Study of a Play-like operator

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Abstract

In this paper, we consider a play-like hysteresis operator defined by an n^{th} order rate-independent differential system. We investigate the properties of the operator for $n = 1$ and $n = 2$. We show that the operator for $n = 2$ satisfies a one-step wiping out property. This result can be extended to show that the n^{th} order operator satisfies an $(n-1)$-th step wiping out property. Thus the new family of operators fall between the first-order differential equation models that do not satisfy any wiping-out properties and the Preisach-type operator that can show, in general, a countably infinite-step wiping out property. We will show that the "backlash-like" operator defined by Su, Stepanenko, Svoboda and Leung (SSSL) is a special case of our operator for $n = 1$.

Key words: hysteresis, Duhem operators, play-like operators, PKP-type operators

1. Introduction

Hysteresis representation by integral operators or differential equations dates back to Duhem's model in 1897 [3]. These operators reflect the observation that hysteresis curves for physical systems are monotone except when the input changes direction. Most differential equation hysteresis operators are first–order differential equations [3–6].

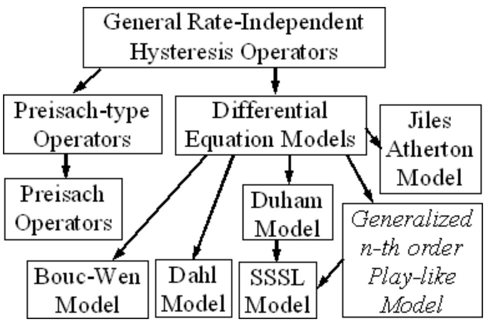

Fig. 1. Relationship between hysteresis models.

Figure 1 shows the relation between some of the various hysteresis models. Most differential equation models are rate-independent and satisfy the Volterra property. Hence, they are general hysteresis operators (Prop 2.2.9 [2]). Advantages of using a differential equation operator include simplicity of implementation and a limited number of parameters. However, unlike operators of the Preisach type, the output trajectories of these rate–independent operators do not depend on the previous input extrema, and hence, they do not satisfy Madelung's rules [1] [2,3].

One such first-order differential equation hysteresis operator was proposed by Bouc in 1964 [4,5], with the primary objective of describing forced vibrations of a hysteretic system under periodic excitation. Another first-order hysteresis operator, called "backlash–like", was introduced for the purpose of avoiding the inversion of hysteresis nonlinearity in an adaptive controller design [1]. We refer to this operator as the SSSL–play operator. We will show that the SSSL operator has a serious limitation in parameter selection. This limitation is overcome by our *generalized* n^{th} *order play–like operator* with the additional benefit that the robust, non-inversion type, adaptive controller defined in [1] is also applicable to this operator without change.

In this paper, we consider a play–like operator and the construction of a Preisach-Krasnoselskii-Pokrovskii (PKP) type operator using an n^{th} order differential system. We investigate the cases where $n = 1$ and $n = 2$ and show that

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[1] (i) Any curve C_1 emanating from a turning point A of the input-output graph is uniquely determined by the coordinates of A. (ii) If any point B on the curve C_1 becomes a new turning point, then the curve C_2 originating at B leads back to the point A. (iii) If the curve C_2 is continued beyond the point A, then it coincides with the continuation of the curve C which led to the point A before the $C_1 - C_2$ cycle was traversed.

the play–like operator corresponding to $n = 1$ is the SSSL operator.

2. Generalized n^{th}–order play–like operator

Suppose $\{v_i\}_{i=-2n+2}^{-1}$ is an alternating string of distinct real numbers such that $\{v_{2i}\}_{i=-n+1}^{-1}$ is strictly decreasing and $\{v_{2i+1}\}_{i=-n+1}^{-1}$ is strictly increasing with $v_{-1} < v_{-2}$ (see Figure 2). Consider operator $w(t) = F_{\frac{c}{a_0}}[v; v_{-2n+2}, \cdots, v_{-1}](t)$ defined by

