On the solution of operator equation problems with application to Preisach density estimation

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***Abstract*— In this paper, we study the numerical solution of a linear, compact, integral operator equation with linear inequality constraints on the solution space. The operator equation is approximated by a linear matrix equation via discretization, which may be then solved using a linear least squares L^2 approach. Three methods, including two new methods, for the regularization of the discretized equation without constraints were presented. We compare the sensitivity of the solutions from these methods for perturbations in the data; we also compare the time taken for solution. Next, we present a new algorithm to solve the linear inequality constrained, minimum norm, least squares problem by adapting the solution methods presented for the unconstrained problem. Then we compare it with the MatLab© function *quadprog* with respect to residual error and speed. Finally, we apply the new method to identify the density of a Preisach operator for two electro-active polymers and a magnetostrictive actuator and again show that the new method performs as well or better than *quadprog*.**

I. INTRODUCTION

A. The Preisach density estimation problem

The Preisach operator is a mathematical tool that has been used to model the phenomena of hysteresis for years [1], [2]. Consider a relay $R_{\beta,\alpha}$ which at any given time is at one of two states: +1 or -1. The relay is parametrized by scalars α and β, and its output function $v_{\beta,\alpha}(t)$ depends on the input $u(t)$ and the initial output $v_{\beta,\alpha}(0)$. Consider a continuous, piecewise-monotone input function $u(t):\ t\in[0,T]$ for the relay, with $v_{\beta,\alpha}(0)$ either $+1$ or -1. Mathematically, this elementary hysteresis operator or *hysteron* is defined as [3]:

$$v_{\beta,\alpha}(t)=\begin{cases}-1 & \text{if}\quad u(t)<\beta,\\ +1 & \text{if}\quad u(t)\geq\alpha,\\ -1 & \text{if}\quad \beta\leq u(t)<\alpha,\ \text{and}\,\exists\, t_1:\ u(t_1)<\beta,\\ & \quad\ \text{and}\,\forall\tau\in(t_1,t),\ u(\tau)\notin[\alpha,\infty),\\ +1 & \text{if}\quad \beta\leq u(t)<\alpha,\ \text{and}\,\exists\, t_1:\ u(t_1)\geq\alpha,\\ & \quad\ \text{and}\,\forall\tau\in(t_1,t),\ u(\tau)\notin(-\infty,\ \beta).\end{cases}$$

To construct the Preisach operator from the elementary hysterons when the input is $u(\cdot)$, denote $v_{\beta,\alpha}(\cdot)=R_{\beta,\alpha}[u](\cdot)$. The Preisach operator's input is $u(\cdot)$, and the output is [1]

$$y(t)=\iint\limits_{\alpha\geq\beta}\mu(\beta,\alpha)R_{\beta,\alpha}[u](t)\,d\beta d\alpha, \tag{1}$$

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where $\mu(\cdot,\cdot)\in L^2(K)$ (the space of square integrable functions on K [4]), where K is a compact region in the (β,α) plane with $\alpha\geq\beta$, and support $(\mu)=K$.

For a given μ, we can define a Preisach operator as a map:

$$\Gamma_\mu: C_{pm}[0,T]\to C_{pm}[0,T], \tag{2}$$

where $C_{pm}[0,T]$ denotes the space of piecewise monotone continuous functions on $[0,T]$. For compatibility with experimental evidence, we restrict $\mu(\cdot,\cdot)$ to be a non-negative function. During the identification experiments, the fixed input $u\in C_{pm}[0,T]$ usually only affects a portion of the set K in the Preisach plane. Without loss of generality, we can restrict attention to this portion. *In the following, the set K_u is the subset of K affected by the input u.* We define the set of density functions:

$$\mathcal{K}_u\stackrel{\triangle}{=}\{\mu\in L^2(K_u)\,|\,\mu\geq 0\}. \tag{3}$$

Due to the assumption on K_u, the output at time 0 *given by $y(0)=\Gamma_\mu[u](0)$ is the same for all density functions in $\mathcal{K}_u$.*

The density function μ has to be determined from experiments, by observing outputs $y(\cdot)$ that correspond to inputs $u(\cdot)$. The initial output $w=\Gamma_\mu[u](0)$ is fixed for a given memory curve at time $t=0$. For a given $u(\cdot)\in C_{pm}[0,T]$, we define the operator:

$$\begin{array}{rccl}\Phi_u: & \mathcal{K}_u & \to & L^2[0,T],\\ & \mu(\beta,\alpha) & \mapsto & y=\Phi_u\,\mu(\cdot)=\Gamma_\mu[u](\cdot)-w.\end{array} \tag{4}$$

