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# Micromagnetics with eddy currents

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**Abstract.** In this paper, we study the modified Landau-Lifshitz-Gilbert (LLG) equation for of a conducting, magnetic body. The modified LLG equations include the magnetic field due to eddy currents in the total effective magnetic field. We derive an expression for the magnetic field due to eddy current losses and show that it is well defined. We then show that the work done by the eddy currents in opposing the change of magnetization is a Rayleigh type dissipation function, and derive the modified LLG equations using the calculus of variations. Finally, we show that the modified LLG equations lead to a decrease in the Gibbs energy. This implies that the LLG equations describes a dynamic process proceeding spontaneously forward in time.

## 1. Introduction

Core losses in a magnetic material are typically considered to be due to three sources: hysteresis losses, classical eddy current losses, and excess losses. For obvious reasons, eddy and excess losses are important in induction motors [1], magnetostrictive actuators [2], amorphous magnetic ribbons, guided spin wave devices [3]. The frequency ranges at which the eddy and excess losses have been measured is about 0 - 1 MHz [2, 4, 5]. An interesting question is whether the theory of micromagnetics can explain the observed excess losses. Below microwave frequencies, the classical LLG equations could be used to study the excess losses (at microwave frequencies, the LLG equation needs to be modified [6]). However, the magnetic field due to eddy currents needs to be accounted for in the LLG equations, which is not done in the vast majority of the literature on micromagnetics [7, 8, 9, 10, 11, 12, 13]. The only references that do include the eddy currents [14, 15, 16] simply state them without derivation. As eddy current losses are a form of dissipation, one anticipates that a Rayleigh type dissipation function could be found, and the classical LLG equations modified to include eddy currents. That is the contribution of this paper. The key result that is used is Corollary 2.1, where it is shown that the eddy current field at a point in the magnetic body does not depend on the rate of change of magnetization at that point.

### *1.1. The Landau-Lifshitz equation*

In 1935, Landau and Lifshitz [17] proposed a new theory with which they were able to compute the thickness of walls between magnetic domains, and also explain domain formation in ferromagnetic materials. This theory which now goes by the name of *micromagnetics* [8] has been instrumental in the understanding and development of magnetic memories. Landau and Lifshitz considered the Gibbs energy $\mathcal{G}$ of a magnetic material to be composed of three terms – exchange, anisotropy and Zeeman energies (due to the external magnetic field) [17], and

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postulated that the observed magnetization per unit volume $\boldsymbol{M}$ field would correspond to a local minimum of the Gibbs energy. Later researchers added other terms to $\mathcal{G}$ such as magnetoelastic energy and demagnetization energy [7, 11]. They also derived the Landau-Lifshitz (LL) equation using physical arguments (but not using the calculus of variations):

$$\dot{\boldsymbol{M}} = \gamma_0\, \boldsymbol{M} \times \boldsymbol{H}_{eff} - \frac{\alpha_0}{M_s^2}\, \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}_{eff}), \tag{1}$$

where $\gamma_0 = \frac{e}{m\,c}$ with $m$ the electronic mass, $e$ the electronic charge, $c$ the speed of light; $\alpha_0$ is a positive constant, $\|\boldsymbol{M}\| = M_s$ is assumed to be constant over the material. The effective field $\boldsymbol{H}_{eff} = -\frac{\partial \mathcal{G}}{\partial \boldsymbol{M}}$ is the negative gradient of the Gibbs energy with respect to the magnetization, and can be expressed as: $\boldsymbol{H}_{eff} = \boldsymbol{H}_0 + \boldsymbol{H}_{demag} + \boldsymbol{H}_{anis} + \boldsymbol{H}_{exch} + \boldsymbol{H}_{me}$ where $\boldsymbol{H}_0$ is the external magnetic field, $\boldsymbol{H}_{demag}$ is the demagnetization field, $\boldsymbol{H}_{anis}$ is the field due to the anisotropy of the sample, $\boldsymbol{H}_{exch}$ is the field due to exchange interactions, and $\boldsymbol{H}_{me}$ is the field due to magnetoelastic interactions.