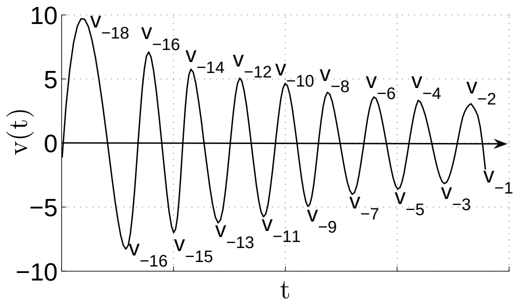


Fig. 2. An alternating string of monotone decreasing and increasing real numbers

$$w^{(n)} + \text{sign}\,(\dot{v})\, a_{n-1} w^{(n-1)} + \cdots + (\text{sign}\dot{v})^{n-1} d(w^{(1)}, w, v) = 0,$$

where $d(w^{(1)}, w, v) = a_1(w^{(1)} - 1) + \text{sign}(\dot{v})\, a_0(w - v) + c$, $w^{(k)} = \dfrac{d^k w}{dv^k}$, $1 \le k \le n$ and $a_m > 0$ for all m. Let $w(v_i) = w_i$ such that $\{w_{2i}\}_{i=-n+1}^{-1}$ is strictly decreasing and $\{w_{2i+1}\}_{i=-n+1}^{-1}$ is strictly increasing with $w_{-1} < w_{-2}$. The solution of (2) is $w = v - \text{sign}(\dot{v})\, \frac{c}{a_0} + c_1 e^{-\text{sign}(\dot{v})\alpha_1 v} + \cdots + c_n e^{-\text{sign}(\dot{v})\alpha_n v}$, where α_1,...,α_n are real, distinct and positive for appropriate choices of coefficients $a_1, \cdots, a_n$. Constants $c_1, \cdots, c_n$ are found using linear algebra from the initial conditions corresponding to $\{v_{2i}\}_{i=-n+1}^{-1} \bigcup \{v_{-1}\}$ for $\dot{v} > 0$ and to $\{v_{2i+1}\}_{i=-n+1}^{-1} \bigcup \{v_{-2}\}$ for $\dot{v} < 0$. Note that $\lim\limits_{v \nearrow \infty} (v - w) = -\dfrac{c}{a_0}$ and $\lim\limits_{v \searrow -\infty} (v - w) = \dfrac{c}{a_0}$. Hence, w asymptotically approaches the output of a play operator with parameter $r = \frac{c}{a_0}$. Note that the operator given by Equations (1)–(2) is rate–independent. By construction, every trajectory passes through n previous turning points, and therefore the operator can close $n - 1$ minor loops, which we refer to as the $(n-1)$–*step wiping–out property.*

Next, we construct a PKP operator using play-like operators, similar to the construction of the Preisach operator from play operators. Suppose $F_r[v(t); \cdot]$ is a play-like hysteresis operator with the $(n-1)$-step wiping-out property. Define operator Γ_r by $\Gamma_r[v(t)] = \int_0^{F_r[v;\cdot](t)} \omega(r, s) ds$, where $\omega \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$. Suppose that $\omega(r, s)$ has compact support. Then, for each r, $\Gamma_r(t)$ can produce saturated outputs with the $(n-1)$-step wiping out property. For $n \ge 2$, these operators satisfy Madelung's first and second rules. However, the third rule cannot be satisfied exactly since we need to allow non-uniqueness of trajectories through a point, which cannot be done using a differential equation with an initial condition. Consider the output map:

$$Q(t) = \int_0^{\infty} \Gamma_r[v(t)] dr \;\text{ with }\; \Gamma_r(v(t)) = \int_0^{F_r[v](t)} \omega(r, s) ds, \quad (1)$$

where $\omega(r, s)$ is the density function and $F_r[v](t)$ is the play-like operator given by Equation (2). A discussion of the properties of $Q(t)$ for an arbitrary n is beyond the scope of this paper. We discuss the properties for $n = 1$ and $n = 2$.

