Extension of Hysteresis operators of Preisach-type to real, Lebesgue measurable functions

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Abstract

Functions in $L^p_{loc}[0,\infty)$ where $1 \leq p \leq \infty$ can be considered as inputs to linear systems. However, hysteresis operators of Preisach type have only been defined on much smaller space of regulated (or Baire) functions. In this paper, we re-define Play operators so that they are well defined for real valued measurable functions. We show that this definition coincides with the older definition for continuous and regulated functions on an interval. Domain extension of hysteresis operators of Preisach type to real, Lebesgue measurable functions is then obtained in the standard manner using the re-defined Play operators.

Key words: Hysteresis operators of Preisach type, Lebesgue measurable functions.
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1 Introduction

The following notation will be used to denote some unusual function spaces:

- $C_u[0,T]$ ($C_l[0,T]$): space of upper (lower) semi-continuous functions on $[0,T]$.
- $R[0,T]$ is the space of regulated functions on $[0,T]$.
- Ψ_0 is the space of Preisach memory curves ([1], page 52).

Recently, there has been a lot of interest in studying systems with hysteresis operators (typically Preisach operators) and differential equations, with the output of the hysteresis operators typically acting as inputs to the differential equations. Thus there is a need to clarify the domain of definition for the hysteresis operators. The largest domain of definition for hysteresis operators so far is the space of regulated functions $R[0,T]$ [2] which is a closed subset of $L^\infty[0,T]$. Hence a density argument will not work for extension to $L^\infty[0,T]$ (this was noted originally by Krejci and Laurencot [3]). Krejci and Laurencot use approximation by functions of bounded variation to define the output of a Play operator – with parameter values greater than a certain critical value – for functions in $L^\infty[0,T]$. The difficulty with this approach is that the critical value mentioned above depends on the input function, and it is not possible to reduce it to 0. A standard approach for the extension problem is to approximate a function in $L^p[0,T]$ (where $1 \leq p < \infty$) by smooth functions – by using convolution with a smooth function with compact support. However, this does not work for hysteresis operators because we are also interested in a continuous extension to the space $L^\infty[0,T]$, and the above mentioned approximation fails for $L^\infty[0,T]$ ([4], pages 10 – 12). Our idea is to use the facts that (i) the essential upper (lower) envelope of a real valued measurable function is an upper (lower) continuous function; (ii) upper (lower) semi-continuous functions can be represented by the infimum (supremum) of a subset of elements in $C[0,T]$. (iii) the Play operator maps continuous functions to continuous functions. By (ii) above, it would map upper (lower) semi-continuous functions to functions of the same type. These steps allow us to define the play operator for real Lebesgue measurable functions. Whether this extension is continuous remains to be seen.

2 Definition of the Play operator for bounded Lebesgue measurable functions

To define the Play operator for essentially bounded Lebesgue measurable functions on $[0,T]$, let us first examine the nature of discontinuities of a measurable function f with domain $[0,T]$. There are four types of discontinuities for f in the interior of $[0,T]$:

(1) D_1 – the set of points where f has limits that exist from the left and the right at all points. This type is the most benign and is typically referred to in the literature in various ways discontinuities of the first-kind. When the right and left limits are identical, the discontinuities are referred to as removable. Functions in $R[0,T]$ only have this kind of discontinuity.

(2) D_2 – the set of points where f only has a limit from the left.

(3) D_3 – the set of points where f only has a limit from the right.

(4) D_4 – the set of points where f does not possess limits from either the left or from the right.

If f has a discontinuity at 0 then it is in the set D_2, while if T is a point of discontinuity then it is in D_3. The set of points $D_2 \cup D_3 \cup D_4$ are also referred to variously as discontinuities of the second kind or essential discontinuities. It is easy to show that the set of discontinuities $D_1 \cup D_2 \cup D_3$ is countable and nowhere dense (see [10] for an easy proof), but D_4 may be the entire set E or a subset of the first or second categories. We give some examples of functions with discontinuities in $D_2 \cup D_3 \cup D_4$.

