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# PROPORTIONAL DERIVATIVE CONTROL OF HYSTERETIC SYSTEMS*

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**Abstract.** We discuss a tracking controller strategy for hysteretic systems, including magnetic and smart actuators, by using two output feedbacks. We show that a dual-loop proportional derivative controller derived from two feedback signals is sufficient to achieve tracking control. We further discuss regularity, well-posedness, and stability and obtain sufficient conditions on controller gains for ultimate bounded tracking control of hysteretic systems.

**Key words.** hysteresis, PD control, magnetic and smart actuators

**AMS subject classifications.** 34C55, 93B52, 93C10, 93D15, 58C07

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## 1. Introduction.

Hysteresis phenomena, caused by magnetism, static friction, elasticity, or mechanical backlash, occur in many physical systems, including magnetic and smart actuators. In recent years, control of hysteretic systems has received much attention due to the growing number of industrial applications of these actuators [1, 2, 3, 4, 5]. However, these actuators are difficult to control to achieve a given objective due to the presence of saturating hysteresis and other nonlinearities. Motivated by such hysteretic systems, in this paper we discuss how to develop a dual-loop proportional derivative (PD) tracking control using two output feedbacks for saturating, nonmonotone hysteretic systems.

Trajectory tracking control methods for hysteretic systems presented in the literature include inverse compensation of hysteresis [6, 7, 8], adaptive control [9, 10, 11], integral control [12, 13, 14, 15], passivity-based control [16, 17], monotonicity-based control [18, 19], and hybrid control, most of which are model dependent [20, 21]. In the case of magnetic and smart actuators, the system has two measured feedbacks that can be used for control: the error output derived from the actuator tip displacement and the induced voltage (magnetomotive force) of the actuator windings, which can be obtained from the input current and voltage (shown in [22]). The use of two feedbacks makes the control design discussed here entirely different from those used in the literature for control of hysteretic systems.

Inverse hysteresis compensation is the most common control method discussed in the literature [6, 8, 23, 24] and is applicable for hysteretic systems at low frequencies [6]. Another widely investigated strategy is integral, proportional, and derivative (PID) control [12, 14, 15], derived from the output error. PID controllers are employed for systems with only hysteresis nonlinearity [12, 15]. Adaptive control is another common method for controlling hysteretic systems [9, 25, 11, 26]. Adaptive controllers are primarily utilized for nonsaturating hysteresis operators without minor-loop closure behavior. Most of these schemes have merit for specific hysteretic

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systems and control operations. Magnetic and magnetostrictive actuators exhibit complex hysteresis, combined with other frequency-dependent nonlinearities such as eddy current and residual losses. These behaviors limit the usefulness of conventional control schemes to low-frequency ranges (less than 200 Hz [6]) and low-amplitude signals, which prevent the actuators from becoming saturated.

Existing feedback control schemes for hysteretic systems use only the output to be regulated to derive the control signal. For example, schemes for magnetostrictive actuators use only tip displacement as input for the control scheme [23]. However, when a current is applied to a magnetostrictive actuator, induced voltage can also be measured. To achieve precision control of these complex systems, it is important to include all measurements that can be easily obtained and that reflect properties of the hysteretic system. Here, we use PD feedbacks derived from the output to be regulated and a proportional feedback derived from a second output. PID controllers derived from the output to be regulated have been shown to be appropriate for hysteretic systems in which hysteresis is the only nonlinearity [12, 15, 27]. For more complicated systems, a PID controller combined with a feedforward loop or with a model-based control signal has been utilized [21, 28, 29, 30, 31]. The objective of this paper is to show that the feedforward loop and model-based controllers can be effectively replaced by adding the second proportional controller loop, and such a controller is appropriate for complex hysteretic systems. Although our primary use is for magnetic and magnetostrictive actuators, the results are applicable to any magnetic system and can be extended to most hysteretic systems.

In this paper, we discuss neither the explicit role of each feedback signal nor stability and tracking in the presence of exogenous disturbances. We restrict our attention to showing that, under the feedback control, the system achieves tracking. In a forthcoming paper, we present the robust control of linear actuators, as well as the explicit role of each feedback to control the system in rated conditions.

**1.1. Electric network model for linear actuators.** Here we obtain the system of equations that governs a linear actuator, the motivating example for the hysteretic system we consider.

Consider the linear actuator connected to a voltage input, $V$, shown in Figure 1.1(a), where the power transfers from the electrical domain to the magnetic domain and then to the mechanical domain, which produces a linear motion of the actuator rod. Variable $i$ is the actuator input current and $y$ is the displacement of the tip.

The corresponding electrical network model is shown in Figure 1.1(b). The model consists of an ideal voltage source, $V$, a linear resistor, $R_L$, a nonlinear resistor, $R_N$,

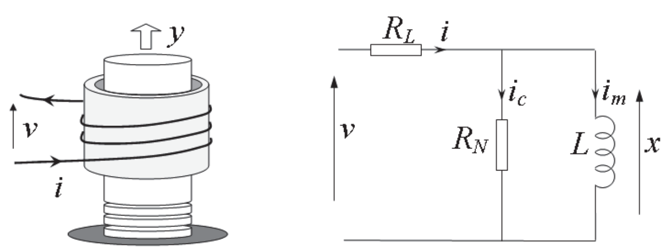


Fig. 1.1. *Electric network model for a magnetic linear actuator.*

and a lossless inductor, $L$. The linear resistor accounts for the Ohmic losses in the winding of the device. All linear actuators exhibit core losses, including eddy current, hysteresis, and residual losses, and the nonlinear resistor in the model corresponds to these losses. Voltage $x$ represents the induced voltage, which opposes the magnetomotive force generated by the changing flux linkage of the winding, assuming a lossless magnetic circuit. From Faraday's and Lenz's laws, induced voltage $x$ is proportional to the rate of change of the flux linkage. Neglecting end effects and assuming a uniform magnetic path, we conclude that $x$ is proportional to the rate of change of magnetic flux density, $B$. That is, $\dot{B} = C_0 x$ for some constant $C_0$. Since $B = \mu_0(H + M)$, $\dot{H} + \dot{M} = C_1 e$, where $C_1$ is a constant and $H$ and $M$ represent the average axial magnetic field and magnetization, respectively.

Instantaneous core losses can be represented as a function of the derivative of flux density and, hence, as a function of induced voltage [32, 33, 34]. Thus, current $i_c$ through the nonlinear resistor can be expressed as a function of $x$, that is, $i_c = \theta(x)$ for some function $\theta$. When the magnitude of the rate of change of flux density increases, both eddy current losses and residual losses increase, as does induced voltage $x$. Hence, $\theta(x)$ is a strictly monotone increasing function of $x$. Magnetization current $i_m$ is proportional to the average axial magnetic field $H$ of the coil, $i_m = \beta H$, where $\beta$ is a constant [35]. Applying Kirchhoff's circuit laws to the model in Figure 1.1(b),

$$i = i_c + i_m = \theta(x) + \beta H. \tag{1.1}$$

Further, for such a system, $M$ relates to $H$ via a rate-independent hysteresis operator such that $M = \Gamma[H\ ;\ \psi_{-1}](t)$.

Motivated by hysteretic systems of linear actuators and the ability to measure induced voltage, we consider the following system.

## 2. System with hysteresis.

Here we consider hysteretic systems modeled by nonlinear functional differential equations of the form

$$\dot{y}(t) + \alpha y(t) = aM^2(t) + bM(t), \tag{2.1}$$

$$\dot{H}(t) + \dot{M}(t) = \gamma x(t), \tag{2.2}$$

$$\theta(x)(t) + \beta H(t) = u(t), \tag{2.3}$$

$$M(t) = \Gamma[H\ ;\ \psi_{-1}](t), \tag{2.4}$$

$$y(0) = y_0, \quad H(0) = H_0, \tag{2.5}$$

where $u(t)$ is the system input, $x(t)$ is an output, and $y(t)$ is the output to be regulated. Functions $H(t)$ and $M(t)$ represent two internal states. For magnetic systems, $x(t)$ corresponds to the induced voltage. Operator $\Gamma[\cdot\ ;\ \psi_{-1}]$ is the hysteresis operator with initial memory state $\psi_{-1}$. Equation (2.2) is equivalent to Faraday's law. Further, $a$, $b$, $\alpha$, $\beta$, and $\gamma$ are positive constants.

For magnetic systems, (2.1) represents the relationship between the output to be regulated (tip displacement) and magnetization, which is quadratic [6, 36]. Magnetostriction, $\lambda$, can be empirically expressed in terms of magnetization: $\lambda = \sum_{i=0}^{\infty} a_i M^{2i}$ [37]. However, terms for $i > 1$ are related to elastic strain and do not play an active role in the magnetomechanical effect [38]. In the case of moving iron controllable actuators, the relationship is linear [39]. To incorporate all such systems, the right-hand side of (2.1) is chosen to be a quadratic function.

