Stabilization of a system with saturating, non-monotone hysteresis and frequency dependent power losses by a PD controller

Stabilization of a system with saturating, non-monotone hysteresis and frequency dependent power losses by a PD controller

D B Ekanayake[1] **and R V Iyer**[2]

[1] Mathematics Department, Western Illinois University, Macomb, IL 61455, USA. [2] Dept. of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409, USA.

E-mail: `DB-Ekanayake@wiu.edu`, `ram.iyer@ttu.edu`

Abstract. We prove the closed loop stability of a PD controller for certain systems with saturating, non-monotone hysteresis and frequency dependent power losses. Most controllers use inverse compensators to cancel out actuator hysteresis nonlinearity. We show that we can achieve stability of the closed-loop system without an explicit inverse computation (using least squares minimization or otherwise).

1. Introduction

Magnetic and smart actuators are smaller and more energy efficient than conventional actuators, and are useful in number of precision control applications [1, 2, 3, 4, 5]. However, they are difficult to control to achieve a given objective due to the occurrence of hysteresis loops, memory effects, and saturation, as well as rate dependent power losses. Motivated by such hysteretic systems, in this paper, we discuss how to develop a proportional derivative (PD) control for saturating, non–monotone hysteretic systems with nonlinear power losses.

The controller strategy discussed in this paper is derived for tracking control, that is, for forcing the system output to follow a specific trajectory. However, we only discuss the stabilization problem in this paper. The system has two measured feedbacks that can be used for control: the error output, which we want to minimize, and another measured output. In the case of magnetic and smart actuators, the other output is the induced voltage (electro-motive force) of the actuator windings. Induced voltage of the actuator windings can be obtained from the input current and voltage (shown in [6, 7]), and the error output is derived from the actuator tip displacement. The use of two feedbacks makes the control design entirely different from those used in the literature for control of hysteretic systems.

Tracking control of hysteretic systems has received wide attention in recent years. Tracking control methods presented in the literature include inverse compensation of hysteresis [9, 10, 11], adaptive control [12, 13, 14], integral control [15, 16, 17], passivity–based control [18, 19], monotonicity-based control [20, 21], and hybrid control, most of which are model–dependent [22, 23].

The most common control method discussed in literature is inverse hysteresis compensation, where an approximate inverse of the hysteresis is employed to supress the hysteresis nonlinearity in the system [8, 9, 11, 24, 25]. These controllers are model–based and applicable for hysteretic

systems at low frequencies [9]. Another widely investigated strategy is integral, proportional, and derivative (PID) control [15, 17], derived from the output error. These controllers have received much attention because of the simplicity and availability of PID modules. A key condition that aids in the development of these controllers is that hysteresis is piecewise monotonic. In literature, these controllers are employed for systems in which hysteresis is the only nonlinearity, or the PID controllers are combined with model–based controllers [22, 23]. Adaptive control is another common method for controlling hysteretic systems. Some adaptive controllers are based on inverse hysteresis compensation [12, 26], while others are robust adaptive control schemes without the hysteresis inverse [14, 27]. They are primarily utilized for non–saturating hysteresis operators without minor–loop closure behavior.

1.1. Novelty of the proposed stabilizing controller

None of the aforementioned control schemes are suitable for non-monotone hysteretic systems showing minor-loop closure and saturation. Magnetic and magnetostrictive actuators exhibit such complex hysteresis with minor-loop closure, as well as other nonlinearities such as eddy current and residual losses, which increase with the operating frequency. These actuators become saturated and some actuators relate the output of the hysteresis to the system output via a square map, such that the system is not monotone. These behaviors limit the usefulness of conventional control schemes. Due to the limitations of both the models and the control schemes, tracking control is limited to low frequency ranges (less than 200 Hz [9]) and low amplitude signals, which prevents the actuators from becoming saturated. Hence, to control these systems in their full frequency and amplitude ranges, a new scheme is required to achieve tracking control.

Existing control schemes for hysteretic systems use only the error output, to be minimized, to derive the control signal. For example, existing control schemes for magnetostrictive actuators use only the actuator tip displacement measurements as input for the control scheme [24]. However, when a current is applied to a magnetostrictive actuator, measurements can be taken for the voltage induced by the actuator, as well as for the displacement of the tip. To achieve precision control of these complex systems, it is important to include all measurements that can easily be obtained and that reflect properties of the hysteretic system. Hence, we use two feedbacks for control. In the following analysis, we show that, given any $\varepsilon > 0$, the output $y(t)$ is ultimately bounded by ε (that is, $\limsup_{t\to\infty} |y(t)| \leq \varepsilon$), in the absence of disturbance.