The operator Φ_u is a *linear* operator between $L^2(K_u)$ and $L^2[0,T]$. Without loss of generality, we can assume $w=0$ because we can subsume w into μ as follows. Define $M=\iint\limits_{\alpha\geq\beta}R_{\beta,\alpha}[u](t)\,d\beta d\alpha$. If we set $\overline{\mu}(\beta,\alpha)=\mu(\beta,\alpha)-\frac{w}{M}$, then (4) is equivalent to $y=\iint\limits_{\alpha\geq\beta}\overline{\mu}(\beta,\alpha)R_{\beta,\alpha}[u](t)\,d\beta d\alpha$. Henceforth, we consider the linear operator Φ_u with $w=0$ in (4). Note that this would necessitate a minor modification to the definition of $\mathcal{K}_u$ as $\overline{\mu}\geq-\frac{w}{M}$.

The adjoint operator $\Phi_u^*: L^2[0,T]\to\mathcal{K}_u$ satisfies the adjoint equation to (4):

$$\tilde{\mu}=\Phi_u^*\,z, \tag{5}$$

$$\tilde{\mu}(\beta,\alpha)=\int_0^T R_{\beta,\alpha}[u](t)\,z(t)\,dt. \tag{6}$$

One can easily check that we have

$$\langle y,z\rangle_{L^2[0,T]}=\langle\mu,\tilde{\mu}\rangle_{\mathcal{K}_u}. \tag{7}$$

By Lemma 3.1 in [5], Φ_u is Hilbert-Schmidt and hence a compact linear operator. Therefore, it does not possess a bounded linear inverse [6], [7].

Given functions u and $y \in L^2[0,T]$, one may consider a least squares formulation of the density estimation problem:

$$\text{Find} \quad \underset{\mu \in \mathcal{K}_u}{\operatorname{argmin}} \frac{1}{2}\|\Phi_u\,\mu - y\|^2. \tag{8}$$

The normal equation corresponding to this problem is

$$\Phi_u^*\,\Phi_u\,\mu = \Phi_u^*\,y. \tag{9}$$

We collect a few facts about Φ_u, Φ_u^* and the normal equation below:

Lemma 1.1: (1) Φ_u^* is a compact linear operator.
(2) $\Phi_u^*\,\Phi_u$ is a self-adjoint, compact linear operator.
(3) Equation (9) is consistent for every $y \in L^2[0,T]$.
Proof:
(1) Φ_u^* is a compact linear operator by Schauder's theorem as Φ_u is a compact linear operator.
(2) $\Phi_u^*\,\Phi_u$ is a compact linear operator as it is a composition of compact linear operators. Then $(\Phi_u^*\,\Phi_u)^* = \Phi_u^*\,\Phi_u$ is easy to check using (7).
(3) The consistency of (9) follows from the Fredholm alternative [6].

■

Although (9) is consistent, it is ill-posed due to the fact that the eigenvalues of $\Phi_u^*\Phi_u$, or equivalently, the singular values of Φ_u converge to zero. Thus the solution of (9) for μ requires a regularization scheme.

There are several regularization strategies including Tikhonov regularization, Landweber iteration, Mollification, Filtering singular systems, and discretization methods like Galerkin projection and collocation [7]. In [5], discretization of the values of the input function and the time interval is used to set up a regularization scheme by truncating singular values of the resulting matrix approximation to Φ_u. In this paper, we study other possibilities for the solution of (9).

B. Regularization strategies for the estimation problem

In [8], two broad regularization schemes for linear ill-posed operator equations are discussed.

(1) *Discretization of time and input values*
Consider the operator equation $y = \Phi_u\,\mu$ (which might be inconsistent). Let $0 = t_1 < \cdots < t_N = T$ be a discretization of time so that we have

$$(\Phi_u\,\mu)(t_j) = y(t_j),\ j = 1, \cdots, N. \tag{10}$$

If the minimum and maximum values of the input signal $u(t)$ are u_{min} and u_{max}, respectively, then one considers the input to take one of the discrete values $u_{min} = u_1 < \cdots < u_n = u_{max}$ (where $u_{i+1} - u_i = \Delta u$ is a constant) at each collocation time t_i. Therefore the memory curves, at the collocation times, have corners that lie on the grid (u_p, u_q) where $u_1 \le u_p \le u_q \le u_n$. Hence, one can consider the density function to be piecewise constant on the rectangles of the grid.
The operator Φ_u can then be considered to be discretized as (β, α) only take values on the grid (u_p, u_q). We denote the piecewise constant density function by μ_n, the discretized operator by $(\Phi_u)_n$ and the time-discretized output by y_n. Thus, we have