We assume hysteresis can be represented by an operator of Preisach type. The Preisach operator has been shown to be a well-suited approximation for magnetic and smart actuators [40, 41, 42] but is more restrictive than an operator of Preisach type. We use the operator of Preisach type defined in Definition 2.4.2 of Brokate and Sprekels [43]. Suppose $\Psi_0$ is the space of admissible memory curves:

$$\Psi_0 := \{\psi \mid \psi : \mathbb{R}_+ \to \mathbb{R},\ |\psi(r) - \psi(\bar{r})| \le |r - \bar{r}|\ \ \forall r, \bar{r} \ge 0, R_{supp}(\psi) < \infty\},$$

where $R_{supp}(\psi) := \sup\{r \mid r \ge 0, \psi(r) \ne 0\}$. Preisach operator $\Gamma[H\ ;\psi_{-1}](t)$, with initial memory curve $\psi_{-1}$, is defined with an output map $Q : \Psi_0 \to \mathbb{R}$ of the form

$$Q(\psi) = \int_0^\infty q(r, \psi(r))dr + w_{00}, \quad \text{where} \quad q(r,s) = 2\int_0^s \omega(r,\sigma)d\sigma, \tag{2.6}$$

with $w_{00} = \int_0^\infty \int_{-\infty}^0 \omega(r,s)dsdr + \int_0^\infty \int_0^\infty \omega(r,s)dsdr$. Here, $\omega \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$ is the Preisach density function.

**3. Statement of the problem.** The control objective is to design a controller such that the output $y$ of system (2.1)–(2.5) approximately follows a specific trajectory, $y_d$. To be precise, for arbitrary $\varepsilon > 0$, we seek an output feedback strategy such that for given $y_d \in W^{2,\infty}[0,\infty)$, the tracking error $r(t) = y(t) - y_d(t)$ is ultimately bounded by $\varepsilon$ (that is, $\limsup_{t\to\infty} |r(t)| < \varepsilon$). To design a control to stabilize system (2.1)–(2.5), the following assumptions are introduced:

- $\mathcal{H}1$: Hysteresis operator of Preisach-type $\Gamma[\cdot\,;\psi_{-1}]$ is counterclockwise dissipative (output-rate dissipative), piecewise monotone, and Lipschitz continuous on $C[0,T]$ [43].
- $\mathcal{H}2$: There exist real numbers $\Gamma_{sat1}$, $\Gamma_{sat2} > 0$ such that $-\Gamma_{sat1} = \inf_{\psi\in\Psi_0} Q(\psi)$ and $\Gamma_{sat2} = \sup_{\psi\in\Psi_0} Q(\psi)$.
- $\mathcal{H}3$: $\theta(\cdot)$ is continuous and strictly monotone increasing with $\theta(0) = 0$.
- $\mathcal{H}4$: $\alpha > 0$.
- $\mathcal{H}5$: The desired trajectory $y_d \in W^{2,\infty}[0,\infty)$ satisfies $\|\ddot{y}_d\|_{2,\infty} \le A$ and $-b^2/4a + \delta < \dot{y}_d(t) + \alpha y_d(t) < a\Gamma_{sat2}^2 + b\Gamma_{sat2} - \delta$ for all $t \in [0,\infty]$, where $\delta > 0$.

Hypothesis $\mathcal{H}1$ is a typical condition which guarantees the thermodynamic consistency of hysteresis operators. In particular, it reflects the fundamental energy dissipation properties of hysteresis. Also, for given $\psi_1, \psi_2 \in \Psi_0$ and for some constant $C > 0$, if $|Q(\psi_1) - Q(\psi_2)| \le C||\psi_1 - \psi_2||_\infty$, then operator $\Gamma$ is Lipschitz continuous on $C[0,T] \times \Psi_0$ (Proposition 2.4.9 in Brokate and Sprekels [43]). Hypotheses $\mathcal{H}2$ represents the saturation property of hysteresis operators, the class of operators in this study. The monotonicity of $\theta$ in hypothesis $\mathcal{H}3$ represents the increase of power losses as the magnitude of $x$ increases. Hypothesis $\mathcal{H}4$ describes the bounds for the desired trajectory. Due to saturation, desired trajectories beyond some limit cannot be achieved. Here $\delta$ represents a "buffer," a very small positive constant that is necessary for the tracking error to be ultimately bounded by an arbitrary $\varepsilon > 0$, seen in section 7.

First we prove that a hysteresis operator satisfying hypotheses $\mathcal{H}1$ and $\mathcal{H}2$ has a output reachability property.

LEMMA 3.1. *For all $y \in (-\Gamma_{sat1}, \Gamma_{sat2})$ and $\psi_{-1} \in \Psi_0$, there exists $H \in C[0,T]$ such that $\Gamma[H;\psi_{-1}](T) = y$.*

*Proof.* By compactness of $\Psi_0$ and the continuity of $Q$ for all $\varepsilon > 0$, $\exists\ \kappa > 0$ such that for all $\psi_1$, $\psi_2 \in \Psi_0$, $\|\psi_1 - \psi_2\|_\infty < \kappa \implies |Q(\psi_1) - Q(\psi_2)| < \frac{\varepsilon}{3}$. For a given $\varepsilon$, fix a value of $\kappa$ so that the above condition is satisfied. By the construction presented in section 3 of [44], there exists $\tilde{\psi}_{-1} \in \Psi_0$ such that

$$\|\tilde{\psi}_{-1} - \psi_{-1}\|_\infty < \kappa. \tag{3.1}$$

Furthermore, there exists a finite alternating sequence $s = (s_0, s_1, \ldots, s_N)$ such that $\mathcal{F}_r[v; 0](T) = \tilde{\psi}_{-1}(r)$, where $\mathcal{F}_r$ is the play operator with parameter $r$ and $v$ is the linear interpolation of $s$ on $[0, T]$. The sequence $s$ is a fading sequence with respect to Preisach ordering (see p. 82 of [43]). Let $s^* = \max_{i\in\{0,\ldots,N\}} |s_i|$. Then there exists an alternating sequence $\bar{s} = (\bar{s}_0, \ldots, \bar{s}_M)$ with $\bar{s}_0 > s^*$ such that

$$\phi(r) = \mathcal{F}_r[w; \tilde{\psi}_{-1}](T) \text{ and } \|\phi\|_\infty < \kappa, \tag{3.2}$$

where $w$ is the linear interpolation of $\bar{s}$ on $[0, T]$.

Next, let $\eta = \min\{\Gamma_{sat2} - y, y + \Gamma_{sat1}\}$. By $\mathcal{H}2$, there exist $\psi_*$ and $\psi^* \in \Psi_0$ such that $-\Gamma_{sat1} \le Q(\psi) < -\Gamma_{sat1} + \frac{\eta}{2}$ and $\Gamma_{sat2} - \frac{\eta}{2} < Q(\psi^*) \le \Gamma_{sat2}$. As $y \in [-\Gamma_{sat1} + \frac{\eta}{2}, \Gamma_{sat2} - \frac{\eta}{2}]$, there exists at least one $\psi \in \Psi_0$ such that $Q(\psi) = y$ by continuity of $Q$ on $\Psi_0$. Now, $\exists\, \tilde{\psi} \in \Psi_0$ such that $\|\tilde{\psi} - \psi\|_\infty < \kappa$, by the constructive proof presented in section 3 of [44]. Furthermore, there exists a finite alternating sequence $m = (m_0, m_1, \ldots, m_L)$ such that $\mathcal{F}_r[x; 0](T) = \tilde{\psi}(r)$, where $x$ is the linear interpolation of $m$ on $[0, T]$. Consider the function $\tilde{H} \in C_{pm}[0, T]$ defined by

$$\tilde{H}(t) = \begin{cases} v(3\,t), & 0 \le t < \frac{T}{3}, \\ w(3\,t - T), & \frac{T}{3} \le t < \frac{2\,T}{3}, \\ x(3\,t - 2\,T), & \frac{2\,T}{3} \le t \le T. \end{cases}$$

Let $H$ be the function on $[0, T]$ defined by $H(t) = \tilde{H}\left(\frac{2}{3}\,t + \frac{T}{3}\right)$. Due to rate independence of $\Gamma$,

$$\Gamma[\tilde{H}\,\big|_{[\frac{T}{3},T]}; \tilde{\psi}_{-1}](T) = \Gamma[\tilde{H}; 0](T) \implies \Gamma[H; \tilde{\psi}_{-1}](T) = \Gamma[\tilde{H}; 0](T).$$

By (3.1), $\|\Gamma[H; \psi_{-1}](T) - \Gamma[H; \tilde{\psi}_{-1}](T)\|_\infty < \frac{\varepsilon}{3}$. Finally,

$$\begin{aligned} |y - \Gamma[H; \psi_{-1}](T)| &\le |Q(\psi) - \Gamma[H|_{[\frac{T}{2},T]}; 0](T)| + |\Gamma[H|_{[\frac{T}{2},T]}; 0](T) - \Gamma[H|_{[\frac{T}{2},T]}; \phi](T)| \\ &\quad + |\Gamma[H|_{[\frac{T}{2},T]}; \phi](T) - \Gamma[H; \psi_{-1}](T)| \\ &\le |Q(\psi) - Q(\tilde{\psi})| + |Q(0) - Q(\phi)| + |\Gamma[H; \tilde{\psi}_{-1}](T) - \Gamma[H; \psi_{-1}](T)| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}. \end{aligned}$$

As $v, w, x$ are piecewise continuous functions, $H$ is a piecewise continuous function as well. As $C_{pm}[0, T]$ is dense in $C[0, T]$ and $Q$ is continuous on $C[0, T]$, the claim follows. ∎

**4. Preliminaries.** In this section we introduce definitions and theorems needed to precisely formulate the problem. First, we explicitly state notation and terminology. Define $\mathbb{R}_+ := [0, \infty)$. Let $I \subset \mathbb{R}_+$ be a compact interval. Spaces $C(I)$, $C^1(I)$, $BC(I)$, and $C_{pm}[I]$ denote continuous, continuously differentiable, bounded continuous, and piecewise monotone continuous functions on $I$, respectively. Further, denote

$W^{1,1}(I)$: Space of absolutely continuous functions on $I$.
$W^{1,\infty}(I)$: Space of Lipschitz continuous functions on $I$.
$W^{2,\infty}(I)$: Space of twice differentiable bounded functions with locally absolutely continuous first derivatives and essentially bounded second derivatives.