2. System with Hysteresis

In this paper, we consider hysteretic systems modelled for almost every $t \in [0, \infty)$ by nonlinear functional differential equations of the form [28]

$$\dot{y}(t) + \alpha y(t) \;=\; aM^2(t) + bM(t) + \Delta(t) \tag{1}$$

$$\frac{d}{dt}[u + M](t) \;=\; \gamma x(t) \tag{2}$$

$$\theta(x)(t) + \beta u(t) \;=\; \widehat{u}(t) \tag{3}$$

$$M(t) \;=\; \Gamma[u \,;\, \psi_{-1}](t) \tag{4}$$

$$y(0) = y_0, \qquad u(0) = u_0, \tag{5}$$

where $\widehat{u}(t)$ is the system input, $x(t)$ is an output and $y(t)$ is the output to be regulated. Functions $u(t)$ and $M(t)$ represent two internal states. The system of equations (1-5) may be used to model a magnetostrictive actuator with inertia ignored (set $m = 0$ in the model of [28]). In this case, $x(t)$ corresponds to the induced voltage, M represents the average magnetization along the axis, y corresponds to the displacement of the actuator tip, u corresponds to the axial average magnetic field, and $\widehat{u}$ corresponds to the applied voltage. Equation (2) is equivalent to Faraday's Law. Operator $\Gamma[\cdot \,;\, \psi_{-1}]$ is the hysteresis operator with initial memory state ψ_{-1}. The function

Δ represents a actuator disturbance signal, while θ is a nonlinear function which represents eddy current and excess losses for magnetic systems.

We assume hysteresis can be represented by an operator of Preisach type. In the literature, Presach operator has been shown to be a well–suited approximation for magnetic and smart actuators [29, 30, 31]. The following definition of Preisach operator may be found in Brokate and Sprekels [32]. Suppose Ψ_0 is the space of admissible memory curves:

$$\Psi_0 := \{\psi \mid \psi : \mathbb{R}_+ \to \mathbb{R}, \ |\psi(r) - \psi(\bar{r})| \leq |r - \bar{r}| \ \forall r, \bar{r} \geq 0, R_{supp}(\psi) < \infty\},$$

where $R_{supp}(\psi) := \sup\{r \mid r \geq 0, \psi(r) \neq 0\}$. The Preisach operator $\Gamma[u \, ; \psi_{-1}](t)$, with the initial memory curve ψ_{-1}, is defined with an output map $Q : \Psi_0 \to \mathbb{R}$ of the form:

$$Q(\psi) = \int_0^\infty q(r, \psi(r))dr + w_{00}, \quad \text{where} \quad q(r, s) = 2\int_0^s \omega(r, \sigma)d\sigma, \tag{6}$$

with $w_{00} = \int_0^\infty \int_{-\infty}^0 \omega(r, s)dsdr + \int_0^\infty \int_0^\infty \omega(r, s)dsdr$. Here, $\omega \in L_{loc}^1(\mathbb{R}_+ \times \mathbb{R})$ is the Preisach density function. For piecewise monotone functions u, the Preisach operator is defined to be: $\Gamma[u; \psi_{-1}(t) \overset{\triangle}{=} Q(\psi(t)(r))$, where $\psi(t)(r) = \mathcal{F}_r[u(t); \psi_{-1}(r)]$ and $\mathcal{F}_r$ is the Play operator with parameter r [32].

3. Statement of the problem

The objective is to design a controller such that all the internal signals u, x, y of the system (1-5) are bounded for all $t \geq 0$ even in the presence of the disturbance Δ. Furthermore, all signals must be uniquely specified. Finally, when the disturbance $\Delta = 0$, for arbitrary $\varepsilon > 0$, the controller must have the property that $|y(t)|$ is ultimately bounded by ε (that is, $\limsup_{t \to \infty} |y(t)| < \varepsilon$) by proper choice of parameters. We introduce the following assumptions.

$\mathcal{H}1$:   The Preisach operator $\Gamma[\cdot \, ; \psi_{-1}]$ is output-rate dissipative (counter-clockwise dissipative), piecewise monotone, and Lipschitz continuous on $C[0, T]$ and $W^{1,1}(0, T)$ where $T > 0$ is any positive real number.

$\mathcal{H}2$:   There exist real numbers $\Gamma_{sat1}, \Gamma_{sat2} > 0$ such that $-\Gamma_{sat1} = \inf_{\psi \in \Psi_0} Q(\psi)$ and $\Gamma_{sat2} = \sup_{\psi \in \Psi_0} Q(\psi)$.

$\mathcal{H}3$: $\theta(\cdot)$ is continuous and strictly monotone increasing; $\theta(0) = 0$.

$\mathcal{H}4$: $a \geq 0$, while b, α, β, and γ are positive. The disturbance signal satisfies $\Delta \in L^\infty[0, \infty)$. Furthermore, $\|\Delta\|_\infty \leq \eta$, and $\eta < \frac{b^2}{4a}$. The initial value parameter y_0 lies in a known bounded set.

Hypothesis $\mathcal{H}1$ is a typical condition which guarantees the thermodynamic consistency of hysteresis operators. In particular, it reflects the fundamental energy dissipation properties of hysteresis. Hypotheses $\mathcal{H}2$ states that the hysteresis operators in this study are allowed to saturate. The next hypothesis, $\mathcal{H}3$, describes the behavior of function θ. The monotonicity represents the increase of power losses as the magnitude of x increases. This hypothesis is the result of modeling frequency dependent excess losses in magnetic systems [6, 28].