$$(\Phi_u)_n\,\mu_n = y_n. \tag{11}$$

The above equation is a matrix equation. The corresponding normal equation is

$$(\Phi_u)_n^T(\Phi_u)_n\mu_n = (\Phi_u)_n^T y_n, \tag{12}$$

which is solved using Cholesky factorization assuming full rank for $(\Phi_u)_n$,

$$(\Phi_u)_n^T(\Phi_u)_n = L_n L_n^T. \tag{13}$$

(2) *Discretization of time and output values*
This method has not been tried in the context of Preisach density estimation. Let Q_n denote projection of the function y to a finite dimensional subspace of step functions discretized by value. Then, formally, the operator equation $y = \Phi_u\,\mu$ (which might be inconsistent) yields

$$Q_n\Phi_u\mu = Q_n y. \tag{14}$$

After discretizing time, one arrives at the matrix equation (possibly inconsistent):

$$Q_n(\Phi_u\,\mu)(t_j) = Q_n y(t_j),\ j = 1, \cdots, N. \tag{15}$$

The normal equation to (15) is solved using a Cholesky decomposition [8]. As the discretization of u is a consequence of the discretization of y, the step function approximation to the density function μ is more difficult to calculate. Therefore, we do not discuss this method in what follows.

II. The unconstrained discretized problem

A. Solution methods

In this section, we study the solution of the equation (11) without the constraint $\mu \ge 0$. In the remarks after Lemma 1.1, we saw that the matrix $(\Phi_u)_n$ could have some very small singular values. While applying the truncated singular value decomposition (TSVD) regularization method, the singular values are set below a certain threshold value to zero. If we apply a method other than TSVD regularization, we are led to consider the least squares problem:

$$\text{Find} \quad \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \|A\,x - b\|_2^2, \tag{16}$$

where $A \in \mathbb{R}^{m \times n}$ with $m > n$ and $\text{rank}(A) = p < n$. As $\text{rank}(A) < \min\{m, n\}$, this problem requires regularization.

The two broad approaches to regularization are:

(a) The *Tikhonov* regularization approach:
Consider

$$\text{Find} \quad \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \|A\,x - b\|_2^2 + \lambda\|x\|_2^2, \tag{17}$$

where λ is a parameter that needs to be chosen. Methods to choose λ are cross-validation, generalized cross-validation, and L-curve method [9], [10], [11]. The solution to (17) satisfies the normal equation

$$(A^T\,A + \lambda\,I)\,x^* = A^T\,b. \tag{18}$$

For $\lambda > 0$, the above equation has a unique solution as $A^T A + \lambda I$ is positive definite.

(b) *Rank estimation* methods:
Let $X = \{x \in \mathbb{R}^n \,|\, \text{argmin } \|A\,x - b\|_2^2\}$. The minimum-norm, least squares problem is

$$\text{Find} \quad \underset{x \in X}{\text{argmin}} \, \|x\|_2^2. \tag{19}$$

As $x \in X$, it satisfies the normal equation to (16):

$$A^T A\, x = A^T b. \tag{20}$$

As rank$(A^T A)$ = rank(A), solutions to the above equation lie on an affine subspace or hyperplane of $\mathbb{R}^n$. Therefore, the problem is to find x that minimizes the cost function in (19) while satisfying the normal equation (20) as its constraint.
All the methods in this category involve the estimation of the rank of the matrix A or $A^T A$, and the subsequent solution of the projected version of the equation $A\,x = b$. The most well known of these methods is the TSVD method. However, as the computation of SVD is quite slow, alternative methods to estimate rank using the pivoted QR decomposition [12] or the more recent pivoted Cholesky decomposition [13] could be considered. Note that for some matrices, rank determination by the pivoted QR decomposition can fail [12], although in general, it can be considered an efficient way of estimating rank compared to the SVD method. The pivoted Cholesky decomposition fares worse [13] and is not recommended for rank determination. The truncation criterion may be chosen according to a generalized cross-validation criterion [9], [10].