### *1.2. The Landau-Lifshitz-Gilbert equation and variational formulation*

Equation (1) was modified by Gilbert [7] who derived the equation:

$$\dot{\boldsymbol{M}} = \gamma_0\, \boldsymbol{M} \times (\boldsymbol{H}_{eff} - \eta\, \dot{\boldsymbol{M}}), \tag{2}$$

where $\eta > 0$, and which showed a better match with certain experimental results [18]. He also showed that the equation:

$$\dot{\boldsymbol{M}} = \gamma'\, \boldsymbol{M} \times \boldsymbol{H}_{eff} - \frac{\alpha'}{M_s^2}\, \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}_{eff}), \tag{3}$$

can be obtained from the LLG equation (2), if one chose $\gamma' = \frac{\gamma_0}{1+\gamma_0^2\,\eta^2\,M_s^2}$ and $\alpha' = \frac{\eta\,\gamma_0^2\,M_s^2}{1+\eta^2\,\gamma_0^2\,M_s^2}$. Gilbert also showed that the second term on the right hand side of (2) could be obtained from a Rayleigh dissipation function:

$$R_G(\dot{\boldsymbol{M}}) = \frac{\eta}{2} \int_\Omega \dot{\boldsymbol{M}} \cdot \dot{\boldsymbol{M}},$$

if one interpreted (2) as Euler-Lagrange equations with a Lagrangian of the type: Kinetic energy minus Potential energy. His contribution were limited by two issues: (i) the kinetic energy was not described, and (ii) eddy currents were neglected. The magnetic field due to eddy currents $\boldsymbol{H}_{eddy}$ is missing as eddy currents were neglected. It should be noted that as the eddy currents do not contribute to the Gibbs energy, it is not straight forward to see how they could be added to the total magnetic field $H_{eff}$ described earlier.

Brown [8] showed the existence of a kinetic energy - a limiting form of the classical rigid body kinetic energy - which along with the Gibbs energy $\mathcal{G}$, and Gilbert's damping function $R_G(\dot{\boldsymbol{M}})$ could be used to derive the LLG equation (2). Brown did not include eddy currents in his theory either.

### *1.3. The LLG equations with eddy currents*

A suggestion to add a magnetic field term $\boldsymbol{H}_{eddy}$ to $\boldsymbol{H}_{eff}$ is found in Bertotti [14], though a derivation was not presented. Following [14], Torres et al. [15] included the eddy current field in their computations, but ignored it in their numerical computations. Serpico, Mayergoyz and Bertotti [16] include the eddy current field in their study of a thin magnetic film, again without providing the justification for modifying the LLG equation.

In this paper, we derive an expression for the magnetic field due to eddy current losses and show that it is well defined. We then show that the modified LLG equations lead to a decrease in the Gibbs energy. This implies that the LLG equations describes a dynamic process proceeding spontaneously forward in time. Finally, we show that the work done by the eddy currents in opposing the change of magnetization is a Rayleigh type dissipation function, and derive the modified LLG equations using the calculus of variations.

## 2. Eddy Currents in a conducting magnetic material

Consider a magnetic material with magnetization per unit volume $\boldsymbol{M}$, occupying a compact, connected set $\Omega$ in $\mathbb{R}^3$, with a Lipschitz continuous boundary. We assume that an external magnetic field $\boldsymbol{H}_0$ is imposed by an external source. The free charge in the material is assumed to be zero.

Maxwell's equations are given by (neglecting $\frac{\partial \boldsymbol{D}}{\partial t}$) :

$$\begin{array}{rclcrcl} \nabla\cdot\boldsymbol{B} &=& 0 & \quad & \boldsymbol{B}_n^+ - \boldsymbol{B}_n^- &=& 0 \\ \nabla\times\boldsymbol{H} &=& \boldsymbol{J} & & \boldsymbol{H}_t^+ - \boldsymbol{H}_t^- &=& \boldsymbol{J}_s\times\boldsymbol{n} \\ \nabla\times\boldsymbol{E} &=& -\frac{\partial \boldsymbol{B}}{\partial t} & & \boldsymbol{E}_t^+ - \boldsymbol{E}_t^- &=& \boldsymbol{0} \\ \nabla\cdot\boldsymbol{D} &=& 0, & & \boldsymbol{D}_n^+ - \boldsymbol{D}_n^- &=& \boldsymbol{0}, \end{array} \tag{4}$$