4. Feedback Stabilization

The following result is an extension of Corollary 2.4.12 in Brokate and Sprekels [32]. It establishes the Lipschitz continuity of the operator $\Gamma \circ (I + \Gamma)^{-1} : W^{1,1}(0, T) \to W^{1,1}(0, T)$. It is useful to prove the existence and uniqueness of solutions to (1-5). If I is an interval in $\mathbb{R}$, $C_I[0, T] = \{u \in C[0, T] \mid u(t) \in I; \ t \in [0, T]\}$. We assume that for $\psi_{-1} \in \Psi_0$, if $u \in C_I[0, T]$, then $\Gamma[u; \psi_{-1}] \in C_J[0, T]$.

Lemma 4.1 *Suppose $\Gamma[u; \psi_1]$ is a Preisach operator generated by a nonnegative bounded density function $\omega(r,s)$ with compact support, where $\psi_{-1} \in \Psi_0$. Furthermore, suppose that $\frac{\partial}{\partial s}\omega(r,s)$ is measurable and bounded on bounded subsets of $\mathbb{R}_+ \times \mathbb{R}$. Then, for any $T > 0$, the following assertions hold:*

(a) $\Gamma[\cdot; \psi_{-1}]$ is Lipschitz continuous on $C[0,T]$ and on bounded subsets of $W^{1,1}(0,T)$.

(b) $(\mathcal{I} + \Gamma)[\cdot; \psi_{-1}] : C_I[0,T] \to C_J[0,T]$ is invertible. $(\mathcal{I} + \Gamma)^{-1}[\cdot; \psi_{-1}]$ is Lipschitz continuous on $C_J[0,T]$ and on bounded subsets of $W^{1,1}(0,T)$.

(c) $\Gamma \circ (\mathcal{I} + \Gamma)^{-1}[\cdot; \psi_{-1}]$ is Lipschitz continuous on $C_J[0,T]$ and on bounded subsets of $W^{1,1}(0,T)$.

The proof of the above result may be shown using the results proved in [32].

The stabilizing control signal can be derived either using a PI controller or a PD controller. In this paper, we restrict our attention only to the PD controller configuration. Control input $\widehat{u}(t)$ is derived from signals y and x:

$$\widehat{u}(t) = -\widehat{k}_P x - k_P y - k_D \dot{y}. \tag{7}$$

Next we obtain the closed loop system as follows. Equation (2) can be expressed in integral form as

$$u + \Gamma[u\,; \psi_{-1}] = f(x), \tag{8}$$

where, $f(x) := \gamma \int_0^t x d\tau + u_0 + \Gamma_0$. By the invertibility of $\mathcal{I} + \Gamma$, $u = (\mathcal{I} + \Gamma)^{-1} f(x)$. Hence $\Gamma[u\,; \psi_{-1}]$ can be rewritten in terms of x as $W[f(x)\,; \psi_{-1}] := \Gamma\left[(\mathcal{I} + \Gamma)^{-1} f(x)\,; \psi_{-1}\right]$. From Equation (8),

$$u = \gamma \int_0^t x d\tau - W[f(x)\,; \psi_{-1}] + u_0 + \Gamma_0. \tag{9}$$

Then, Equation (9) combined with system of equations (1-5) yields the closed loop system:

$$\dot{y} + \alpha y \;=\; g\left(W[f(x)\,; \psi_{-1}]\right) + \Delta(t) \tag{10}$$

$$\varphi(x(t)) + \beta\gamma \int_0^t x(\tau)d\tau \;=\; (k_D\alpha - k_P)y(t) + G(t_0; x)(t), \tag{11}$$

where

$$\varphi(x) \;=\; \widehat{k}_P x + \theta(x),$$
$$g(\cdot) \;=\; a(\cdot)^2 + b(\cdot), \qquad \text{and}$$
$$G(t_0; x)(t) \;=\; \beta W[f(x)\,; \psi_{-1}](t) - k_D g\left(W[f(x)\,; \psi_{-1}](t)\right) - \beta(u_0 + \Gamma_0) - k_D \Delta(t).$$

Due the semi-group property satisfied by Γ, we also have for any $0 \le t_1 \le t$,

$$\varphi(x) + \beta\gamma \int_{t_1}^t x(\tau)d\tau = (k_D\alpha - k_P)y + G(t_1; x)(t),$$
$$G(t_1; x)(t) = \beta W[f(x); \psi_{-1}](t) - k_D g\left(W[f(x); \psi_{-1}]\right) - \beta(u(t_1) + \Gamma[u; \psi_{-1}](t_1)) - k_D \Delta(t).$$

This observation is useful in Section 4.1.

