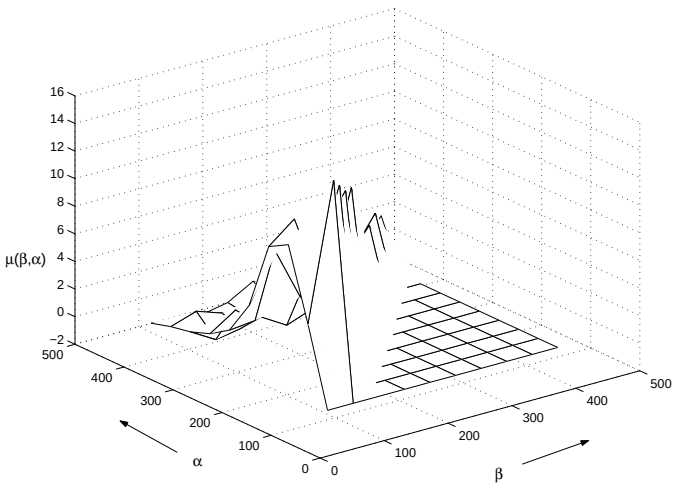
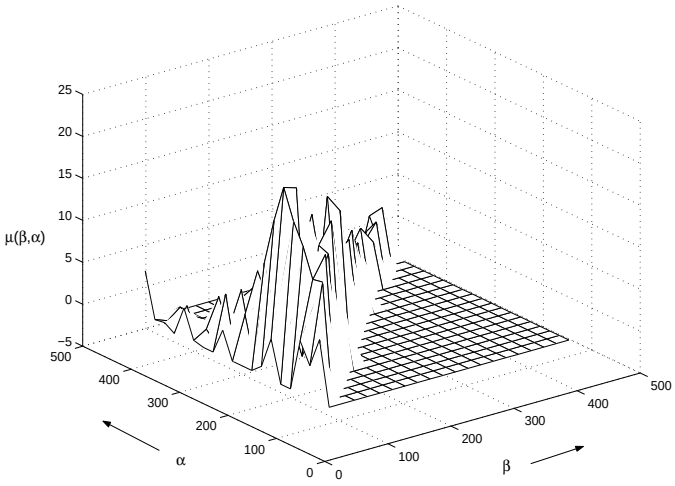
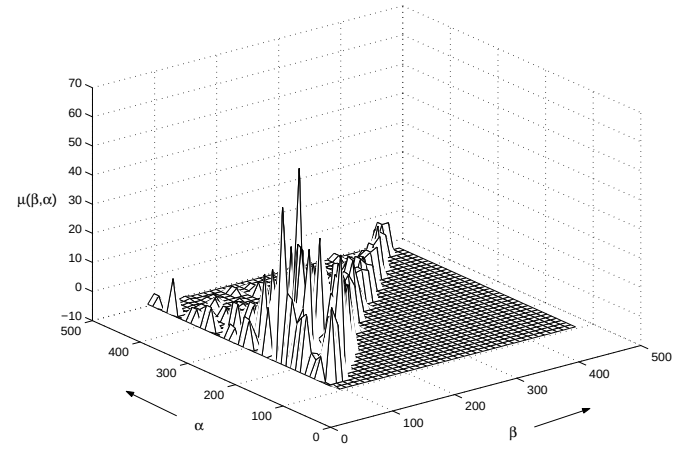