Next we state necessary theorems and lemmas and recall some established properties of hysteresis operators of Preisach type.

THEOREM 4.1 (inverse function theorem). *If a function is continuous and strictly monotone on an interval, then it has a single-valued inverse function, which is strictly monotone and continuous on that interval.*

LEMMA 4.1 (Theorem 2.11.20 in Brokate and Sprekels [43]). *Let $\mathcal{W}[u\,;\psi_{-1}]$ be a piecewise strictly monotone continuous Preisach operator with domain $I=[a,b]$. Suppose $u(t)\in I$ for all $t\in[0,T]$. Define $\mathcal{X}(\cdot)$ on $[0,b-a]$ as*

$$\mathcal{X}_I(x) := \inf\{|W[u;\psi_{-1}](T)-W[u:\psi_{-1}](0)| : \psi_{-1}\in\Psi_0,\ u\in C_m[0,T],\\ |u(T)-u(0)|=x\}.$$

*If $\mathcal{X}_I(x)\geq\hat{\gamma}x$ for all $x>0$ and for some constant $\hat{\gamma}>0$, then $W: C_I[0,T]\to C[0,T]$ is invertible and $W^{-1}: C_J[0,T]\to C_I[0,T]$ is Lipschitz continuous, where $C_J[0,T]$ is the image of $C_I[0,T]$ under $\Gamma$.*

LEMMA 4.2. *Suppose $\Gamma$ is a hysteresis operator of Preisach type satisfying hypotheses $\mathcal{H}1$ and $\mathcal{H}2$, and let $\psi_{-1}\in\Psi_0$. Then, $(\mathcal{I}+\Gamma)[u;\psi_{-1}]$ is invertible and $(\mathcal{I}+\Gamma)^{-1}$ is Lipschitz continuous on $C[0,T]$.*

*Proof.* The claim follows from the observation that for $(\mathcal{I}+\Gamma)[u;\psi_{-1}]$, $\mathcal{X}_I(x)\geq x$ on any closed interval $I$ and from Lemma 4.1. ∎

## 5. Feedback control.

Control input $u(t)$ is derived from both output error feedback $r=y-y_d$ and output feedback $x$:

$$u(t) = -\widehat{k}_P x - k_P r - k_D\dot{r}. \tag{5.1}$$

Although the signal obtained from $r(t)$ can be generated by a PI, PD, or PID controller (see [45]), we restrict our attention to PD controllers. The proportional control signal $k_P x$ effectively replaces the nonlinear model-based control signals, such as those used in hybrid PI/PID controllers in the existing literature [28, 29, 30, 31]. Subsequent analysis shows that such a dual-loop feedback controller has a stability region that extends tracking to almost the full range. Stability region refers to the set of values for controller parameters that yields stability in closed-loop systems [46].

Next we obtain the closed-loop system. Equation (2.2) can be expressed as

$$(\mathcal{I}+\Gamma)[H\,;\psi_{-1}] = \gamma\int_0^t x\,d\tau + H_0 + \Gamma_0 := f(x). \tag{5.2}$$

By the invertibility of $\mathcal{I}+\Gamma$, $H=(\mathcal{I}+\Gamma)^{-1}f(x)$. Hence $\Gamma[H\,;\psi_{-1}]$ can be rewritten in terms of $x$ as $W[f(x)\,;\psi_{-1}] := \Gamma\left[(\mathcal{I}+\Gamma)^{-1}f(x)\,;\psi_{-1}\right]$. Then, (5.2) combined with (2.1)–(2.5) yields the closed-loop system

$$\dot{y}+\alpha y = g\left(W[f(x)\,;\psi_{-1}]\right), \tag{5.3}$$

$$\varphi(x(t)) + \beta\gamma\int_0^t x(\tau)d\tau = (k_D\alpha - k_P)y(t) + G(y_d,x)(t), \tag{5.4}$$

where $\varphi(x)=\widehat{k}_P x+\theta(x)$, $g(\cdot)=a(\cdot)^2+b(\cdot)$, and

$$G(y_d,x)(t) = \beta W[f(x);\psi_{-1}](t)\\ - k_D g\left(W[f(x);\psi_{-1}](t)\right) + k_P y_d(t) + k_D\dot{y}_d(t) - \beta(H_0+\Gamma_0).$$

Due to the semigroup property satisfied by $\Gamma$, we also have

$$\varphi(x) + \beta\gamma\int_{t_1}^t x(\tau)d\tau = (k_D\alpha-k_P)y + G(y_d,x)(t),\\ G(y_d,x)(t) = \beta W[f(x)\,;\psi_{-1}](t) - k_D g\left(W[f(x)\,;\psi_{-1}]\right) + k_P y_d(t) + k_D\dot{y}_d(t)\\ - \beta(H(t_1)+\Gamma[H;\psi_{-1}](t_1)),$$

where $0\leq t_1\leq t$. This observation is useful in section 6.

Next we show that if hypotheses $\mathcal{H}1-\mathcal{H}5$ are satisfied, then system (5.3)–(5.4) possesses a solution such that $(x, y) \in BC[0,\infty)\times W^{1,\infty}[0,\infty)$. Existence of a solution for system (5.3)–(5.4) guarantees existence of a bounded continuous $x(t)$ satisfying (2.2) almost everywhere. We note that for a given $H \in C_{pm}[0,\infty)$, the derivative of $M(t) = \Gamma[H;\psi_{-1}](t)$ can be discontinuous at every peak. However, this analysis shows that for $y_d$ satisfying hypothesis $\mathcal{H}5$, $x$ is continuous and thus both left and right derivatives of $H(t)$ become zero at each peak. Hence, $\dot{M}(t)$ can be extended to a continuous function. This continuity can be observed in the simulations in section 8, which also demonstrate that failure to satisfy the conditions in $\mathcal{H}5$ results in a discontinuous $x$. Further, if $y_d \in W^{1,\infty}[0,\infty)$, then there exists a unique solution for $M \in W^{1,\infty}[0,\infty)$ (Theorem 6.1) and hence there exists a unique essentially bounded $x(t)$ which needs not be continuous everywhere.

**6. Existence and stability.** First, we determine specific properties required of $\varphi(x)$. The proof of the following lemma is a direct consequence of Theorem 4.1.

LEMMA 6.1. *For each $x \in \mathbb{R}$, the function $\varphi := \widehat{k}_P x + \theta(x)$ is strictly monotone. Further, $\varphi^{-1}$ is well defined, strictly monotone, and Lipschitz continuous with Lipschitz constant $1/\widehat{k}_P$.*

To establish existence of a solution for the closed-loop system, we need to show that there exists $(x, y) \in BC[0,\infty) \times W^{1,\infty}[0,\infty)$ satisfying (5.3)–(5.4).

LEMMA 6.2. *$W[f(x)\,;\psi_{-1}]$ and $G(y_d, x)$ are locally Lipschitz continuous with respect to $x$. That is, whenever $(x_i, \psi_{-1,i}) \in C[0,T] \times \Psi_0$, $i = 1, 2$, then, for some positive constants $C$, $D$, and $E$,*

$$\|W[f(x_1);\psi_{-1,1}] - W[f(x_2);\psi_{-1,2}]\|_\infty \le C(\max\{DT\|x_1 - x_2\|_\infty, \|\psi_{-1,1} - \psi_{-1,2}\|_\infty\}),$$
$$\|G(y_d, x_1) - G(y_d, x_2)\|_\infty \le C(\max\{ET\|x_1 - x_2\|_\infty, \|\psi_{-1,1} - \psi_{-1,2}\|_\infty\}).$$

*Proof.* From Lemma 4.2,

$$\begin{aligned}&\|W[f(x_1)\,;\psi_{-1,1}] - W[f(x_2)\,;\psi_{-1,2}]\|_\infty\\ &\quad\le C(\max\{\|[\mathcal{I}+\Gamma]^{-1}[f(x_1);\psi_{-1,1}] - [\mathcal{I}+\Gamma]^{-1}[f(x_2);\psi_{-1,1}]\|_\infty, \|\psi_{-1,1} - \psi_{-1,2}\|_\infty\}.\end{aligned}$$

Define $V_i = [\mathcal{I}+\Gamma]^{-1}[\cdot,\psi_{-1,i}]$, where $i = 1, 2$. Since $[\mathcal{I}+\Gamma]^{-1}$ is Lipschitz continuous, there exists positive constant $C_1$ such that

$$\begin{aligned}\|V_1[f(x_1)] - V_2[f(x_2)]\|_\infty &\le C_1\ \max\{\|f(x_1) - f(x_2)\|_\infty, \|\psi_{-1,1} - \psi_{-1,2}\|_\infty\}\\ &= C_1\ \max\left\{\gamma \sup_{t\in[0,T]}\left|\int_0^t (x_1 - x_2)d\tau\right|, \|\psi_{-1,1} - \psi_{-1,2}\|_\infty\right\}\\ &\le C_1\ \max\left\{\gamma \sup_{t\in[0,T]} \|x_1 - x_2\|_\infty t, \|\psi_{-1,1} - \psi_{-1,2}\|_\infty\right\}\\ &\le C_1\ \max\{\gamma T\|x_1 - x_2\|_\infty, \|\psi_{-1,1} - \psi_{-1,2}\|_\infty\},\end{aligned}$$

from which we conclude the Lipschitz continuity of $W[f(x)\,;\psi_{-1,1}]$. Lipschitz continuity of $G(y_d, x)$ then follows from the Lipschitz continuity of $W[f(x)\,;\psi_{-1,1}]$, the local Lipschitz continuity of $g(\cdot)$, and the boundedness of $g(W[f(x)\,;\psi_{-1,1}])$. ∎

LEMMA 6.3. *Suppose hypotheses $\mathcal{H}1$–$\mathcal{H}5$ hold. For a given $y \in W^{1,\infty}[0,T]$, and $\psi_{-1} \in \Psi_0$, there exists a solution $x \in C[0,T]$ for* (5.3)–(5.4).