The difficulty with the Tikhonov approach is that for small values of λ, we still have a rank estimation problem, as $A^T A$ is rank-deficient. Therefore, we focus on the rank-estimation based methods below, and their sensitivity to perturbations (with the latter three in detail):

(1) *TSVD* of A: "skinny" SVD $A = \tilde{U}\,\tilde{S}\,\tilde{V}^T$ yields $x^* = \tilde{V}\,\tilde{S}^{-1}\,\tilde{U}^T\,b$ [14];
(2) *Complete Orthogonal Factorization (COF) method*: rank estimation by QR decomposition of pivoted A followed by second QR decomposition to obtain the solution [12];
(3) *QR-Cholesky (QRC) method*: rank estimation by QR decomposition of pivoted A followed by a Cholesky decomposition to obtain the solution [15];
(4) *QR-Cholesky of $A^T A$ (QRCH) method*: rank estimation by QR decomposition of pivoted $A^T A$ followed by a Cholesky decomposition to obtain the solution;
(5) *Cholesky-Cholesky (CHCH) method*: rank estimation by pivoted Cholesky decomposition of $A^T A$ followed by a second Cholesky decomposition to obtain the solution.

As far as we are aware, the QRCH and CHCH methods are novel.

QRC METHOD

Suppose $A\,P = Q\,R$ where Q is an $m \times p$ matrix with orthonormal columns, R is an upper-triangular $p \times n$ matrix, P is a permutation matrix, and rank$(A) = p$. The rank estimation is achieved using the pivoted QR equation above. Then (20) reduces to $(R\,R^T)\,v = Q^T\,b$ and $x = P\,R^T\,v$.

We compute a Cholesky factorization of the reduced normal equations. The matrix $R\,R^T$ is a non-singular $p \times p$ matrix. Therefore, we may compute a Cholesky factorization $L\,L^T = R\,R^T$ and proceed to solve for v using forward and back substitutions. Once v is found, x is computed.

This method is not common in the literature because the recommended method in the literature is the COF method [16]. However, in our studies [15], the QRC method should be preferred as the errors and sensitivities from both methods are identical while QRC is significantly faster.

QRCH METHOD

If $m \gg n$, the pivoted QR decomposition of A in the QRC method is very computationally expensive compared to the pivoted QR decomposition of $A^T A$. However, the squaring of the condition number is a concern, so we expect the method to be successful for matrices A with low to moderate condition numbers.

We start with the pivoted QR decomposition of $A^T A$.

$$A^T A\,P = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} = Q_1 \begin{bmatrix} R_1 & R_2 \end{bmatrix}, \tag{21}$$

where Q_1 is a $n \times p$ matrix with orthonormal columns, R_1 is a $p \times p$ upper diagonal and nonsingular matrix, and P is a permutation matrix. Note that $Q_1^T\,Q_1 = I_{p \times p}$.

Let $v = P^T\,x$. From the normal equation (20), we have

$$Q_1 \begin{bmatrix} R_1 & R_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = Q_1\,(R_1\,v_1 + R_2\,v_2) = A^T\,b. \tag{22}$$

Let us write $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T = w + n$ where $n = \begin{bmatrix} n_1 & n_2 \end{bmatrix}^T$ is a vector in the null space of $\begin{bmatrix} R_1 & R_2 \end{bmatrix}$, and w belongs in the range space of $\begin{bmatrix} R_1 & R_2 \end{bmatrix}^T$. Then $R_1\,n_1 + R_2\,n_2 = 0$ and $w = \begin{bmatrix} R_1 & R_2 \end{bmatrix}^T\,\theta$ for some vector θ. As R_1 is invertible, we have

$$n_1 = -R_1^{-1}\,R_2\,n_2. \tag{23}$$

Denote $C = R_1^{-1}\,R_2$. Note that it is easy to compute R_1^{-1} as R_1 is upper triangular. The minimum norm problem is to minimize

$$\|x\|_2^2 = \|w + n\|_2^2 = \|w_1 + n_1\|_2^2 + \|w_2 + n_2\|_2^2$$

subject to (22), which may be simplified to:

$$(R_1\,R_1^T + R_2\,R_2^T)\,\theta = Q_1^T\,A^T\,b \tag{24}$$

and solved for θ. The minimum norm solution is then $x = P\,w$ which may be seen as follows.