Next we need to show that the above system possesses a solution such that $y \in W^{1,\infty}[0,\infty)$ satisfies the differential equation almost everywhere. Further, all the states must remain bounded for all $t \in [0,\infty)$. The next section establishes existence, uniqueness and stability of a solution to Equations (10)-(11). We then show that for the disturbance-free case $\Delta = 0$, given any $\varepsilon > 0$, we can make $|y(t)|$ ultimately bounded by ε by careful choice of parameters $\widehat{k}_P, k_P, k_D$.

4.1. Existence and uniqueness results

First, we determine the specific properties required of function $\varphi(x)$. Proof of the following lemma is a direct consequence of the inverse function theorem.

Lemma 4.2 *For each $x \in \mathbb{R}$, the function $\varphi := \widehat{k}_P x + \theta(x)$ is strictly monotone. Further, φ^{-1} is well–defined, strictly monotone, and Lipschitz continuous with Lipschitz constant $\dfrac{1}{\widehat{k}_P}$.*

This lemma shows that by selecting $\widehat{k}_P > 0$, we obtain the Lipschitz continuity of φ, which is important to prove existence and uniqueness of the closed-loop system.

Since function $g(M(t)) = aM^2(t) + bM(t)$ is non-monotone (Figure 1), two values of $M(t)$ achieve the same value of $g(M(t))$ and, hence, the same output $y(t)$. However, to avoid ambiguity in the control design, the value of $M(t)$ must be restricted to an interval such that a unique value of M corresponds to each value of $g(M)$. If $b = 0$, then $g(M)$ can only take on non-negative values and it can quickly be seen that (1) cannot be stabilized, that is $\limsup_{t \to \infty} |y(t)|$ cannot be made to be less than any given $\varepsilon > 0$. If $a = 0$, then the graph of $g(M)$ is monotone on $[-\Gamma_{sat1}, \Gamma_{sat2}]$. If $a > 0$ and $b > 0$, then the initial value of M, that is, Γ_0 must be away from the turning point $-\frac{b}{2a}$ in order that we have local uniqueness.
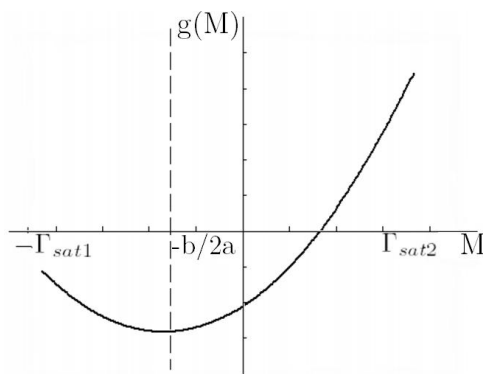

Figure 1. Function $g(M)$

Theorem 4.1 (Local Existence and Uniqueness) *Suppose hypotheses $\mathcal{H}1$ - $\mathcal{H}4$ hold for all $T > 0$. Further suppose that $\Gamma_0 \in (\max\{-\frac{b}{2a}, -\Gamma_{sat1}\}, \Gamma_{sat2}]$. There exists $\tau > 0$ such that the system (1) - (5) and (7) has a unique solution $(B, y) \in W^{1,\infty}(0,\tau) \times W^{1,\infty}(0,\tau)$, where $B = u + M$.*

Proof With $B = u + M$, $\varphi(x) = \widehat{k}_P\,x + \theta(x)$, $V = (I + \Gamma)^{-1}[\cdot; \psi_{-1}]$ and $g(x) = ax^2 + bx$, we have $u = V(B)$. The combined system (1) - (5) and (7) is:

$$
\begin{aligned}
\dot{B} &= \gamma\,\varphi^{-1}\left(-\beta\,V(B) - (k_P - k_D\alpha)y - k_D\,g((\Gamma \circ V)(B)) - k_D\,\Delta(t)\right) \\
\dot{y} &= -\alpha y + g((\Gamma \circ V)(B)) + \Delta(t).
\end{aligned}
$$

The above equations are in the form $\dot{w} = F(t, w, V[\mathcal{S}(w)], W[\mathcal{S}(w)])$ where $w = (B, y) \in \mathbb{R}^2$, $\mathcal{S}[w] = B$ is a projection operator, and $W = \Gamma \circ V$. Note that F is Lipschitz continuous as function of the last three arguments, and measurable as a function of t. As V and $\Gamma \circ V$ are Lipschitz continuous from $W^{1,1}(0,T)$ to $W^{1,1}(0,T)$ by Lemma 4.1, the claim follows from the same argument as in Theorem 3.1.1 in [32]. □

Theorem 4.1 concludes the existence of a local solution on a sufficiently small interval $[0, \tau)$. Our goal is to show existence and uniqueness on $[0, \infty)$. However, extension of the solution presents a problem because $M(t)$ might pass through the turning point $-\frac{b}{2a}$ after which multiple solutions might exist. The following two lemmas together show that this cannot happen by proper choice of parameters k_P and k_D.