Example 1 *(1) Let $f : [0, T] \to \mathbb{R}$ be given by: $f(x) = (-1)^{n+1}$; $x \in [1 - \frac{1}{n}, 1 - \frac{1}{n+1})$ for $n \in \mathbb{N}$; $f(1) = 0$. It has discontinuities in D_1 at the points $1 - \frac{1}{n}$ for $n \geq 2$; $n \in \mathbb{N}$ and in D_3 at 1.*

(2) Let $g : [0, T] \to \mathbb{R}$ be given by: $g(x) = \sin\left(\frac{1}{x - \frac{T}{2}}\right)$. Then g has a discontinuity in D_4 at $x = \frac{T}{2}$.

(3) The Thomae function $t(x) = \frac{1}{q}$; if $x = \frac{p}{q} \in \mathbb{Q} \cap [0, T]$, and $t(x) = 0$; $x \in \mathbb{Q}^c \cap [0, T]$, has $D_4 = \mathbb{Q} \cap [0, T]$.

(4) The Dirichlet function $d(x) = 1$; $x \in \mathbb{Q} \cap [0, T]$, and $d(x) = 0$; $x \in \mathbb{Q}^c \cap [0, T]$, has $D_4 = [0, T]$.

(5) The indicator function for the Cantor middle-thirds ternary set $f(x) = 1_C$ (C denotes the Cantor ternary set) has $D_4 = C$.

As the set D_4 could be of the second category for a general measurable function f, we consider the *essential upper and lower envelopes* of f which are known to be upper and lower semicontinuous respectively ([5], page 183). Some desirable properties of the envelopes are: (i) they can be represented by the supremum or infimum of a subset of $C[0, T]$; (ii) their points of continuity form a residual set or a set of second category within $[0, T]$ ([11]). Such functions are referred to as *point-wise discontinuous* by Hobson ([5], page 241); (iii) The points of discontinuity of these functions forms a nowhere dense set

([5], page 317). However, this set need not have measure 0. (iv) the operator $\mathcal{Q}_u : L^\infty[0,T] \to C_u[0,T]$ (respectively $\mathcal{Q}_l : L^\infty[0,T] \to C_l[0,T]$) taking a function to its essential upper (lower) envelope is a (nonlinear) projection operator.

We describe the envelopes below when the domain is $\mathbb{R}$. When the domain is $[0,T]$ one simply has to ensure that the sequences lie within the set, with obvious adjustments made for the end points. For $t \in \mathbb{R}$, consider any strictly monotone decreasing sequence $\{\epsilon_n\}$ converging to zero. Define:

$$u_n(t) = \operatorname*{ess\,sup}_{\tau \in (t-\epsilon_n, t+\epsilon_n)} f(\tau), \qquad f_u(t) \triangleq \inf_n u_n(t), \tag{1}$$

$$l_n(t) = \operatorname*{ess\,inf}_{\tau \in (t-\epsilon_n, t+\epsilon_n)} f(\tau), \qquad f_l(t) \triangleq \sup_n l_n(t), \tag{2}$$

The main difficulties in directly defining the output of the Play for an upper semi-continuous function are discontinuities of type D_3 and D_4. As at these points the function is not continuous from the right, it is difficult to attempt a definition along the lines of what was done for a continuous, monotone function. However, there is a way using the characterization of semi-continuous functions using continuous functions, and then defining the Play using this characterization.