3. SSSL-play operator

The backlash-like operator proposed in [1] – which we refer to as the SSSL-play operator – is a relation $w = W[v; w_0]$ between a function $v \in C^1[0, T]$ and a continuous function w, defined by the differential equation $\frac{dw}{dt} = \alpha \left|\frac{dv}{dt}\right| (v - w) + \beta\, \frac{dv}{dt}$, with $w(v_0) = w_0$, where parameters α and β are positive. If $a_0 = \alpha$ and $c = 1 - \beta$, the above equation can be rearranged as

$$\text{sign}(\dot{v}) \left[\left(\frac{dw}{dv} - 1 \right) + c \right] + a_0\,(w - v) = 0, \quad (2)$$

which is the operator corresponding to $n = 1$ in Equation (2). It is easy to verify that the SSSL operator is a hysteresis operator using Prop. 2.2.9 in [2]. We show that for an appropriate choice of c, the operator is strictly monotone [2] .

Lemma 1 *Suppose* $a_0, c > 0$ *and initial state* $w_0 \in \left(v_0 - \frac{c}{a_0}, v_0 + \frac{c}{a_0}\right)$. *Then the trajectories of the SSSL-play operator corresponding to increasing inputs are convex and those corresponding to decreasing inputs are concave. Furthermore, if* $c \in (0, 1/2)$, *then the SSSL-operator is strictly monotone.*

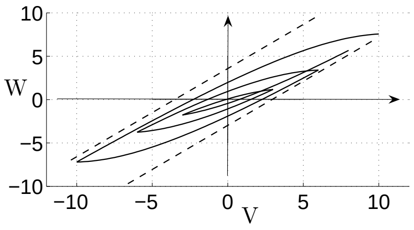
Proof: Suppose $c \in (0, 1/2)$. Since $w_0 - v_0 + \frac{c}{a_0} > 0$, for $\dot{v} > 0$, we have $c_1 = \left(w_0 - v_0 + \frac{c}{a_0}\right) e^{a_0 v_0} > 0$ and $\frac{d^2 w}{dv^2} = a_0^2 c_1 e^{-a_0\, v} > 0$. Also $\frac{dw}{dv}(v_0) = 1 - c - a_0(w_0 - v_0) > 1 - 2c > 0$. Similarly, for $\dot{v} < 0$, $\frac{d^2 w}{dv^2} < 0$ and $\frac{dw}{dv}(v_0) > 0$. □

For a given c, we construct the corresponding PKP operator as in Equation (1). Suppose $c = \frac{1}{4}$. Consider density function $\omega(r, s) = (5 - |s|)U_r$, for $|s| < 5$, where $U_r = 1$ for $0 < r < 5$ and $U_r = 0$ otherwise. Corresponding outputs of the SSSL and PKP-type operators are given in Figure 3, which shows that these operators do not possess the wiping out property. This is not the case for $n = 2$, as we will see in the next subsection.

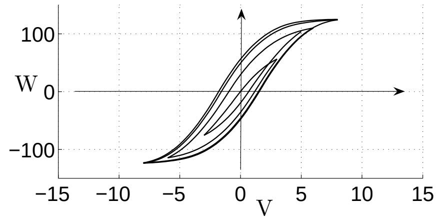
Next we consider operator $w(t) = F_r[v; \cdot](t)$ for $n = 2$. Letting $a_0 = ab$ and $a_1 = a + b$, the differential equation corresponding to Equation (2) is

$$\frac{d^2 w}{dv^2} + \text{sign}(\dot{v}) \left[(a+b) \left(\frac{dw}{dv} - 1 \right) + c \right] + ab(w - v) = 0, \quad (3)$$

[2] If a hysteresis operator satisfies sign $\left(\frac{dw}{dt}\right)$ = sign $\left(\frac{dv}{dt}\right)$ whenever the derivatives exist, then it is called a strictly monotone hysteresis operator. See also the definition (11.22) on page 111 in [2].



(a) Output of SSSL operator for $r = 3$



(b) Output of SSSL-PKP operator

Fig. 3. An illustration that SSSL operators do not possess any wiping out property.