Lemma 4.3 *Suppose that the conditions of Theorem 4.1 hold. Let a unique solution to (10) - (11) exist on the interval $[0, \tau)$. Then, there exists bounds independent of parameters k_P and k_D for both y and x on $[0, \tau)$.*

Proof By Theorem 4.1, there exists a solution on a sufficiently small interval $[0, \tau)$. From Equation (10), for $t \in [0, \tau)$:

$$
\begin{aligned}
\dot{y}(t) + \alpha y(t) &= g(W[f(x)])(t) + \Delta(t) \\
\implies \quad y(t) &= e^{-\alpha t} \left[\int_0^t e^{\alpha \tau} g(W[f(x)])(\tau) d\tau + \int_0^t e^{\alpha \tau} \Delta(\tau) d\tau + y_0 \right].
\end{aligned}
$$

Since W is saturating (by Hypothesis $\mathcal{H}2$) and Δ is essentially bounded on $[0, \infty)$ (by Hypothesis $\mathcal{H}4$),

$$
|y(t)| \leq e^{-\alpha t} \int_0^t N e^{\alpha \tau} d\tau + e^{-\alpha t} |y_0|,
$$

for some constant N. Hence,

$$
\forall \, t \in [0, \tau), \qquad |y(t)| \leq N/\alpha + |y_0|. \tag{12}
$$

With $z = \varphi(x)$, (11) becomes:

$$
z(t) = G\left(0; \varphi^{-1}(z)\right)(t) - \beta \gamma \int_0^t \varphi^{-1}(z(\tau)) d\tau + (k_D \alpha - k_p) y(t), \quad t \in [0, \tau). \tag{13}
$$

Due to Hypotheses $\mathcal{H}2$ and $\mathcal{H}4$, there exists $L > 0$ such that $\forall \, t \in [0, \tau)$, $|G(0; \varphi^{-1}(z))(t)| < L$. Since y is bounded (by Claim 1), there exists M such that $|(k_D \alpha - k_P) y(t)| < M$ for all $t \in [0, \tau)$. By (11), $z(0) < L + M$.

We claim that $|z(t)| < L + M$ for all $t \in [0, \tau)$. Suppose that this is not the case. Then there exists $t^* \in [0, \tau)$ such that $|z(t^*)| \geq L + M$. Consider the case $z(t^*) \geq L + M$. Suppose $z(t) > 0$ for all $t \in (t^{**}, t^*]$, where $z(t^{**}) = 0$. Since $\varphi^{-1}(0) = 0$ and φ^{-1} is strictly monotone increasing, $x(t) = \varphi^{-1}(z)(t) > 0$ for $t \in (t^{**}, t^*]$. By the semi-group property of Γ (see the comment following (11)),

$$
\begin{aligned}
L + M \leq z(t^*) &= G\left(t^{**}; \varphi^{-1}(z)\right)(t^*) - \beta \gamma \int_{t^{**}}^{t^*} \varphi^{-1}(z(\tau)) d\tau + (k_D \alpha - k_P) y(t^*) \\
&< L + M,
\end{aligned}
$$

which is a contradiction. By a similar argument, we can obtain $z(t) > -L - M$, concluding the proof of the claim. Hence,

$$
\forall \, t \in [0, \tau), \qquad |x(t)| \leq \varphi^{-1}(L + M). \square \tag{14}
$$

For the case, $a > 0$ and $b > 0$, we may assume $b/2a < \Gamma_{sat1}$, since otherwise $[-\Gamma_{sat1}, \Gamma_{sat2}] \subset [-b/2a, \Gamma_{sat2}]$ and $M(t)$ is uniquely determined as desired.

Denote $\pi = [-b/2a, \Gamma_{sat2}]$. Lemma 4.4 establishes that appropriate choices for the gain parameters force the controller to maintain $M(t) \in [-b/2a, \Gamma_{sat2}]$.

Lemma 4.4 *Suppose that the conditions of Theorem 4.1 hold. Let a unique solution to (10) - (11) exist on the maximal interval $[0, \tau)$. Then $\forall\, k_0 > 0\ \exists\, k_{P0} > 0 \ni$ if k_P, k_D satisfy $k_P \geq k_{P0}$ and $k_D \in [\frac{k_P}{\alpha}, k_0 + \frac{k_P}{\alpha}]$, then $M(t) \in \pi$ for all $t \in [0, \tau]$. Furthermore, if $M(\tau) = -\frac{b}{2a}$ then $\dot{M}(\tau) \geq 0$.*

Proof By the conditions of Theorem 4.1, $M(0) \in \text{Interior}(\pi)$. Given $k_0 > 0$, pick $k_{P0} = \dfrac{\alpha(\beta d_1 + k_0 d_2)}{\frac{b^2}{4a} - \eta}$. Suppose k_D satisfies $0 \leq k_D - \frac{k_P}{\alpha} \leq k_0$.

As $[0, \tau)$ is the maximal interval of existence and uniqueness, we have either a loss of existence (solution becomes unbounded) or loss of uniqueness as $t \to \tau$. Lemma 4.3 shows that all signals exist at $t = \tau$. Hence, the only possibility is that there is a loss of uniqueness for $t > \tau$. This may happen if $M(\tau) = -\frac{b}{2a}$. If $\dot{M}(\tau)$ does not exist, then M is increasing at τ. Suppose $\dot{M}(\tau)$ exists. We only need to show that we can find parameters k_P and k_D such that $\dot{M}(\tau) \geq 0$.