(a) Discretization = 40 Oe

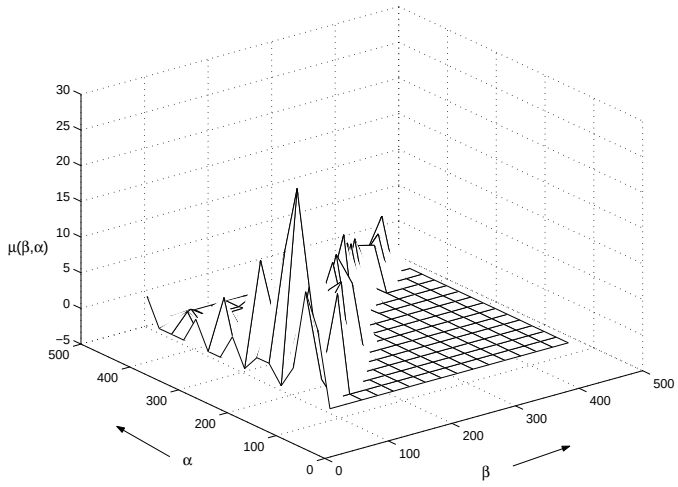


(b) Discretization = 20 Oe

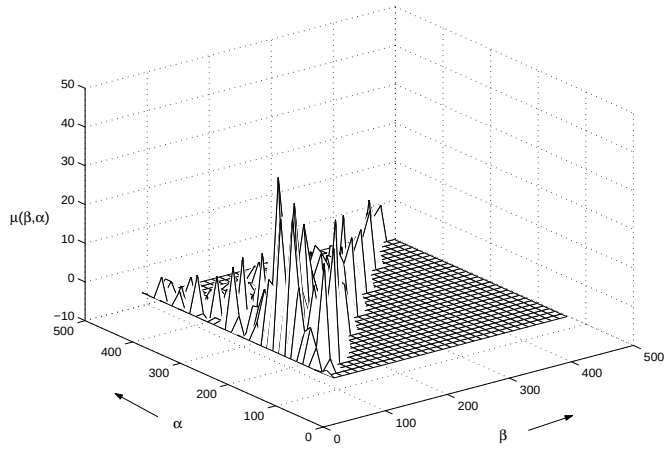


(c) Discretization = 10 Oe

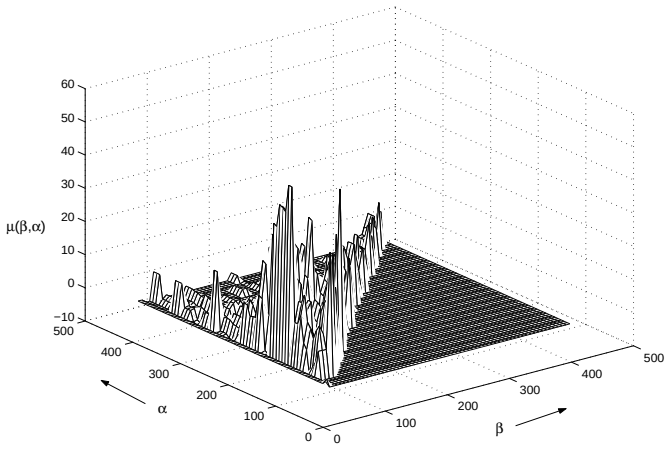
Fig. 4. Identification experiments for a commercial magnetostrictive actuator [9] with different magnetic field discretizations