*Proof.* From the hypotheses, $y_d$, $\dot{y}_d$, $W[f(x)]$, and $g(W[f(x)])$ are bounded. Therefore, there exists a bound $B$ such that $|G(y_d, x)| < B$ for all $t \in [0,\infty)$.

Further, from Lemma 6.2, since $\psi_{-1} \in \Psi_0$ is given, whenever $x_1, x_2 \in C[0,T]$, $\|G(y_d, x_1) - G(y_d, x_2)\|_\infty \le L\|x_1 - x_2\|_\infty$ for some positive constant $L$. Define $z = \varphi(x)$. From Lemma 6.1, $\varphi^{-1}$ is well defined and Lipschitz continuous. Note that $C[0,T]$ is convex and a Banach space under the norm $\|x\|_\infty = \sup_{t\in[0,T]} |x(t)|$. Define operator $R$ by

$$Rz(t) := G\left(y_d, \varphi^{-1}(z)\right) - \beta\gamma \int_0^t \varphi^{-1}(z)d\tau + (k_D\alpha - k_P)y.$$

*Claim* 1. $R : C[0,T] \to C[0,T]$.

To verify this, consider $t_1, t_2 \in [0,T]$ such that $t_2 \ge t_1$:

$$\begin{aligned}|Rz(t_1) - Rz(t_2)| \le{}& \left|G\left(y_d, \varphi^{-1}(z)\right)(t_1) - G\left(y_d, \varphi^{-1}(z)\right)(t_2)\right| \\ &+ |\beta\gamma| \left|\int_{t_1}^{t_2} \varphi^{-1}(z)d\tau\right| + |k_D\alpha - k_P||y(t_1) - y(t_2)|.\end{aligned}$$

By the Lipschitz continuity of $G(y_d, x)$ in the $x$ variable, and the Lipschitz continuity of $\varphi^{-1}$,

$$\begin{aligned}|Rz(t_1) - Rz(t_2)| \le{}& C \sup_{\tau\in[t_1,t_2]} |z(\tau) - z(t_1)| + |\beta\gamma| \left|\sup_{\tau\in[t_1,t_2]} \varphi^{-1}(z)(\tau)\right| \left|\int_{t_1}^{t_2} d\tau\right| \\ &+ |k_D\alpha - k_P||y(t_1) - y(t_2)|\end{aligned}$$

for some positive constant $C$. From the given hypothesis, $y \in C[0,T]$. From the continuity of $z(t)$, $|Rz(t_1) - Rz(t_2)| \le C_1|t_1 - t_2| \to 0$ as $t_1 \to t_2$, where $C_1$ is a positive constant, concluding the claim.

*Claim* 2. $R : C[0,T] \to C[0,T]$ is continuous.

Consider a sequence $\{z_n\} \subset C[0,T]$ such that $z_n \to z$ in $C[0,T]$. Then, by a similar argument to that in the proof of Claim 1,

$$\begin{aligned}\|Rz_n - Rz\|_\infty \le{}& \left\|G\left(y_d, \varphi^{-1}(z_n)\right) - G\left(y_d, \varphi^{-1}(z)\right)\right\|_\infty \\ &+ |\beta\gamma| \sup_{t\in[0,T]} \left|\int_0^t \varphi^{-1}(z_n)d\tau - \int_0^t \varphi^{-1}(z)d\tau\right| \\ \le{}& C\|z_n - z\|_\infty + |a| \sup_{t\in[0,T]} \int_0^t \left|\varphi^{-1}(z_n) - \varphi^{-1}(z)\right| d\tau.\end{aligned}$$

Since $\varphi^{-1}$ is continuous, $\varphi^{-1}(z_n) \to \varphi^{-1}(z)$ as $z_n \to z$. Then, as $z_n \to z$, $Rz_n \to Rz$ for all $t \in [0,T]$, which concludes the proof that $R$ is continuous.

*Claim* 3. $R : C[0,T] \to C[0,T]$ is completely continuous.

Let $S$ be a bounded set in $C[0,T]$; that is, there exists a constant $C_0$ such that $\sup_{t\in[0,T]} |z(t)| \le C_0$ for all $z \in S$. We need $R(S)$ to be relatively compact. Since $\varphi^{-1}(0) = 0$, from Lemma 6.1,

$$\|\varphi^{-1}(z)\|_\infty = \sup_{t\in[0,T]} |\varphi^{-1}(z)(t) - \varphi^{-1}(0)(t)| < \sup_{t\in[0,T]} \frac{1}{\widehat{k}_P}|z(t) - 0| = \frac{C_0}{\widehat{k}_P}.$$

For all $z \in S$,

$$\begin{aligned}|Rz(t)| &\le \sup_{z\in S} |G(y_d, \varphi^{-1}(z))| + \sup_{z\in S} \left|\beta\gamma \int_0^t \varphi^{-1}(z)d\tau\right| + |k_D\alpha - k_P||y(t)| \\ &\le B + \frac{|\beta\gamma|TC_0}{\widehat{k}_P} + |k_D\alpha - k_P|\|y\|_\infty.\end{aligned}$$

Then $R(S)$ is uniformly bounded. Also, for any given sequence $\{z_n\} \in C[0,T]$,

$$\|Rz_{m1} - Rz_{m2}\|_\infty \leq \left\|G\left(y_d, \varphi^{-1}(z_{m1})\right) - G\left(y_d, \varphi^{-1}(z_{m2})\right)\right\|_\infty + \sup_{t\in[0,T]} |\beta\gamma| \left|\int_0^t \varphi^{-1}(z_{m1}) - \varphi^{-1}(z_{m2}) d\tau\right|.$$

By the Lipschitz continuity of $G\left(y_d, \varphi^{-1}(z)\right)$ and $\varphi^{-1}(z)$, we have $\|Rz_{m1} - Rz_{m2}\|_\infty \leq (C_1 + |\beta\gamma|TC_2)\|z_{m1} - z_{m2}\|_\infty$ for some constants $C_1$ and $C_2$. Hence $R(S)$ is equicontinuous. Then from the Arzela–Ascoli theorem, $R(S)$ is relatively compact.

*Proof of the lemma.* Since $Rz(t)$ is completely continuous, it follows from the Schauder fixed point theorem that equation $z = Rz(t)$ has a fixed point in $C[0,T]$. Since $z = \varphi(x)$ and $\varphi^{-1}(z)$ is Lipschitz continuous and strictly monotone increasing, there exists a solution $x \in C[0,T]$ satisfying (5.3)–(5.4). □

Lemma 6.3 concludes the existence of a local solution for $x(t) \in C[0,T]$. The next lemma establishes global existence of a solution for a given $y$. We follow [47] and the existence principle discussed by Lee and O'Regan [48].

LEMMA 6.4. *Suppose hypotheses $\mathcal{H}1$–$\mathcal{H}5$ hold. Let $y \in W^{1,\infty}[0,\infty)$ and $\psi_{-1} \in \Psi_0$ be given. Then there exists a solution $x \in BC[0,\infty)$ for* (5.3)–(5.4).

*Proof.* Consider the sequence of functions $\{z_n\}$ given by

$$z_n = G\left(y_d, \varphi^{-1}(z_n)\right) - \beta\gamma \int_0^t \varphi^{-1}(z_n) d\tau + (k_D\alpha - k_p)y, \quad t \in [0, t_n], \tag{6.1}$$

where $0 < t_1 < t_2 < \cdots < t_n < \cdots$ with $t_n \to \infty$ as $n \to \infty$. Suppose $\{\psi_n(t)\}$ is the set of memory evolutions of hysteresis operator $W$ corresponding to the input sequence for the hysteresis operator $\left\{(\mathcal{I} + \Gamma)^{-1} f(\varphi^{-1}(z_n))\right\}$ at each $t \in [0, t_n]$, where $\psi_{-1} = \psi_n(0)$ is fixed for all $n$. From Lemma 6.3, (6.1) has a solution $z \in C[0, t_n]$ for each $n$. However, the solutions may not be identical. Hence, even though $\psi_n(0) = \psi_{-1}$ for all $n$, $\psi_n(t)$ need not be same for each $n$.

*Claim* 1. For each $j = 1, 2, 3, \ldots$, the sequence $\{z_n\}_{n\geq j}$ is uniformly bounded on $[0, t_j]$.