From (3) and (7),

$$\beta\, u(t) = -\widehat{k}_P x(t) - \theta(x)(t) - k_P y(t) - k_D \dot{y}(t).$$

It was shown in the proof of Lemma 4.3 that $x(\tau)$, $y(\tau)$ are bounded and the bounds are independent of parameters $\widehat{k}_P$, k_P and k_D. It follows that $\forall\, t \in [0, \tau]$, $|u(t)| \leq d_1$ for some constant d_1 that is independent of gain parameters.

Now, $\forall\, t \in [0, \tau]$, $M(t) \in \pi$, which implies that $|\dot{y}(t)| < d_2$ for some constant d_2 that is independent of parameters $\widehat{k}_P$, k_P and k_D. As $M(\tau) = -b/2a$,

$$\dot{y}(\tau) + \alpha y(\tau) = -b^2/4a + \Delta(\tau).$$

Denote as before, $\varphi(x)(\tau) = \widehat{k}_P x(\tau) + \theta(x)(\tau)$.

We rewrite (3) and (7) at time τ by adding and subtracting $-\frac{k_P}{\alpha} \dot{y}(\tau)$ to the right hand side, and collecting similar terms:

$$
\begin{aligned}
\varphi(x)(\tau) &= -\beta\, u(\tau) - \frac{k_P}{\alpha} \dot{y}(\tau) - k_P y(\tau) - k_D \dot{y}(\tau) + \frac{k_P}{\alpha} \dot{y}(\tau) \\
&= -\beta\, u(\tau) - \frac{k_P}{\alpha} \left(\dot{y}(\tau) + \alpha y(\tau) \right) + \left(-k_D + \frac{k_P}{\alpha} \right) \dot{y}(\tau) \qquad (15) \\
&\geq -\beta\, d_1 + \frac{k_P}{\alpha} \left(\frac{b^2}{4a} - \eta \right) - \left(k_D - \frac{k_P}{\alpha} \right) d_2. \\
&\geq -\beta\, d_1 + \frac{k_P}{\alpha} \left(\frac{b^2}{4a} - \eta \right) - k_0\, d_2 \qquad (16) \\
&= \frac{1}{\alpha} \left(\frac{b^2}{4a} - \eta \right) (k_P - k_{P0}) \quad \geq 0 \qquad (17)
\end{aligned}
$$

By Hypothesis $\mathcal{H}3$, $x(\tau) \geq 0$. By Hypothesis $\mathcal{H}1$ and Equation (2), $\dot{M}(\tau) \geq 0$. $\qquad \square$

Having shown that M is uniquely determined even if it reaches the boundary of π, we can now prove the following theorem.

Theorem 4.2 (Global Existence and Uniqueness) *Suppose hypotheses $\mathcal{H}1$-$\mathcal{H}4$ hold for all $T > 0$ and $\Gamma_0 \in (\max\{\frac{-b}{2a}, -\Gamma_{sat1}\}, \Gamma_{sat2}]$. Suppose the parameters k_0, k_P and k_D satisfy the conditions specified in Lemma 4.4. Then, for any $T > 0$, (10), and (11) have a unique solution $(B, y) \in W^{1,\infty}(0, T) \times W^{1,\infty}(0, T)$, where $B = u + M$. Furthermore, $x \in BC[0, \infty)$, $y \in W^{1,\infty}(0, \infty)$ and $\|y\|_\infty + \|\dot{y}\|_\infty \leq K$ for some $K < \infty$ independent of parameters k_P, k_D, and $\widehat{k}_P$.*

Proof Suppose $[0, \tau)$ is the maximal interval of existence and uniqueness of the solution to (10) and (11), where $\tau < \infty$. We will prove a contradiction.

By Lemma 4.3 $y(t)$ and $x(t)$ are uniformly bounded on $[0, \tau)$.
   Therefore, (8) yields,

$$\forall\, t \in [0, \tau), \qquad |B(t)| \leq \tau\, \gamma\, \varphi^{-1}(L + M) + |u_0 + \Gamma_0|. \tag{18}$$

Using Theorem 4.1, we see that the solution may be extended to an interval larger than $[0, \tau)$, and the extension is unique by Lemma 4.4. This contradicts the maximality of that interval $[0, \tau)$. Hence, the solution exists on the half-line $[0, \infty)$. We also see that for any $T > 0$, $(B, y) \in W^{1,\infty}(0, T) \times W^{1,\infty}(0, T)$.
   The claim that $x \in BC[0, \infty)$ follows from the proof of Lemma 4.3. The claim that $y \in W^{1,\infty}[0, \infty)$ follows immediately from Equation (10) by observing that,

$$\|\dot{y}\|_\infty \leq N + \alpha|y_0| + a \max\{\Gamma_{sat1}^2, \Gamma_{sat2}^2\} + b \max\{\Gamma_{sat1}, \Gamma_{sat2}\} + \eta.$$