(a) Discretization = 25 Oe

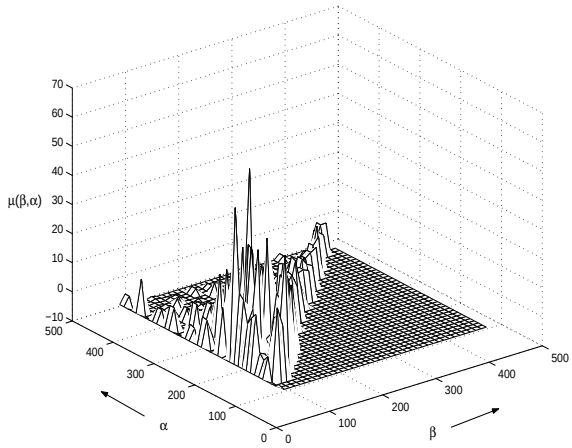


(b) Discretization = 12.5 Oe

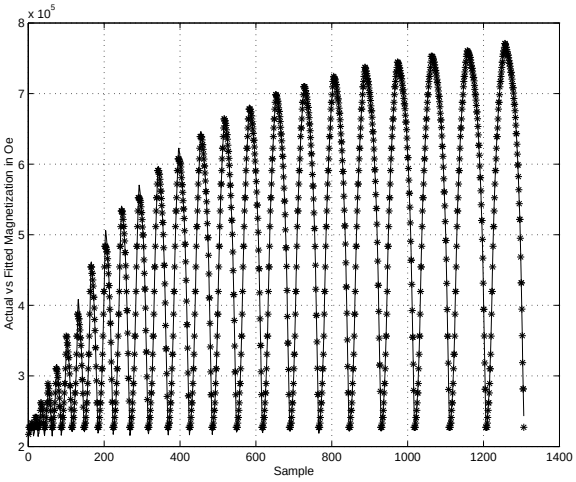


(c) Discretization = 6.25 Oe

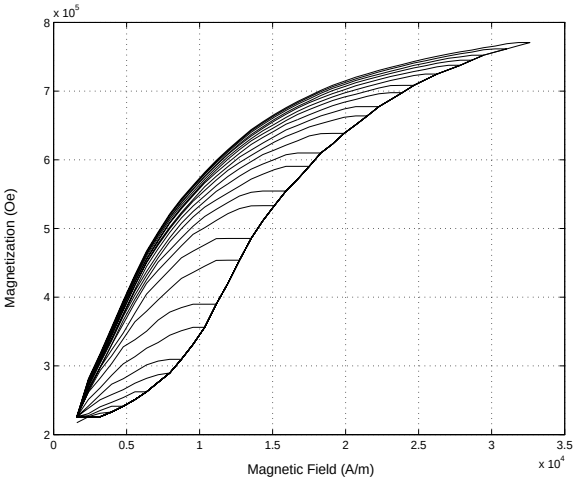
Fig. 5. Identification experiments for a commercial magnetostrictive actuator [9] with different magnetic field discretizations



(a) The computed Preisach density function

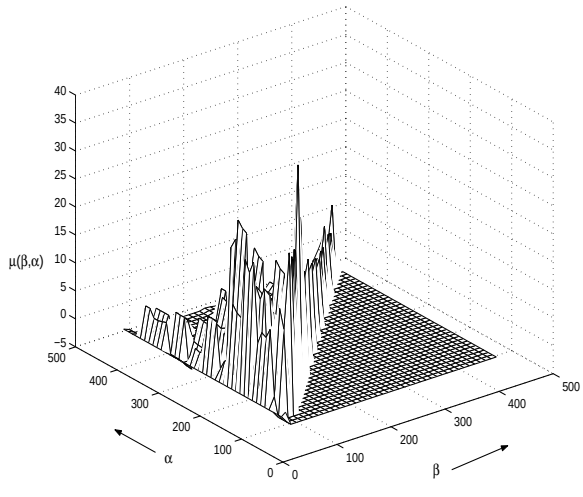


(b) Comparison between fitted and actual Magnetization versus Magnetic field (* indicates the fitted values).

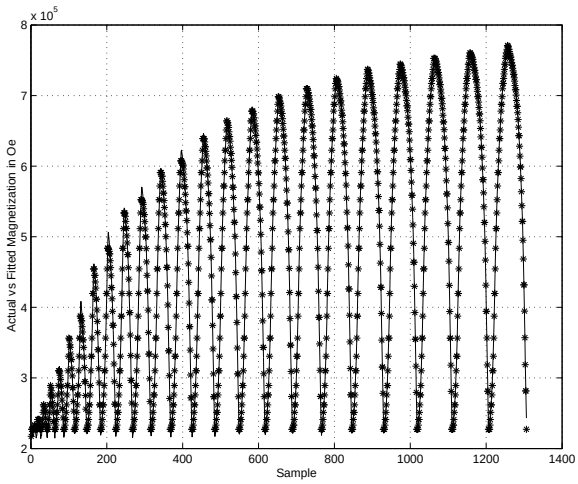


(c) Fitted Magnetic field versus Magnetization characteristic.

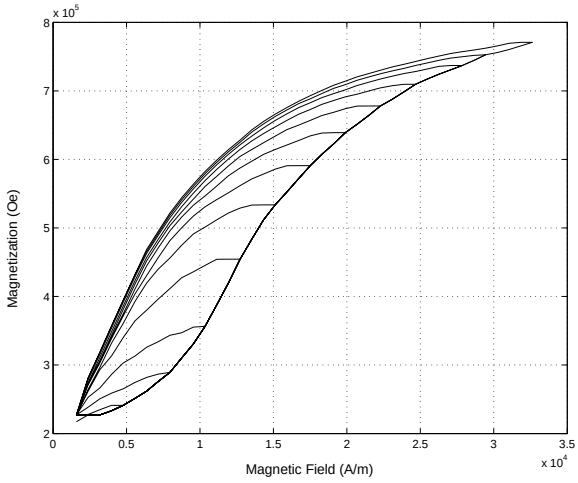
Fig. 6. Identification experiments for a commercial magnetostrictive actuator [9] with a magnetic field discretization of $10\,Oe$.