Since n_1 is related to n_2 according to (23), the problem reduces to varying n_2 alone in order to find the minimum norm solution. Consider

$$f(n_2) = \|w_1 - C\,n_2\|_2^2 + \|w_2 + n_2\|_2^2. \tag{25}$$

We minimize this function with respect to n_2. Setting the first derivative equal to zero, we obtain $(C^T\,C + I)\,n_2 = C^T\,w_1 - w_2$. But, $w = \begin{bmatrix} R_1 & R_2 \end{bmatrix}^T\,\theta$, and so $(C^T\,C + I)\,n_2 = 0$, yielding $n_2 = 0$ and also $n_1 = 0$.

We have the following:

$$(R_1\,R_1^T + R_2\,R_2^T)\,\theta = Q_1^T\,A^T\,b, \tag{26}$$

$$w_1 = R_1^T\,\theta, \tag{27}$$

$$w_2 = R_2^T\,\theta, \tag{28}$$

and the minimum-norm, least-squares solution is given by

$$x^* = P\,w. \tag{29}$$

Equation (26) is solved using a Cholesky decomposition and forward and back substitutions.

CHCH METHOD

In the QRCH method, the pivoted QR step is used as a rank determination step. In [8], the truncated Cholesky decomposition is proposed for some linear operator equation problems. Below, we study a truncated Cholesky followed by another Cholesky decomposition. Consider the Lagrangian for problem (19):

$$f(x,\lambda) = \frac{1}{2}\,\|x\|_2^2 + \mu^T\,(A^T A\,x - A^T\,b). \tag{30}$$

The solution to (19) is the unconstrained minimizer of (30). The minimizer (x,μ) of (30) satisfies

$$A^T A\,x = A^T\,b \text{ and } A^T A\,\mu = x. \tag{31}$$

Again, as $\text{rank}(A^T A) = p$, the solution μ to (31) lies in an affine subspace of $\mathbb{R}^n$. Let (P, L) be the *pivoted Cholesky* decomposition of $A^T A$, that is,

$$P\,A^T\,A\,P^T = L\,L^T = \begin{bmatrix} L_{11} & 0 \\ L_{21} & 0 \end{bmatrix} \begin{bmatrix} L_{11}^T & L_{21}^T \\ 0 & 0 \end{bmatrix}, \tag{32}$$

where L_{11} is lower triangular and nonsingular $p \times p$ matrix. Let $\mu = P^T\,\theta$ for some θ, so that (31) becomes

$$P\,A^T\,A\,P^T\,\theta = P\,x \quad\Longleftrightarrow\quad L\,L^T\,\theta = P\,x. \tag{33}$$

Now, $L^T\,\theta$ has the form:

$$L^T\,\theta = \begin{bmatrix} L_{11}^T & L_{21}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \triangleq \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

If we can solve for a, then we can find x using (33).

Let $x = P^T\,z$ for some z. Then, (20) becomes

$$P\,A^T\,A\,P^T\,z = P\,A^T\,b \quad\Longleftrightarrow\quad L\,L^T\,z = P\,A^T\,b. \tag{34}$$

We also have

$$L^T\,z = \begin{bmatrix} y \\ 0 \end{bmatrix}, \tag{35}$$

for some y. From (34), we have

$$\begin{bmatrix} L_{11}^T & L_{21}^T \end{bmatrix} \begin{bmatrix} L_{11} \\ L_{21} \end{bmatrix} y = \begin{bmatrix} L_{11}^T & L_{21}^T \end{bmatrix} (P\,A^T\,b). \tag{36}$$

Denote $\tilde{L} \triangleq \begin{bmatrix} L_{11} \\ L_{21} \end{bmatrix}$. Then, (36) may be written as

$$\tilde{L}^T\,\tilde{L}\,y = \tilde{L}^T\,P\,A^T\,b, \tag{37}$$

which can be solved for y. Next, from (35), we see that $\tilde{L}^T\,z = y$. As $z = P\,x$, we have $z = \tilde{L}\,a$. Hence, by (35),

$$\tilde{L}^T\,\tilde{L}\,a = y. \tag{38}$$

As $\text{rank}(\tilde{L}^T\,\tilde{L}) = p$, we can solve for a using a regular Cholesky decomposition, and then find x using $x = P^T\,\tilde{L}\,a$.

B. Numerical Results

We compare the accuracy and speed of computation of the above methods. Let the solution obtained by the five methods be denoted x_{SVD}, x_{COF}, x_{QRC}, x_{QRCH}, and x_{CHCH}, respectively. We compare each method with the SVD method. The relative error of the COF method is defined as $e_{COF} = \|x_{COF} - x_{SVD}\|/\|x_{SVD}\|$. The relative errors e_{QRC}, e_{QRCH}, and e_{CHCH} are similarly defined.