Select

$$K = (\frac{1}{\alpha} + 1)\, (N + \alpha|y_0|) + a \max\{\Gamma_{sat1}^2, \Gamma_{sat2}^2\} + b \max\{\Gamma_{sat1}, \Gamma_{sat2}\} + \eta\}.$$

$\square$

4.2. Proof of Stabilization
The next objective is to show that the control signal given by Equation (7) achieves stabilization. In the previous subsection, we saw that by selecting $\widehat{k}_P > 0$, we can prove existence and uniqueness of solutions to the closed loop system. In this subsection, we derive the necessary conditions on gain parameters k_D and k_P so that given $\varepsilon > 0$ and $\Delta = 0$, we obtain $\limsup_{t \to \infty} |y(t)| \leq \varepsilon$.

Lemma 4.5 *Suppose the conditions in Lemma 4.4 are satisfied. Then*

$$\int_{t_0}^{t} (2aM(\tau) + b)u(\tau)\frac{dM}{d\tau}\, d\tau \geq 0.$$

Proof Define $g : \pi \to \mathbb{R}$ by $g(M(t)) := aM^2(t) + bM(t)$. As g is monotone increasing and differentiable on π and the hysteresis operator is output-rate dissipative, we have $\int_0^t ug'(M(t))dt \geq 0$, which proves the claim. $\square$

Finally, we show that stabilization can be achieved by proper choice of parameters in the absence of disturbance Δ.

Theorem 4.3 *Suppose that the conditions in Lemma 4.4 are satisfied, and $\Delta = 0$. Then, for any $\varepsilon > 0$, there exists $\bar{k}_{P0} \geq k_{P0}$ such that, if $k_P > \bar{k}_{P0}$ and $\alpha k_D - k_P > 0$, then $\limsup_{t \to \infty} |y(t)| \leq \varepsilon$.*

Proof We employ a Lyapunov stability argument using La Salle's invariance principle. By Theorem 4.2, B is absolutely continuous. By Theorem 4.1, the operator $(I + \Gamma)^{-1}[\cdot; \psi_{-1}]$ is Lipschitz continuous on bounded subsets of $W^{1,1}(0, T)$. Hence, u is absolutely continuous on $(0, T)$ for any $T > 0$. Then, by Equation (9), we see that $W[f(x); \psi_{-1}]$ is absolutely continuous

on $(0, T)$ for any $T > 0$. Equation (10) implies that $\dot{y}$ is locally absolutely continuous, that is, for almost every t, $\ddot{y}(t)$ exists.

Consider Equation (15) at time t,

$$\varphi(x)(t) + \beta u(t) = -\frac{k_P}{\alpha} (\dot{y} + \alpha y)(t) - \left(k_D - \frac{k_P}{\alpha}\right) \dot{y}(t). \tag{19}$$

By multiplying (19) with $\ddot{y} + \alpha \dot{y}$, and collecting terms, we find:

$$\frac{k_P}{\alpha} (\dot{y} + \alpha y)(\ddot{y} + \alpha \dot{y})(t) + \left(k_D - \frac{k_P}{\alpha}\right) \ddot{y}(t)\dot{y}(t)$$
$$= \left(-\theta(x) - \beta u - \widehat{k}_P x\right)(\ddot{y} + \alpha \dot{y})(t) - (\alpha k_D - k_P) \dot{y}^2(t). \tag{20}$$

Consider the Lyapunov function

$$V(t) = \frac{k_P}{2\alpha} (\dot{y} + \alpha y)^2(t) + \left(k_D - \frac{k_P}{\alpha}\right) \frac{\dot{y}^2(t)}{2} + \beta \int_0^t (2aM + b)u\frac{dM}{d\tau}d\tau, \tag{21}$$

Then $V(t) \geq 0$. By (20) and (21), for almost every $t \in [0, \infty)$,

$$\dot{V}(t) = \left(-\theta(x) - \beta u - \widehat{k}_P x\right)(\ddot{y} + \alpha \dot{y})(t) - (\alpha k_D - k_P) \dot{y}^2(t) + \beta (2aM(t) + b) u(t)\dot{M}(t). \tag{22}$$

From Equation (1), for almost every $t \in [0, \infty)$,

$$(\ddot{y} + \alpha \dot{y})(t)\beta u(t) = (2aM(t) + b) \beta u(t)\dot{M}(t),$$

and $\qquad (\ddot{y} + \alpha \dot{y})(t) \left(\theta(x)(t) + \widehat{k}_P x(t)\right) = (2aM(t) + b) \dot{M}(t) \left(\theta(x)(t) + \widehat{k}_P x(t)\right).$

From Equation(22)

$$\dot{V}(t) = -(2aM(t) + b) \dot{M}(t) \left(\theta(x) + \widehat{k}_P x\right)(t) - (\alpha k_D - k_P) \dot{y}^2(t).$$

From Lemma 4.4, $2aM(t)+b > 0$ and, since $\dot{M}(t)$, $x(t)$, and $\theta(x)(t)$ have the same sign, $\dot{V}(t) \leq 0$. Thus $\dot{V}(t) \to 0$ as $t \to \infty$ for almost every t.