As we have already seen in Lemma 6.3, $|G(y_d, \varphi^{-1}(z_n))| < B$. Since $y$ is bounded from the lemma hypotheses, there exists $M$ such that $|(k_D\alpha - k_P)y| < M$. We show that $\{z_n\}$ is uniformly bounded. Notice that $z_n(0) < B + M$. We claim that $|z_n| < 2B + 2M$ for all $n$. Assume, by contradiction, that this is not the case. Then there exists $t^* \in [0,\infty)$ such that $|z_n(t^*)| \geq 2B + 2M$. Consider the case $z_n(t^*) \geq 2B + 2M$. Suppose $z_n(t) > 0$ for all $t \in [0, t^*]$. Since $\varphi^{-1}(0) = 0$ and $\varphi^{-1}$ is strictly monotone increasing, $\varphi^{-1}(z_n) > 0$. Then,

$$\begin{aligned} 2B + 2M < z_n(t^*) &= G\left(y_d, \varphi^{-1}(z_n)\right) - \beta\gamma \int_0^{t^*} \varphi^{-1}(z_n) d\tau + (k_D\alpha - k_P)y(t^*) \\ &\leq B + M - a_1 < B + M \end{aligned}$$

for some constant $a_1 > 0$, which is a contradiction. Therefore, there exists $t_0 < t^*$ such that $z_n(t_0) \leq 0$. Because of the continuity of $z_n(t)$, there exists $t \in [t_0, t^*]$ such that $z_n(t) = 0$. Without loss of generality, assume $z_n(t_0) = 0$ and $z_n(t) > 0$ for $t \in (t_0, t^*]$. Subsequently,

$$\begin{aligned}2B+2M &\le z_n(t^*)-z_n(t_0)\\ &= G\left(y_d,\varphi^{-1}(z_n)\right)(t^*) - \beta\gamma\int_0^{t^*}\varphi^{-1}(z_n)d\tau + (k_D\alpha-k_P)y(t^*)\\ &\quad -G\left(y_d,\varphi^{-1}(z_n)\right)(t_0) + \beta\gamma\int_0^{t_0}\varphi^{-1}(z_n)d\tau - (k_D\alpha-k_P)y(t_0)\\ &\le 2B+2M-\beta\gamma\int_{t_0}^{t^*}\varphi^{-1}(z_n)d\tau \le 2B+2M-a_2 < 2B+2M\end{aligned}$$

for some $a_2>0$. This is a contradiction, which implies $z_n(t^*)<2B+2M$. By a similar argument, we can show that $z_n(t)>-2B-2M$, concluding the proof of the claim.

*Claim* 2. For each $j=1,2,3,\ldots$, the sequence $\{z_n\}_{n\ge j}$ is equicontinuous on $[0,t_j]$.

Fix $t,t'\in[0,t_j]$ with $t<t'$. Then

$$\begin{aligned}|z_n(t)-z_n(t')| &\le \left|G\left(\varphi^{-1}(z_n(t))\right)-G\left(\varphi^{-1}(z_n(t'))\right)\right|\\ &\quad + \beta\gamma\int_t^{t'}\left|\varphi^{-1}(z_n)\right|d\tau + \left|\,[k_D\alpha-k_P][y(t')-y(t)]\,\right|.\end{aligned}$$

Since $y_d,\dot{y}_d,x\in C[0,t_n]$ for each $n$, then $G(y_d,\varphi^{-1}(z_n))\in C[0,t_n]$. Also, from the hypotheses of the lemma, $y\in C[0,t_n]$ for all $n$, and from Claim 1, $|\varphi^{-1}(z_n)|<(2B+2M)/\widehat{k}_P$ . Then we have $|z_n(t)-z_n(t')|\to 0$ as $t\to t'$ in $[0,t_j]$, and hence, for each $j=1,2,3,\ldots$, the sequence $\{z_j\}_{n\ge j}$ is equicontinuous on $[0,t_j]$.

From the Arzela–Ascoli theorem, there exists a subsequence $N_1\subset N^+$ and a function $\widehat{z}_1\in C[0,t_1]$ such that $z_n\to\widehat{z}_1$ in $C[0,t_1]$ as $n\to\infty$ in $N_1$. Define $N_1^*=N_1\setminus\{1\}$. A similar argument yields a subsequence $N_2$ of $N_1^*$ and a function $\widehat{z}_2\in C[0,t_2]$ such that $z_n\to\widehat{z}_2$ in $C[0,t_2]$ as $n\to\infty$ in $N_2$, where $N_2\subseteq N_1$. Hence, by induction, we can obtain sequences $\{N_k\}$ and $\{\widehat{z}_k\}$ such that $N_1\supseteq N_2\supseteq\cdots N_k\supseteq\cdots$ and $z_n\to\widehat{z}_k$ uniformly on $[0,t_k]$ as $n\to\infty$ in $N_k$, where $\widehat{z}_k\in C[0,t_k]$. Further, $\widehat{z}_k=\widehat{z}_j$ on $[0,t_j]$ for $j\le k$. To show that Preisach memory curves $\{\widehat{\psi}_k(t)\}$ satisfy $\widehat{\psi}_k(t)=\widehat{\psi}_j(t)$ for all $t\in[0,t_j]$, consider two memory curves $\psi_{M1}(t)$ and $\psi_{M2}(t)$ such that $M1,M2\in N_j$:

$$\|\psi_{M1}-\psi_{M2}\|_\infty \le \|(\mathcal{I}+\Gamma)^{-1}f(\varphi^{-1}(z_{M1}))-(\mathcal{I}+\Gamma)^{-1}f(\varphi^{-1}(z_{M2}))\|_\infty.$$

From the definition of $f(x)$,

$$\|f(\varphi^{-1}(z_{M1}))-f(\varphi^{-1}(z_{M2}))\|_\infty \le \gamma\max_{t\in[0,t_j]}\int_0^t|\varphi^{-1}(z_{M1})-\varphi^{-1}(z_{M2})|d\tau.$$

Since $z_n$ is Cauchy for $n\in N_i$ and $\varphi^{-1}$ is continuous, for any $\varepsilon>0$, there is $\widehat{N}$ such that for all $M1,M2>\widehat{N}$, we have $\|\psi_{M1}-\psi_{M2}\|_\infty<\varepsilon$. Hence, the sequence $\psi_n(t)$ is Cauchy for $n\in N_j$ and it converges to a function $\widehat{\psi}_j(t)$ for $t\in[0,t_j]$. This argument is true for all $j\le k$ and, since $N_k\subseteq N_j$, $\widehat{\psi}_k(t)=\widehat{\psi}_j(t)$ for all $t\in[0,t_j]$. Next we show that $\widehat{z}_k$ satisfies (6.1). Since $\varphi^{-1}$ is bounded, from the dominant convergence theorem,

$$\begin{aligned}\lim_{n\to\infty}f(\varphi^{-1}(z_n)) &= \lim_{n\to\infty}\left\{H_0+\Gamma_0+\gamma\int_0^t\varphi^{-1}(z_n)d\tau\right\},\\ f(\varphi^{-1}(\widehat{z}_k)) &= H_0+\Gamma_0+\gamma\int_0^t\varphi^{-1}(\widehat{z}_k)d\tau\end{aligned}$$

for $n \in N_k$. Since the initial memory curves are the same, from Lemma 4.2, $W[f(\varphi^{-1}(z_n))]$ converges to $W[f(\varphi^{-1}(\widehat{z}_k))]$. Therefore, if $n \in N_k$, then

$$\lim_{n\to\infty} G\left(y_d, \varphi^{-1}(z_n)\right) = G\left(y_d \varphi^{-1}(\widehat{z}_k)\right).$$

From (6.1),

$$\widehat{z}_k = \lim_{n\to\infty} z_n = \lim_{n\to\infty}\left\{G\left(y_d, \varphi^{-1}(z_n)\right) - \beta\gamma\int_0^t \varphi^{-1}(z_n)d\tau + (k_D\alpha - k_P)y\right\}$$

$$= G\left(y_d, \varphi^{-1}(\widehat{z}_k)\right) - \beta\gamma\int_0^t \varphi^{-1}(\widehat{z}_k)d\tau + (k_D\alpha - k_P)y \tag{6.2}$$

for all $t \in [0, t_k]$. Here, we have constructed a sequence such that each function $\widehat{z}_k$ satisfies $\widehat{z}_k|_{t\in[0,t_{k-1}]} = \widehat{z}_{k-1}|_{t\in[0,t_{k-1}]}$. Define a function $\widehat{z} : [0,\infty) \to \mathbb{R}$ by $\widehat{z}(t) = \widehat{z}_k(t)$ on $[0, t_k]$ for all $k$. Taking the limit of (6.2) as $k \to \infty$,

$$\widehat{z} = G\left(y_d, \varphi^{-1}(\widehat{z})\right) - \beta\gamma\int_0^t \varphi^{-1}(\widehat{z})d\tau + (k_D\alpha - k_P)y, \quad t \in [0,\infty). \tag{6.3}$$

As $|\widehat{z}|$ is bounded by $2B + 2M$, $\widehat{z} \in BC[0,\infty)$. Hence, $x = \varphi^{-1}(\widehat{z}) \in BC[0,\infty)$ is the global solution of (5.4). ∎

Next we establish the existence of a unique solution for the hysteretic system given by (5.3)–(5.4).

THEOREM 6.1 (uniqueness on bounded interval). *Suppose hypotheses $\mathcal{H}1$–$\mathcal{H}4$ hold and $y_d \in W^{1,\infty}[0,T]$. Then the system* (5.3)–(5.4) *has a unique solution* $(H, M, y) \in W^{1,\infty}[0,T] \times W^{1,\infty}[0,T] \times W^{1,\infty}[0,T]$.