(a) The computed Preisach density function

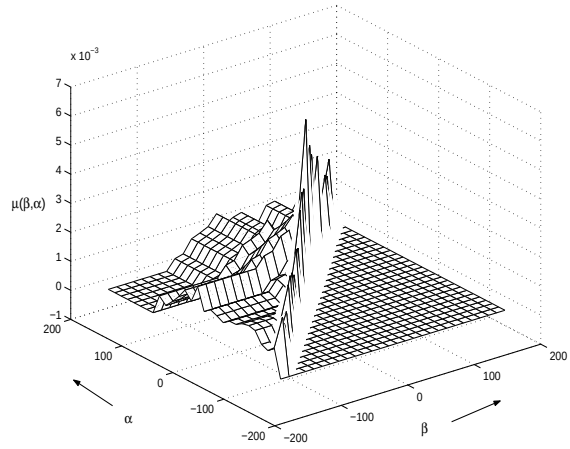


(b) Comparison between predicted and actual Magnetization versus Magnetic field (* indicates the predicted values).

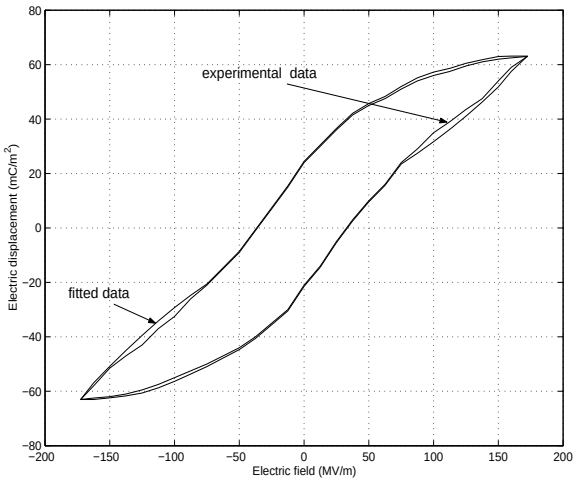


(c) Fitted Magnetic field versus Magnetization characteristic.

Fig. 7. Identification experiments for a commercial magnetostrictive actuator [9] with a magnetic field discretization of $10\,Oe$ and only some of the input signal used.

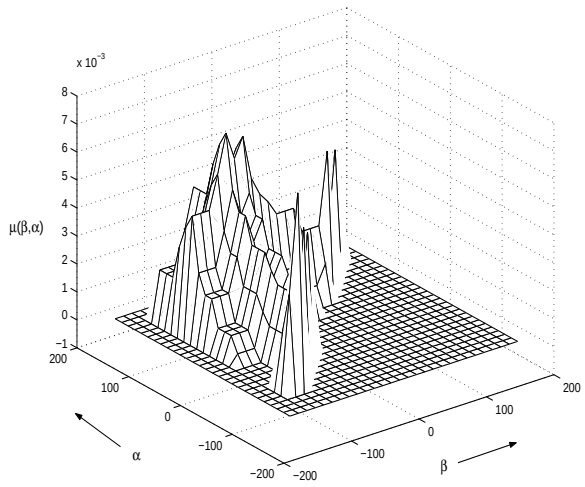


(a) The computed Preisach density function

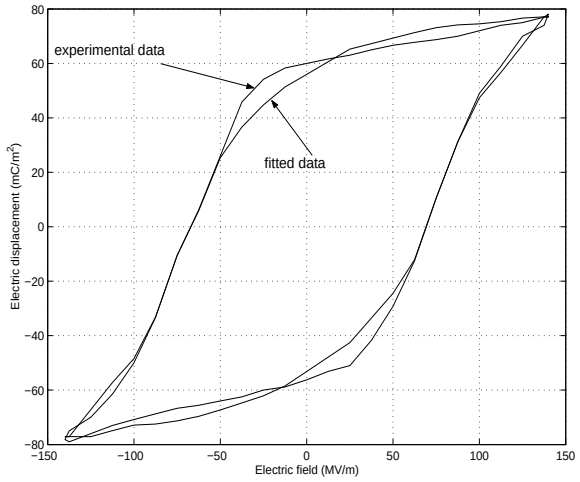


(b) Comparison between fitted and actual electric displacement versus electric field data

Fig. 8. Identification experiments for a VDF/TrFE/HFP electro-active polymer [19] .



(a) The Preisach density calculated for a VDF/HFP (5 %) electro-active polymer.



(b) Comparison between fitted and actual charge versus electric field data for a VDF/HFP(5 %) electro-active polymer.

Fig. 9. Identification experiments for a VDF/HFP (5 %) electro-active polymer [20].