Consider a parameter $r > 0$, initial value $w_{-1} \in \mathbb{R}$, and a function $f \in L^\infty[0,T]$. Let f_u and f_l be its upper and lower envelopes respectively. Then there exists K_u, $K_l \subset C[0,T]$ ([8], page 16), such that $\forall\, x \in [0,T]$, we have: $f_u(x) = \inf\{g_\alpha(x) \,|\, g_\alpha \in K_u\}$ and $f_l(x) = \sup\{h_\beta(x) \,|\, h_\beta \in K_l\}$. A concrete representation is $K_u = \{g \in C[0,T] \,|\, f_u \le g\}$ and $K_l = \{g \in C[0,T] \,|\, g \le f_l\}$ ([9], page 28). Denote the Play operator on the space of continuous functions by $\mathcal{F}_r[\cdot\,; w_{-1}] : C[0,T] \to C[0,T]$. Define:

$$\mathcal{F}_r^u[f_u;w_{-1}](x) = \inf\{\mathcal{F}[g_\alpha;w_{-1}](x) \mid g_\alpha \in K_u\} \tag{3}$$
$$\mathcal{F}_r^l[f_l;w_{-1}](x) = \sup\{\mathcal{F}[h_\beta;w_{-1}](x) \mid h_\beta \in K_u\}. \tag{4}$$

The functions $\mathcal{F}_r^u[f_u;w_{-1}]$ and $\mathcal{F}_r^l[f_l;w_{-1}]$ are well defined, and upper, lower semi-continuous respectively.

Definition 1 *(Play operator) Let f be a real Lebesgue measurable function on $[0,T]$ and $w_{-1} \in \mathbb{R}$. Define: $\mathcal{P}_r[f;w_{-1}] = \frac{1}{2}\left(\mathcal{F}_r^u[f_u;w_{-1}] + \mathcal{F}_r^l[f_l;w_{-1}]\right)$ as the output of the play operator with parameter $r > 0$.*

Theorem 1 *Let $f \in C[0,T]$. Let $r \geq 0$ and $w_{-1} \in \mathbb{R}$. Then: $\mathcal{P}_r[f;w_{-1}] = \mathcal{F}_r[f;w_{-1}]$. Furthermore, if $f \in R[0,T]$, then $\mathcal{P}_r[f;w_{-1}] = \mathcal{F}_r[f;w_{-1}]$.*

Proof. It is straight forward to see that the definition of the play operator agrees with the earlier definition for continuous functions in [1]. As regulated functions have discontinuities of type D_1, it is clear that the new definition yields identical outputs for such functions. Another way to see this, is to realize that regulated functions are pointwise limits of continuous functions. □

Example 2 *For the Thomae function t in Example 1.1 (3), $f_u = 0$, $f_l = 0$; $w(0) = w_{-1} = 0$ and $y(0) = 0$. Hence: $\mathcal{F}_r[t;w_{-1}](x) = 0$ for all $x \in [0,1]$. Similar outputs can be found for the Dirichlet and the indicator function for the Cantor set.*

The above outputs are what we would expect from an engineering point of view because the input functions mentioned have no “power”. It is straightforward to define the outputs of hysteresis operators of Preisach-type for real, Lebesgue measurable functions on $[0,T]$.

Definition 2 *(Hysteresis operators of Preisach type (HOPT)) Suppose* $Q : \Psi_0 \to \mathbb{R}$ *is an output mapping and let* $\psi_{-1} \in \Psi_0$*. Let* f *be a real Lebesgue measurable function on* $[0, T]$*. For* $t \in [0, T]$*, define:* $\mathcal{W}[f; \psi_{-1}](t) = Q \circ \psi(t)$ *as the output of a hysteresis operator of Preisach type , where* $\psi(t)(r) = \mathcal{P}_r[f(t); \psi_{-1}(r)]$.

Using Theorem 1 it is easy to see that the new definition of HOPT coincides with the old one for $C[0, T]$ and $R[0, T]$.

3 Conclusion

In this paper, we have presented an extension of the domain of the Play operator and hence hysteresis operators of Preisach type to real, Lebesgue measurable functions. We have shown that this extension agrees with the older definition for the space of continuous ($C[0, T]$) and regulated functions ($R[0, T]$).

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