*Proof.* With $y_1 = H + M$, $y_2 = y$, $\varphi(x) = \widehat{k}_P\, x + \theta(x)$, $V = (I + \Gamma)^{-1}[\cdot;\psi_{-1}]$, and $g(x) = ax^2 + bx$, the combined system is

$$\dot{y_1} = \gamma\,\varphi^{-1}\left(-\beta V(y_1) - (k_P - k_D\alpha)y_2 - g((\Gamma\circ V)(y_1)) + k_P y_D + k_D\dot{y}_d\right),$$
$$\dot{y_2} = -\alpha y_2 + g((\Gamma\circ V)(y_1)).$$

As $\varphi^{-1}$, $V$, $g$, and $\Gamma$ are Lipschitz continuous from $W^{1,1}[0,T]$ to $W^{1,1}[0,T]$, the theorem follows from the same argument as in Theorem 3.1.1 in [43]. ∎

Note that the above solution does not guarantee that $x(t) = \dot{B}(t)/\gamma$ is continuous. However, we have already shown the existence of $x(t) \in BC[0,\infty)$. Hence, the system has a unique solution $(x, y) \in BC[0,T] \times W^{1,\infty}[0,T]$. Next we establish the existence of a unique bounded global solution.

THEOREM 6.2. *Suppose $\alpha > 0$ and hypotheses $\mathcal{H}1$–$\mathcal{H}4$ hold. Then system* (5.3)–(5.4) *possesses a unique global solution $(x, y) \in BC[0,\infty) \times W^{1,\infty}[0,\infty)$. Furthermore $\|y\|_\infty + \|\dot{y}\|_\infty \le K$ for some $K < \infty$ independent of gain parameters $k_P$, $k_D$, and $\widehat{k}_P$.*

*Proof.* We first prove that $y$ is bounded. From (5.3), $\dot{y} + \alpha y = g(W[f(x)])$, and hence

$$y = e^{-\alpha t}\left[\int_0^t e^{\alpha\tau} g(W[f(x)])(\tau)d\tau + y_0\right].$$

Since $W$ is saturating, $|y(t)| \leq e^{-\alpha t} \int_0^t a_3 e^{\alpha\tau} d\tau + e^{-\alpha t}|y_0|$ for some constant $a_3$. Hence, $\|y\|_\infty \leq a_3/\alpha + |y_0|$. From ODE theory, we can extend the solution given in Theorem 6.1 to $t \in [0, \infty)$. Since $y \in W^{1,\infty}[0,\infty)$, $x$ also has a global solution in $BC[0,\infty)$. Furthermore, $\|\dot{y}\|_\infty \leq a_3 + \alpha|y_0| + a\max[\Gamma_{sat1}, \Gamma_{sat2}]^2 + b\max[\Gamma_{sat1}, \Gamma_{sat2}]$, which completes the proof. ∎

*Remark* 6.1. Any continuous bounded input disturbance $\Delta(t)$ can be combined with $G(y_d, x)$ and Lemma 6.2 will still hold. Further, if hypotheses $\mathcal{H}1$–$\mathcal{H}4$ hold, then all the subsequent theorems still follow, and hence the system possesses a unique global solution $(x, y) \in BC[0,\infty) \times W^{1,\infty}[0,\infty)$.

## 7. Tracking.

The next objective is to show that the control signal given by (5.1) achieves output tracking. In this section, we derive the necessary conditions on gain parameters $k_D$, $k_P$, and $\widehat{k}_P$ such that for a given desired output $y_d \in W^{2,\infty}[0,\infty)$, the output error can be made sufficiently small. That is, for a given $\varepsilon > 0$, $\limsup_{t\to\infty} |y - y_d| \leq \varepsilon$. Since function $g(M(t)) = aM^2(t) + bM(t)$ represents a squaring function, two values of $M(t)$ can achieve the same value of $g(M(t))$ and hence the same output $y$. However, to achieve tracking, the PD controller must operate in a monotone region. Lemma 7.1 establishes that appropriate choices for the gain parameters force the controller to maintain $M(t) \in [-b/2a, \Gamma_{sat2}]$.

Lemma 7.1. *Suppose hypotheses $\mathcal{H}1$–$\mathcal{H}4$ hold. Consider the closed-loop system given by*

$$\theta(x)(t) + \beta H(t) = -\widehat{k}_P x(t) - k_P r(t) - k_D \dot{r}(t), \tag{7.1}$$

$$\dot{r}(t) + \alpha r(t) = aM^2(t) + bM(t) - (\dot{y}_d + \alpha y_d)(t), \tag{7.2}$$

$$\dot{M}(t) + \dot{H}(t) = \gamma x(t), \tag{7.3}$$

$$M(t) = \Gamma[H\,; \psi_{-1}](t). \tag{7.4}$$

*Let $\pi = [-b/2a, \Gamma_{sat2}]$. Then there exist $k_{P0}$ and $k_0$ such that if $M(t_0) \in Interior(\pi)$, $k_p \geq k_{P0}$, and $0 \leq \left(k_D - \frac{k_P}{\alpha}\right) \leq k_0$, then $M(t) \in \pi$ for all $t \in [t_0, \infty)$.*

*Proof.* We may assume $b/2a < \Gamma_{sat1}$. (Otherwise $[-\Gamma_{sat1}, \Gamma_{sat2}] \subset [-b/2a, \Gamma_{sat2}]$ and $M(t) \in \pi$, as desired.) Then, from hypothesis $\mathcal{H}2$, it follows that $|u(t)| < d_1$ for some constant $d_1$ that is independent of gain parameters. From Theorem 6.2, system (7.1)–(7.4) has a unique solution for $x \in BC[t_0,\infty)$ and hence for $\dot{M} \in BC[t_0,\infty)$. Consequently, we only need to show that if $M(t) = -b/2a$, then we can find gain parameters such that $\dot{M}(t) > 0$. Note that if $\dot{M}(t)$ does not exist when $M(t) = -b/2a$, then $M(t)$ is increasing as desired. From Theorem 6.2 we also have $|\dot{r}(t)| < d_2$ for some constant $d_2$ that is independent of gain parameters. When $M(t) = -b/2a$, $\dot{r}(t) + \alpha r(t) = -b^2/4a - (\dot{y}_d + \alpha y_d)(t)$. From hypothesis $\mathcal{H}4$, $(\dot{y}_d + \alpha y_d)(t) > -b^2/4a + \delta$ and hence $\dot{r}(t) + \alpha r(t) < -\delta$. From (7.1),

$$\begin{aligned} x(t) + \frac{\theta(x)(t)}{\widehat{k}_P} &= -\frac{\beta H(t)}{\widehat{k}_P} - \frac{k_P}{\alpha \widehat{k}_P}(\dot{r} + \alpha r)(t) - \left(k_D - \frac{k_P}{\alpha}\right)\frac{\dot{r}(t)}{\widehat{k}_P} \\ &> -\frac{\beta d_1}{\widehat{k}_P} + \frac{k_P \delta}{\alpha \widehat{k}_P} - \frac{k_0 d_2}{\widehat{k}_P}. \end{aligned}$$

By setting $k_{P0} = \frac{\alpha(\beta d_1 + k_0 d_2)}{\delta}$, the quantity $x(t) + \frac{\theta(x)(t)}{\widehat{k}_P} > 0$. Function $\theta(x)$ is a continuous monotone function of $x$ and $\theta(0) = 0$. Subsequently, at $M(t) = -b/2a$, $x(t)$ is positive. Now from hypothesis $\mathcal{H}1$ combined with (7.3), $\dot{M}(t)$ is also positive. ∎

For the remaining analysis, we use the counterclockwise dissipativity of the hysteresis operator. Since the circulation of the hysteresis loops is counterclockwise, $\int_0^t H\dot{M}\,dt \geq 0$ for all $t \geq 0$. Then from Lemma 7.1, we have $\int_{t_0}^t (2aM(\tau) + b) H(\tau)\frac{dM}{d\tau}\,d\tau \geq 0$. Using this counterclockwise dissipativity property, we will first prove that tracking is possible for a constant tracking signal.

THEOREM 7.1. *Suppose $y_d$ is a constant and the conditions in Lemma 7.1 are satisfied. Then, for any $\varepsilon > 0$, there exists $\bar{k}_{P0} \geq k_{P0}$ such that if $k_P > \bar{k}_{P0}$ and $\alpha k_D - k_P > 0$, then $\limsup_{t\to\infty} |r(t)| \leq \varepsilon$, where $r(t) = y(t) - y_d$.*

*Proof.* From (7.1),

$$\theta(x)(t) + \beta H(t) + \widehat{k}_P x(t) = -\frac{k_P}{\alpha}\left(\dot{r} + \alpha r\right)(t) - \left(k_D - \frac{k_P}{\alpha}\right)\dot{r}(t), \quad \text{which implies}$$

$$\begin{aligned}&\frac{k_P}{\alpha}\left(\dot{r} + \alpha r\right)\left(\ddot{r} + \alpha \dot{r}\right)(t) + \left(k_D - \frac{k_P}{\alpha}\right)\ddot{r}(t)\dot{r}(t)\\ &\quad = \left(-\theta(x) - \beta H - \widehat{k}_P x\right)\left(\ddot{r} + \alpha\dot{r}\right)(t) - \left(\alpha k_D - k_P\right)\dot{r}^2(t).\end{aligned} \tag{7.5}$$

Consider the function

$$V(t) = \frac{k_P}{2\alpha}\left(\dot{r} + \alpha r\right)^2(t) + \left(k_D - \frac{k_P}{\alpha}\right)\frac{\dot{r}^2(t)}{2} + \beta\int_0^t (2aM + b)H\frac{dM}{d\tau}d\tau.$$