Experiment 1: Vectors b were generated from $\mathcal{U}[0,1]$. Matrices A of size $100 \times n$ with singular values $\sigma_k = k^t + k\,r_k$ were constructed, where $t \in \{2,3,4\}$, and r_k was a random number drawn from $\mathcal{U}[0,1]$, $k = 1, \cdots, p$, where $p = 70$. The rest of the singular values of A were set to 0. A was computed using $A = USV$, where S was the diagonal matrix of singular values, and the entries of U were IID from $\mathcal{N}(0,1)$. The columns of U were then orthonormalized using Gram-Schmidt to a tolerance of 10^{-10}, and V was created similarly. For each of the five methods, 100 trials were conducted. The relative error for each trial was computed. Similar computations and comparisons were made for A matrices with condition numbers in three different ranges: low (1-10^3), moderate (10^3-10^5), and high (10^5-10^{11}).

The results of Experiment 1 appear in Table I. For each method, Table I shows the median relative error and the median computation time for 100 trials. These summaries show that for low and moderate condition numbers, all four errors are relatively small. For high condition numbers, e_{COF} and e_{QRC} are relatively small, so the solutions of these two methods are practically identical to the solution we get from the SVD method. Moreover, the QRC method is significantly faster than the COF method.

TABLE I

STATISTICAL SUMMARIES FROM EACH OF THE FIVE METHODS FOR THE SOLUTION OF AN UNCONSTRAINED, MINIMUM NORM, LEAST SQUARES PROBLEM. THE MEDIAN RESIDUAL ERRORS (E) AND COMPUTATION TIMES (T) ARE DISPLAYED.

κ		SVD	COF	QRC	QRCH	CHCH
1-10^3	e	0.0E+00	1.0E-12	1.0E-12	5.2E-10	1.1E-09
	t	2.0E-03	1.6E-03	9.0E-04	9.0E-04	8.0E-04
10^3-10^5	e	0.0E+00	6.2E-11	6.2E-11	2.1E-06	5.5E-06
	t	1.9E-03	1.6E-03	9.0E-04	1.0E-03	8.0E-04
10^5-10^{11}	e	0.0E+00	4.3E-09	4.3E-09	7.0E+00	5.8E-01
	t	2.8E-03	1.6E-03	8.9E-04	1.4E-03	8.6E-04

Next, we compare the solution from each of the five methods with that obtained by perturbing the matrix A. As A is rank-deficient, the perturbation of A will most likely render A with full-rank. The question investigated is whether this changes the solution significantly. Let the perturbed solution obtained by the five methods be denoted x'_{SVD}, x'_{COF}, x'_{QRC}, x'_{QRCH}, and x'_{CHCH}, respectively. We compute the sensitivity of the SVD method by $e'_{SVD} = \|x'_{SVD} - x_{SVD}\|/\|x_{SVD}\|$. The sensitivities e'_{COF}, e'_{QRC}, e'_{QRCH}, and e'_{CHCH} are similarly defined.

Experiment 2: Matrices A with $m = 100$ rows, $n = 90$ columns and $p = 70$ rank and vectors b were both generated similarly to those in Experiment 1. Perturbation

δA comprised of IID entries drawn from $\mathcal{N}(0,t)$, a Gaussian distribution with mean zero and standard deviation $t = 10^{-6}$. For each of the five methods, 100 trials were conducted and the sensitivity of each method was computed for each trial.

For each method, Table II shows min, max, and median of sensitivities of the solutions for 100 trials. For low condition numbers, all five sensitivities are relatively small. For moderate condition numbers, e'_{SVD}, e'_{COF}, e'_{QRC}, and e'_{QRCH} are relatively small. For high condition numbers, only e'_{SVD}, e'_{COF}, and e'_{QRC} are relatively small.

The results of Experiments 1 and 2 are summarized in Table III. *It suggests that the QRC method is superior to SVD and COF in terms of accuracy of solutions, sensitivity to perturbations of matrix A, and speed of computation.*

TABLE II

STATISTICAL SUMMARIES FOR THE PERTURBED MATRIX A INCLUDING MIN, MAX AND MEDIAN OF RELATIVE ERRORS FOR EACH OF THE FIVE METHODS WITH THREE DIFFERENT TYPES OF CONDITION NUMBERS κ.