with initial memory $w_{-2} = w(v_{-2})$ and $w_{-1} = w(v_{-1})$, and satisfying $0 < \dfrac{w_{-1} - w_{-2}}{v_{-1} - v_{-2}} < 1$, $|w_{-1} - v_{-1}| < \frac{c}{ab}$, and $|w_{-2} - v_{-2}| < \frac{c}{ab}$. The solution is

$$w(t) = c_1 e^{-\text{sign}(\dot{v})av} + c_2 e^{-\text{sign}(\dot{v})bv} + v - \text{sign}(\dot{v})r, \quad (4)$$

where $r = \frac{c}{ab}$. An argument similar to the one for the SSSL operator shows that this operator is a hysteresis operator. We again desire that increasing trajectories be convex and decreasing trajectories be concave. Observe that for $\dot{v} > 0$, if c_1 and c_2 are positive in Equation (4), then $\frac{d^2 w}{dv^2} > 0$. Similarly for $\dot{v} < 0$, if c_1 and c_2 are both negative then $\frac{d^2 w}{dv^2} < 0$, as desired.

We need the following two lemmas.

Lemma 2 *Define* $\Delta_1 = \frac{1}{v_1 - v_0} \ln\left(\frac{r+w_0-v_0}{r+w_1-v_1}\right)$, $\Delta_2 = \frac{1}{v_1-v_0} \ln\left(\frac{r+v_1-w_1}{r+v_0-w_0}\right)$, $\Delta_3 = \frac{1}{r+w_0-v_0}$, *and* $\Delta_4 = \frac{1}{r+v_1-w_1}$. *Suppose the initial memory* v_0, v_1, w_0, w_1 *satisfies* $v_1 - v_0 > 0$, $0 < \frac{w_1 - w_0}{v_1 - v_0} < 1$, $|w_1 - v_1| < r$, *and* $|w_0 - v_0| < r$. *Then i. if* $\Delta_2 \leq \Delta_1$, *then* $\Delta_3 \leq \Delta_4$. *ii. if* $\Delta_1 \leq \Delta_2$, *then* $\Delta_4 \leq \Delta_3$.

Proof: Let $\Delta_2 \leq \Delta_1$. Then $(v_1 - w_1)^2 \leq (w_0 - v_0)^2$. Suppose $w_0 - v_0 < 0$. Since $v_1 - v_0 > 0$ and $\dfrac{w_1 - w_0}{v_1 - v_0} < 1$, $w_1 - v_1 < w_0 - v_0 < 0$. Then $(v_1 - w_1)^2 > (w_0 - v_0)^2$, which leads to a contradiction. Hence $w_0 - v_0 \geq 0$ and $v_1 - w_1 \leq w_0 - v_0$. Furthermore, $0 < r + v_1 - w_1$ and thus $\dfrac{1}{r + w_0 - v_0} \leq \dfrac{1}{r + w_0 - v_0}$. The proof of the second statement is similar but with the inequality reversed. □

Lemma 3 *Suppose the initial memory satisfies the conditions in Lemma 2. Let* $\dfrac{w_1 - w_0}{v_1 - v_0} = k < 1$. *If* $r > \max\left\{\frac{2-k}{k}(w_0 - v_0), \frac{2-k}{k}(v_1 - w_1)\right\}$, *then* $\min\{\Delta_3,\ \Delta_4\} \geq \max\{\Delta_1, \Delta_2\}$.

Proof: Let $r > \frac{2-k}{k}(w_0 - v_0)$. Substituting the value of k and simplifying, we obtain $\frac{1}{v_1 - v_0} \frac{(v_1-v_0)-(w_1-w_0)}{r+v_0-w_0} < \frac{1}{r+w_0-v_0}$. Using the inequality $\ln x \leq x - 1$ and $\frac{r+v_1-w_1}{r+v_0-w_0} > 0$, $\ln\left(\frac{r+v_1-w_1}{r+v_0-w_0}\right) \leq \frac{r+v_1-w_1}{r+v_0-w_0} - 1 = \frac{(v_1-v_0)-(w_1-w_0)}{r+v_0-w_0} \leq \frac{v_1-v_0}{r+w_0-v_0}$. Thus $\Delta_1 \leq \Delta_3$. Similarly, $r \geq \frac{2-k}{k}(v_1 - w_1)$ implies $\Delta_2 \leq \Delta_4$. Combining these results with Lemma 2, we have $\min\{\Delta_3,\ \Delta_4\} \geq \max\{\Delta_1, \Delta_2\}$. □

Now we can state and prove the desired theorem.