Then $V(t) \geq 0$. Further, from (7.5),

$$\begin{aligned}\dot{V}(t) &= \left(-\theta(x) - \beta H - \widehat{k}_P x\right)\left(\ddot{r} + \alpha\dot{r}\right)(t)\\ &\quad - \left(\alpha k_D - k_P\right)\dot{r}^2(t) + \beta\left(2aM(t) + b\right)H(t)\dot{M}(t).\end{aligned} \tag{7.6}$$

From (7.2) with constant $y_d$, $\left(\ddot{r} + \alpha\dot{r}\right)(t) = 2aM(t)\dot{M}(t) + b\dot{M}(t)$. Hence, $\left(\ddot{r} + \alpha\dot{r}\right)(t)\beta H(t) = \left(2aM(t) + b\right)\beta H(t)\dot{M}(t)$ and $\left(\ddot{r} + \alpha\dot{r}\right)(t)\big(\theta(x)(t) + \widehat{k}_P x(t)\big) = \left(2aM(t) + b\right)\dot{M}(t)\big(\theta(x)(t) + \widehat{k}_P x(t)\big)$. Then from (7.6)

$$\dot{V}(t) = -\left(2aM(t) + b\right)\dot{M}(t)\left(\theta(x) + \widehat{k}_P x\right)(t) - \left(\alpha k_D - k_P\right)\dot{r}^2(t).$$

From Lemma 7.1, $2aM(t) + b > 0$ and, since $\dot{M}(t)$, $x(t)$ and $\theta(x)(t)$ have the same sign, $\dot{V}(t) \leq 0$ almost everywhere. Thus $\dot{V}(t) \to 0$ as $t \to \infty$.

The invariant set for $V$ is the set $(x, \dot{r}) = (0, 0)$. The largest invariant subset of this set for the system (7.1)–(7.4) is given by $\beta H = -k_P r$. Hence, $|r| \leq \frac{\beta|H|}{k_P}$, which shows that for any $\varepsilon > 0$ we may choose $k_P > \frac{\beta(2B+2M)}{\varepsilon}$ and the largest invariant set for $\dot{V}$ will have the property $|r| < \varepsilon$. ∎

In light of Theorem 7.1, $k_P$ must be sufficiently large. Also, to obtain an appropriate value for $k_D$, one must first obtain an estimate for $\alpha$, but discussion of this is beyond the scope of this paper. Next we discuss tracking for any desired output. To simplify the discussion, we only consider the case $\alpha k_D = k_P$ and show that tracking is possible even with this restriction.

THEOREM 7.2. *Suppose hypotheses $\mathcal{H}1$–$\mathcal{H}5$ hold. Further suppose that $\frac{b}{2a} < \Gamma_{sat1}$ and hysteresis operator $\Gamma[H,;\psi_{-1}]$ satisfies $\mathcal{X}_I(x) \geq cx$ for all $x > 0$. Let $M(0) \in \pi$. Then, for a given $\varepsilon > 0$, there exists $\bar{k}_{P0} \geq k_{P0}$ such that if $k_P > \bar{k}_{P0}$, $\alpha k_D = k_P$, and $\widehat{k}_P > 0$, then $\limsup_{t\to\infty} |r(t)| \leq \varepsilon$.*

*Proof.* Note that the conditions for Lemma 7.1 are satisfied, and hence $M(t) \in \pi$. We will utilize the fact that the functions given in (7.1)–(7.4) are continuous.

Suppose at a given $t \in [0, \infty)$, $\dot{r}(t) + \alpha r(t) > \varepsilon$. Then from (7.1) with $\alpha k_D = k_P$, $\varphi(x)(t) + \beta H(t) < -\frac{k_P \varepsilon}{\alpha}$, where $\varphi(x)(t) = \theta(x)(t) + \widehat{k}_P x(t)$. Then $\varphi(x)(t) < -\frac{k_P \varepsilon}{\alpha} - \beta H_*$, where $H_* = \min\{H \mid M(t) \in \pi\}$. $H_*$ exists because $M \geq -\frac{b}{2a} > -\Gamma_{sat1}$ when $M \in \pi$. As $\Gamma$ is strictly increasing, $H$ has a minimum as well when $M \in \pi$. Therefore, $\lim_{k_P \to \infty} \varphi(x)(t) = -\infty$. Since $\theta(x)$ is a continuous monotone function of $x$ with $\theta(0) = 0$ and $\widehat{k}_P > 0$, $\lim_{k_P \to \infty} x(t) = -\infty$. Then for a given $\varepsilon > 0$ and $\delta_1 > 0$, there exists $\bar{k}_{P0} > k_{P0}$ such that if $k_P > \bar{k}_{P0}$, then $x < -\delta_1$. Without loss of generality, assume $M(t) < \Gamma_{sat2}$. (Otherwise $\dot{M}(t) = 0$ and, from hypothesis $\mathcal{H}5$, $\dot{r}(t) + \alpha r(t) > \delta$ and $\dot{H}(t) = \gamma x(t) - \dot{M}(t) < -\gamma \delta_1$. Then there exists $t^* > t$ such that $M(t^*) < \Gamma_{sat2}$.) If $C$ is the Lipschitz constant of $\Gamma$, define $c = \frac{1}{C}$. From hypothesis $\mathcal{H}1$, $-\delta_1 > x = \frac{1}{\gamma}[\dot{H} + \dot{M}] \geq \frac{1}{\gamma}[c\dot{M} + \dot{M}]$ for every $t$ where $\dot{M}(t)$ exists ($\dot{M}(t)$ exists for almost every $t$). Thus, for such $t$ where $\dot{M}$ exists, $\dot{M}(t) < -\frac{\gamma \delta_1}{1+c} = \delta_2$. Further, from (7.2), $aM^2(t) + bM(t) - (\dot{y}_d + \alpha y_d)(t) > \varepsilon$. From hypothesis $\mathcal{H}5$, $\dot{y}_d(t) + \alpha y_d(t) > -b^2/4a$. But then $aM^2(t) + bM(t) + b^2/4a > \varepsilon$, that is, $\left(M(t) + \frac{b}{2a}\right)^2 > \frac{\varepsilon}{a}$. Since $M(t) \in \pi$, $M(t) > -b/2a + \sqrt{\varepsilon/a}$, which implies $2aM(t) + b > \sqrt{4a\varepsilon}$. Differentiating (7.2), we get $\ddot{r}(t) + \alpha \dot{r}(t) = (2aM(t) + b)\dot{M}(t) - (\ddot{y}_d + \alpha \dot{y}_d)(t)$. Using the bounds for $2aM(t) + b$, $\dot{M}(t)$, and $\ddot{y}_d(t) + \alpha \dot{y}_d(t)$ (from $\mathcal{H}5$), we get $\ddot{r}(t) + \alpha \dot{r}(t) < -\delta_2 \sqrt{4a\varepsilon} + A$. By picking $\bar{k}_{P0}$ sufficiently large, we can make $\delta_2$ sufficiently large and hence $\ddot{r}(t) + \alpha \dot{r}(t) < 0$. A similar argument can be used to show that there exists $k_P$ sufficiently large such that if $\dot{r}(t) + \alpha r(t) < -\varepsilon$, then $\ddot{r}(t) + \alpha \dot{r}(t) > 0$. Now if $(\dot{r}(t) + \alpha r(t))^2 > \varepsilon^2$, then $(\dot{r}(t) + \alpha r(t))(\ddot{r}(t) + \alpha \dot{r}(t)) < 0$. Therefore, $\limsup_{t \to \infty} |\dot{r}(t) + \alpha r(t)| \leq \varepsilon$. To complete the proof, since $\alpha > 0$, if $\limsup_{t \to \infty} |\dot{r}(t) + \alpha r(t)| \leq \varepsilon$, then $\limsup_{t \to \infty} |r(t)| \leq \varepsilon/\alpha$. ∎

Throughout the proofs, we only require $\widehat{k}_p > 0$. In a forthcoming paper we will discuss the explicit role of $\widehat{k}_p$ in the presence of exogenous disturbances.

**8. Examples.** We present simulation results for a hysteretric system derived from a magnetostrictive actuator. The corresponding hysteretic system is given by [22]

$$\theta(x) + \beta H = i, \tag{8.1}$$

$$\dot{H} + \dot{M} = \gamma x, \tag{8.2}$$

$$M(\cdot) = \Gamma[H(\cdot); \psi_0], \tag{8.3}$$

$$\dot{y} + \alpha y = aM^2 + bM - c, \tag{8.4}$$

$$H(0) = H_0, \quad M(0) = M_0, \tag{8.5}$$

where $i$ is the input current and $x$ is the induced voltage. $H$ and $M$ represent the average magnetic field and magnetization in the magnetostrictive actuator rod, respectively. $H_0$ and $M_0$ are corresponding initial conditions. A classical Preisach operator is employed to represent rate-independent hysteresis, where $\Gamma[\cdot\,; \psi_0]$ is the hysteresis operator and $\psi_0$ is the initial memory curve. An approximation of the function $\theta(x)$, $\theta_m(x)$, is given by [47]: $\theta_m(x) = \frac{R}{R_{classical}} x + R\,\text{sign}(x(t)) \sum_{i=1}^{N} \frac{|x(t)|^{\nu_i}}{C_i}$.