The invariant set for V is the set $(x, \dot{y}) = (0, 0)$. By Equation (7), the largest invariant subset of this set is given by:

$$\beta u = -k_P y.$$

Hence, for almost every t, $|y(t)| \leq \frac{\beta |u|}{k_P}$, which shows that for any $\varepsilon > 0$ we may choose $k_P \geq \bar{k}_P := \max\{k_{P0}, \frac{\beta d_1}{\varepsilon}\}$, where d_1 is the bound on u discussed in Lemma 4.4. Then, the largest invariant set for V will have the property: $|y| < \varepsilon$. $\square$

5. Conclusion

In this paper, a novel control strategy for hysteretic systems associated with magnetic and smart actuators is proposed. The controller is a PD controller derived using two feedback signals. We proved regularity, well-posedness, and stability of the controller and then found conditions on the gain parameters for closed loop stability.

6. References

[1] Dimitriadis E K, Fuller C R and Rogers C A 1991 Journal of vibration and acoustics, **113** (1) 100–107.

[2] Howe D 2000 Sensors and Actuators A: Physical **81** (1-3) 268-274.

[3] Baz A A, Ro J 1996 Smart Materials and Structures **5** (3) 135-144.

[4] Engdahl G 2000 *Handbook of giant magnetostrictive materials* (Academic Press).

[5] Zhang T, Jiang C, Zhang H and Xu H 2004 Smart Material and Structures **13** (3) 473-477.

[6] Ekanayake D B, Iyer R V and Dayawansa W P 2007 American Control Conference 4321-4326.

[7] Ekanayake D B 2006 MS thesis (Texas Tech University, Lubbock, TX).

[8] Tan X and Baras J 2004 Automatica **40** (9) 1469–1480.

[9] Tan X and Baras J S 2005 IEEE Trans. Automatic Control **50** (6) 827–839.

[10] Song G, Zhao J, Zhou X and De Abreu-Garcia J A 2005 IEEE/ASME Trans. Mechatronics **10** (2) 198–209.

[11] Krejci P and Kuhnen K 2001 Proceedings of Control Theory and Applications IEEE **148** (3) 185–192.

[12] Tao G and Kokotovik P V 1995 IEEE Trans. on Automatic Control **40** (2) 200–212.

[13] Shieh H J, Lin F J, Huang P K, and Teng L T 2006 IEEE Trans. Industrial Electronics **53** (3) 905–914.

[14] Su C Y, Stepanenko Y, Svoboda J, and Leung T P 2000 IEEE Transaction on Automatic Control **45** (12) 2427–2432.

[15] Jayawardhana B, Logemann H, Ryan E P 2008 International Journal of Control **81** (8) 1331–1342.

[16] Valadkhan S, Morris K and Khajepour A 2010 International Journal of Robust and Nonlinear Control **20** (4) 460–471.

[17] Logemann H, Ryan E P and Shvartsman I 2007 ESAIM: Control, Optimization and Calculus of Variations **13** (3) 458–483.

[18] Valadkhan S, Morris K A and Khajepour A 2007 SIAM Journal of Applied Mathematics **67** (3) 667–686.

[19] Gorbet R B, Morris K A and Wang D W L 2001 IEEE Trans. Control Systems Technology **9** (1) 5–16.

[20] Ilchmann A, Ryan E P and Townsend P 2007 SIAM J. Control Optimization **46** (1) 210–230.

[21] Angeli D and Sontag E 2003 IEEE Trans. Automatic Control **48** (10) 1684–1699.

[22] Nealis J M and Smith R C 2007 IEEE Trans. Control Systems Technology **15** (1) 22–39.

[23] Ge P and Jouanech M 1996 IEEE Transactions on Control Systems Technology **4** (3) 209–216.

[24] Tan X and Iyer R V 2009 IEEE Control Systems Magazine **29** (1) 26–28.

[25] Cruz-Hernandez J M and Hayward V 2001 IEEE Trans. Control Systems Technology **9** (1) 17–26.

[26] Kuhnen K and Janocha H 1999 Proceedings of European Control Conference paper F 0291.

[27] Su C Y, Wang Q, Chen X and Rakheja S 2005 IEEE Transaction on Automatic Control **50** (12) 2069–2074.

[28] Iyer R V and Manservisi S 2006 Physica B **372**(1-2) 378–382.

[29] Iyer R V and Shirley M 2004 IEEE Transactions on Magnetics **40** (5) 3227–3239.

[30] Gorbet R B, Wang D W L and Morris K A 1998 Proceedings, IEEE international conference on robotics and automation **3** 2161–2167.

[31] Hughes D and Wen J T 1997 Smart materials and structures **6** (3) 287–300.

[32] Brokate M and Sprekels J 1996 *Hysteresis and Phase Transitions* Applied Mathematical Sciences Springer.