κ	r.e.	SVD	COF	QRC	QRCH	CHCH
1-10^3	Min	2E-06	3E-06	3E-06	2E-06	2E-06
	Max	2E-05	4E-05	4E-05	2E-05	2E-05
	Med	9E-06	1E-05	1E-05	9E-06	9E-06
10^3-10^5	Min	2E-06	2E-06	2E-06	3E-06	4E-01
	Max	2E-05	3E-05	3E-05	2E-05	4E-01
	Med	1E-05	2E-05	2E-05	1E-05	4E-01
10^5-10^{11}	Min	5E-06	6E-06	6E-06	2E+00	5E-01
	Max	7E-05	1E-04	1E-04	3E+01	8E-01
	Med	3E-05	4E-05	4E-05	4E+00	6E-01

TABLE III

SUMMARY OF UNPERTURBED AND PERTURBED SOLUTIONS OF FOUR METHODS. G DENOTES GOOD METHOD, B DENOTES LARGER ERROR.

	κ	COF	QRC	QRCH	CHCH
unperturbed	1-10^3	G	G	G	G
	10^3-10^5	G	G	G	G
	10^5-10^{11}	G	G	B	B
perturbed	1-10^3	G	G	G	G
	10^3-10^5	G	G	G	B
	10^5-10^{11}	G	G	B	B

III. SOLUTION METHODS FOR CONSTRAINED DISCRETIZED EQUATIONS

As regularization methods such as generalized cross-validation requires repeated solutions of (12) with the constraint $\mu_n \geq 0$, we study algorithms based on the solutions of unconstrained problems discussed in the last section.

In [17], *lasso*, for "least absolute shrinkage and selection operator", minimizes $\|A\,x - b\|_2$ subject to $\|x\|_1 \leq \lambda$, where λ is a regularization parameter. The one-norm constraint is equivalently posed as a set of linear inequality constraints. Starting with the unconstrained problem, the inequality constraints that are violated in each iteration are converted to equality constraints for the subsequent iteration.

Let $X = \{x \in \mathbb{R}^n \,|\, \text{argmin } \|A\,x - b\|_2^2\}$. We adapt the *lasso* to set up the following maximum negative index algorithm for the constrained minimum-norm problem:

$$\text{Find} \quad \underset{x \in X;\, x \geq 0}{\text{argmin}} \ \|x\|_2^2. \tag{39}$$

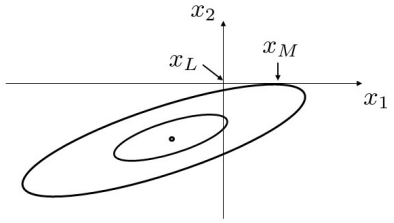

Fig. 1. Level sets of the residual error for a two-dimensional least squares problem with constraint $x \geq 0$. The unconstrained solution has $x_2 < x_1 < 0$. The solution to the constrained problem lies on the positive x_1 axis. The *lasso* solution x_L converges to the origin by setting $x_1 = x_2 = 0$. The maximum negative index method sets $x_2 = 0$ and obtains the correct solution x_M by subsequently minimizing the residual error.

Step (0) *Begin:* Let x_0 be the solution of the unconstrained problem (19).

Step (k) For $k \geq 1$, suppose x_{k-1} is the solution from Step (k-1). Set the index of x_{k-1} that is most negative to zero as an additional equality constraint for Step k. The resulting equality constraint problem is solved using the QRC method. Increment k.

End If x_N is a solution satisfying the constraints, stop.

The sequence $\{x_k\}$ converges because in each step the number of constraints increases by at least one until the final step. Figure 1 shows that this method leads to setting $x_2 = 0$ in Step (1) as the unconstrained solution has $x_2 < x_1 < 0$.

IV. APPLICATION TO PREISACH DENSITY ESTIMATION

We applied the maximum negative index method and *quadprog* to an electro-active polymer (polymer I: VDF/TrFE/HFP $(55.17/42.35/2.46)$ terpolymer 121) at $42^\circ\,C$ [18], a VDF/HFP (5%) electro-active polymer (polymer II) [19], and a commercial magnetostrictive actuator [20]. The discretization of the electric field was chosen to be $12.5\,MV/m$ for both polymers. Polymer I exhibits significant hysteresis in its electric displacement vs electric field characteristic. First, the close to zero singular values were truncated and *quadprog* was used to solve the constrained least squares problem as detailed in [5]. Second, the maximum negative index method was applied to the same data. Figure 2 and Table IV show the results of the experiment. Figure 3 shows the average magnetic field versus magnetization obtained in the experiment, and the fit obtained using the maximum negative index method with an input discretization value of 10 Oe, which may be compared with those obtained in [5]. One can see that the maximum negative index method outperforms *quadprog* for the polymers and is at least as effective as *quadprog* for the magnetostrictive actuator. Some other early work on Preisach density estimation may be found in [21].