For simulation purposes, we use data from an AA-050H series Terfenol-D actuator manufactured by Etrema. The maximum displacement of the actuator is $\pm 25\mu$m. We consider a system with $\alpha = 10000$, $a = 7.5 \times 10^{-13}$, $b = 0$, and $c = 2.2 \times 10^{-5}$, so that the system is capable of producing the given maximum displacement. The following parameters are available from the manufacturer: number of turns of the winding, $N = 1300$; cross-section area of the actuator rod, $A = 2.83 \times 10^{-5}\text{m}^2$; magnetic field due to the permanent magnet, $H_0 = 1.23 \times 10^4\text{A/m}$; and $\beta = 1.54 \times 10^4$. From these data we can find $\gamma$ using $\gamma = \mu_0 NA$. We also have $\theta_m(x) = 0.0076e + 0.0005x^{5/6} + 0.0011x^{2/3} + 0.0068x^{0.5}$ [22]. We use the Preisach representation given in Figure 8.1 for hysteresis. Data is obtained from [49].

*Example* 1. Tracking signal $y_d$ is generated using the sum of three sinusoidal waveforms: $8 \times 10^{-6} \sin(1200\pi t + \pi/3)$, $5 \times 10^{-6} \sin(375\pi t + \pi/8)$, and $6 \times 10^{-6} \sin(100\pi t)$. Thus, tracking is performed for a signal with frequency components 50Hz, 187.5Hz, and 600 Hz. Also, $y_d \in W^{2,\infty}[0, \infty)$ and satisfies all the conditions given in $\mathcal{H}5$.

The corresponding gain parameters are chosen according to the conditions given in Theorem 7.1. We chose $k_P$ sufficiently large and $\widehat{k}_P$ positive: $k_P = 6 \times 10^5$, $k_D = 610$, and $\widehat{k}_P = 0.2$. Note that $k_D - k_P/\alpha = 10$ satisfies the conditions in the theorem. The simulation results are given in Figure 8.2, where the maximum relative error does not exceed 7.5%.

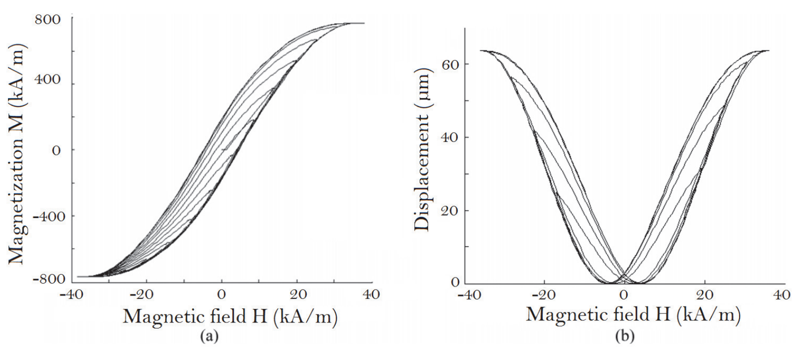


Fig. 8.1. (a) *Magnetic field versus magnetization curves for the magnetostrictive actuator.* (b) *Magnetic field versus displacement curves for the magnetostrictive actuator.*

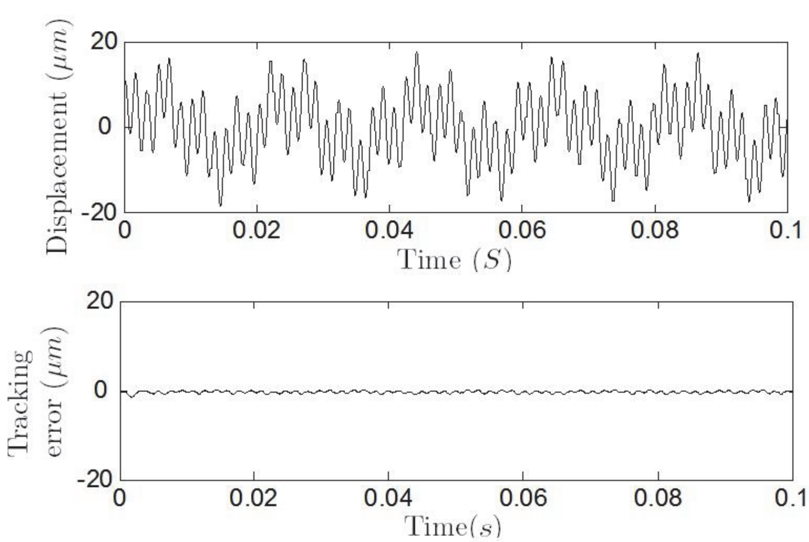


Fig. 8.2. *Displacement and tracking error for Example* 1.

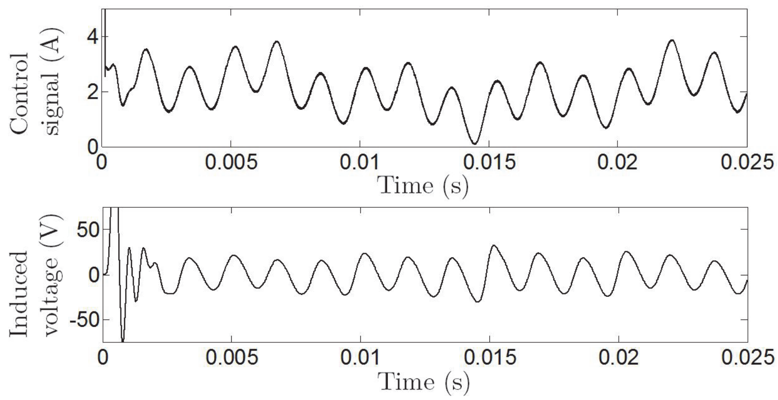


FIG. 8.3. *Control signal and the induced voltage for Example* 1.

Figure 8.3 shows the control signal and the induced voltage for $t \in [0, 0.025]$. The figure demonstrates the continuity of both signals.

*Example* 2. For the next simulations, the tracking signal (Figure 8.4) is generated using the sum of two sinusoidal waveforms, $5 \times 10^{-6} \sin(1200\pi t + \pi/3)$ and $5 \times 10^{-6} \sin(375\pi t + \pi/8)$, and a triangle waveform $\frac{2\times 10^{-5}}{\pi} \sin^{-1}(\sin 800\pi t)$. Notice that $y_d \in W^{1,\infty}[0, \infty)$ and does not satisfy hypothesis $\mathcal{H}5$. In this case, Theorem 6.1 still holds; however, Lemma 6.4 does not hold.

Figure 8.5 shows the decomposed control signal for the two feedback loops. One can easily observe that both the control signal and the induced voltage are discontinuous at each peak of the triangle wave.

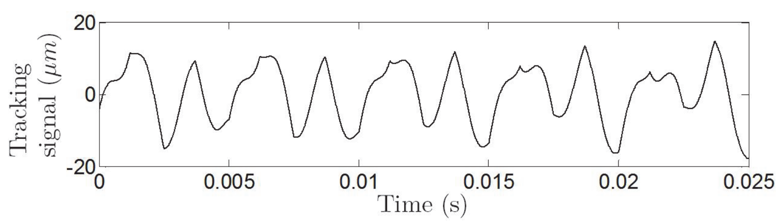


FIG. 8.4. *Tracking signal for Example* 2.

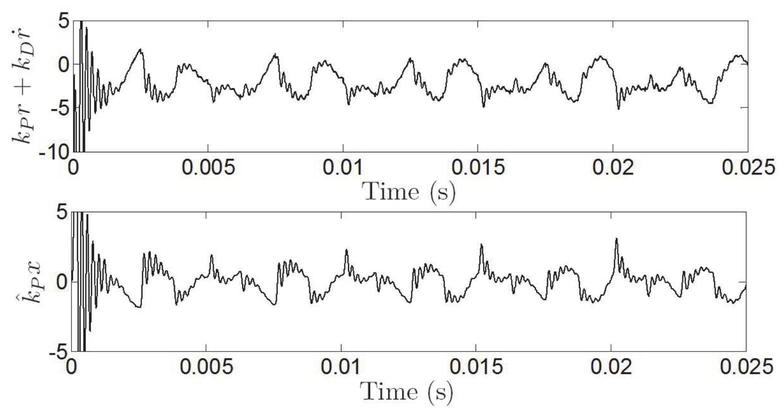


FIG. 8.5. *Decomposed control signal for the two feedback loops for Example* 2.

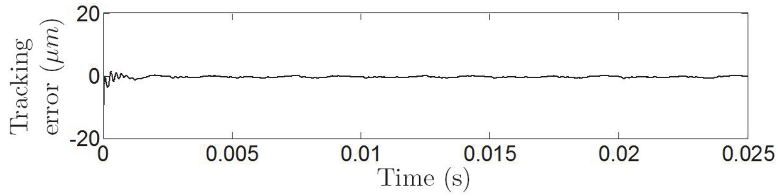


FIG. 8.6. *Tracking error for Example* 2.

It can also be observed that tracking still holds (Figure 8.6). However, the reasons for such observations are beyond the scope of this paper.

## 9. Conclusion.

In this paper, we discussed a controller strategy for hysteretic systems associated with magnetic and smart actuators. The controller is a PD controller derived using two feedback signals. We proved regularity, well-posedness, and stability of the controller and then found conditions on the gain parameters for ultimate bounded tracking for a varying tracking signal. The careful mathematical analysis in this paper establishes tracking for disturbance-free hysteretic systems, which simulations demonstrate. We do not attempt to identify whether tracking fails outside the given gain constraints; rather we demonstrate that within the given gain constraints, the system achieves tracking.

Next it is necessary to extend the results to achieve tracking in the presence of exogenous disturbances, to control the system within the rated conditions, and to relax the conditions given in hypothesis $\mathcal{H}5$ and Theorem 7.2 (since, for some $x > 0$, Priesach operators with bounded density functions cannot achieve the condition $\mathcal{X}_I(x) \geq cx$ [43]). These topics will be discussed in a forthcoming paper.

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