Theorem 1 *Suppose* $a, b, c > 0$ *and* $b > a$ *in Equation (3). For a given initial memory* $w_0 = w(v_0)$ *and* $w_1 = w(v_1)$, *and for a given parameter* $r = c/ab$, *there exists a bounded region* $D \subset \mathbb{R}^2_+$ *such that if* $(a, b) \in D$, *then all the increasing trajectories are convex, all the decreasing trajectories are concave, and the corresponding operator is strictly monotone.*

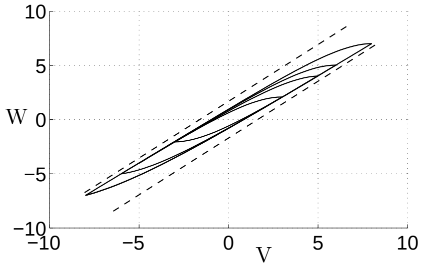
Proof: From Equation (4),
$c_1 = \dfrac{(r + w_1 - v_1)\, e^{bv_1} - (r + w_0 - v_0)\, e^{bv_0}}{e^{(b-a)v_1} - e^{(b-a)v_0}}$ and
$c_2 = \dfrac{(r + w_0 - v_0)\, e^{av_0} - (r + w_1 - v_1)\, e^{av_1}}{e^{(a-b)v_0} - e^{(a-b)v_1}}$ for increasing trajectories. It can be shown that, c_1 and c_2 are positive if $a \leq \Delta_1 \leq b$. Similar calculations for decreasing curves give $a \leq \Delta_2 \leq b$. Therefore, if $b \geq \max\{\Delta_1, \Delta_2\} \equiv \Delta$ and $a \leq \min\{\Delta_1, \Delta_2\} \equiv \overline{\Delta}$, then increasing trajectories are convex and decreasing trajectories are concave.

Next, we find conditions on b for which the operator is strictly monotone. For increasing trajectories, $\dot{w} = -c_1 a e^{-av} - c_2 b e^{-bv} + 1$. Since c_1 and c_2 are positive and $a < b$, we have $ac_1 e^{-av_0} + bc_2 e^{-bv_0} < b\left(c_1 e^{-av_0} + c_2 e^{-bv_0}\right) = b(r + w_0 - v_0)$. If $b \leq \Delta_3$, then $\dot{w} > 0$. Similarly for decreasing trajectories, $b \leq \Delta_4$. So if $b < \min\{\Delta_3,\ \Delta_4\} \equiv \overline{\overline{\Delta}}$, then the operator is strictly monotone. From Lemma 3, $\bar{\bar{\Delta}} > \Delta$. Therefore, if $\Delta < b < \bar{\bar{\Delta}}$ and $0 < a < \bar{\Delta}$, the operator is strictly monotone for both increasing and decreasing curves. □

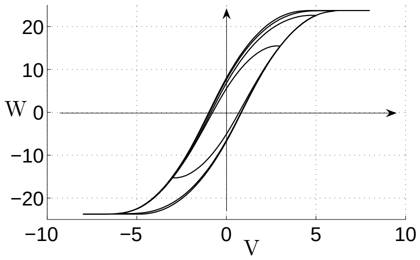
Now we define the PKP-type operator as $Q(t) = \int_0^\infty \int_\Delta^{\bar{\bar{\Delta}}} \int_0^{\bar{\Delta}} \Gamma_r[v(t)]da\ db\ dr$. The results of a numerical simulation for $b = 0.5$ and $a = 0.3$ are given in Figure 4, which illustrates that the operator processes the wiping out property.

4. Conclusion

In this paper, we consider a play-like hysteresis operator defined by an n^{th} order rate-independent differential system. We showed that the "backlash-like" operator defined by Su, Stepanenko, Svoboda and Leung (SSSL) is a special case of our operator for $n = 1$. The generalized play operators for $n \geq 2$ satisfy what we call the $(n-1)$-step wiping property. We have shown that the SSSL operator is the only one in the family that does not satisfy any kind of wiping out property.



(a) Output of play-like operator



(b) Output of PKP-type operator

Fig. 4. An illustration that the play-like operator and the PKP-operator formed from it possesses the wiping out property for $n = 2$.

5. Acknowledgement

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