V. CONCLUSION

In this article, we have presented three methods for the solution of a minimum norm, least squares problem. We presented numerical results comparing the three methods with the standard SVD and COF methods in terms of solution accuracy and sensitivity to perturbations. Building on these results, we presented a new method (*maximum negative*

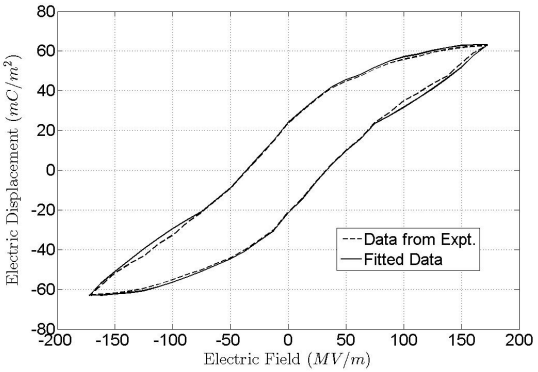

Fig. 2. Maximum negative index method and Matlab© *quadprog* method applied to data from experiment for polymer I (see Table IV for more details). The dashed line is the data from the experiment. The fitted data using the maximum negative index method is indistinguishable from the data from the experiment. The solid line shows the fitted data using *quadprog*. $\|\text{residual error}\|_1 = 10^{-13}\%$ of 1-norm of electric displacement for the maximum negative index method, while $\|\text{residual error}\|_1 = 3\%$ of 1-norm of electric displacement for *quadprog*. The graph for polymer II is similar.

TABLE IV

SUMMARIES FOR THE PREISACH DENSITY ESTIMATION FOR TWO ELECTRO-ACTIVE POLYMERS AND A MAGNETOSTRICTIVE ACTUATOR

Application		Max neg. index with QRC	*Quadprog*
Electroactive polymer 1	discretization	12.5	12.5
	rel. error %	1E-13	2.8
	time	0.13	0.09
	κ	2.9E+15	2.9E+15
Electroactive polymer 2	discretization	12.5	12.5
	rel. error %	0.42	6.2
	time	0.22	0.13
	κ	3.6E+15	3.6E+15
Magnetostriction	discretization	12.5	12.5
	rel. error %	7.5	7.5
	time	61	5
	κ	2.7E+51	2.7E+51
	discretization	10	10
	rel. error %	5.8	5.8
	time	266	15
	κ	1.9E+64	1.9E+64
	discretization	8	8
	rel. error %	6.4	6.4
	time	1143	42
	κ	3.3E+65	3.3E+65

index) for the solution of linear inequality constrained, minimum norm, least squares problem. Numerical results show that for condition numbers approaching $\frac{1}{\sqrt{\epsilon}}$, where ϵ is the machine precision, the new method outperforms *quadprog* of MatLab© with lower residual error. We then applied the maximum negative index method and *quadprog* to identify the Preisach density for two electroactive polymers and a commercial magnetostrictive actuator. The results show that the new method produces a fit to the data that is better than, or at least as good as, *quadprog*.

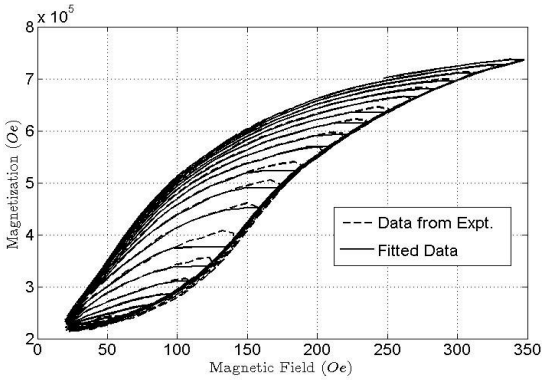

Fig. 3. Results of an identification experiment for a commercial magnetostrictive actuator [20] with a magnetic field discretization of $10\,Oe$ using the maximum negative index method. The results are virtually identical to that obtained using the MatLab© functin *quadprog* in [5]. See Table IV.

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