The relation between the fields are: $\boldsymbol{B} = \mu_0\,(\boldsymbol{H} + \boldsymbol{M})$, $\boldsymbol{D} = \epsilon_0\,\boldsymbol{E}$, $\boldsymbol{J} = \sigma\boldsymbol{E}$, where $\sigma$ is the conductivity of the material. $\sigma$ is required to be a measurable and bounded function. To make physical sense, it is also required to be non-negative.

The (change in the) magnetization $\boldsymbol{M}$ has to be related to the applied field using the Landau-Lifshitz theory of micromagnetics [17, 8]. To this end, let $G$ be the Gibbs free energy given by:

$$\mathcal{G} = E_{demag} + E_{anis} + E_{exch} + E_{me} - \int_{\Omega} \boldsymbol{H}_0\cdot\boldsymbol{M}\,dv,$$

where $E_{demag}$ is the demagnetization energy, $E_{exch}$ is the exchange energy, $E_{anis}$ is the anisotropy energy, and $E_{me}$ is the magnetoelastic energy. The expressions for these energies can be found in Brown [8] (page 38).

*In what follows, we employ CGS units as in Brown [8].* Consider Figure 1. Let $P$ be a point in $\Omega$. Suppose $\epsilon > 0$ is any given number sufficiently small, and consider a small (relative) neighborhood $S$ that is removed from $\Omega$. $S$ is contained in a small ball of radius $\epsilon$ with center at $P$, that is, $S \subset B_\epsilon(P)$. Consider an infinitesimal tube of surface area $dA$ and length $dl$ located at the point $Q$ in $\Omega \cap S^c$ ($S^c$ is the complement set of $S$ with $S^c$ containing points in space that are not in $S$), such that the current $\boldsymbol{J}(Q)\,dA(Q)$ in the tube is along the axis of the tube. Let $\boldsymbol{r}_{PQ}$ be the vector centered at $Q$ and Then, the work done by the current $\boldsymbol{J}(Q)\,dA(Q)$ in opposing the change in the magnetization $\boldsymbol{M}(P)$ at the point $P$ is given by [8] (page 27):

$$\frac{dR}{dt}(P;S) = \int_{\Omega\setminus S} \frac{\mu_0}{4\pi}\,\frac{\boldsymbol{J}(Q)\,dv(Q)\cdot\left(\dot{\boldsymbol{M}}(P)\times\boldsymbol{r}_{PQ}\right)}{\|r_{QP}\|^3} = \int_{\Omega\setminus S} \frac{\mu_0}{4\pi}\,\frac{(\boldsymbol{r}_{PQ}\times\boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\,dv(Q)\;\cdot\;\dot{\boldsymbol{M}}(P), \tag{5}$$

where $dv(Q) = dA\,dl$ is the volume of the infinitesimal tube located at the point $Q$.

**Definition 2.1** *(Absolute convergence of the eddy current integral)*

*Let $P \in \Omega$. If the last integral in (5) exists for all $\epsilon > 0$ and $S \subset B_\epsilon(P)$, converges absolutely as $\epsilon \to 0$, and if the limit is independent of the shape of $S$, then we say that the eddy current field $\mathbf{H}_{eddy}$ exists at $P$, and*

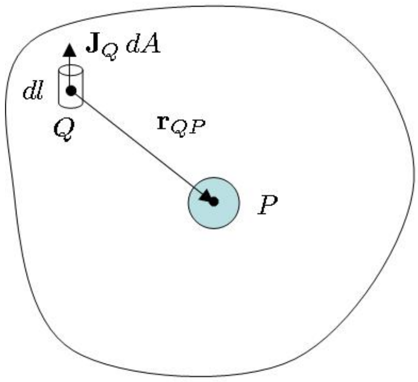


**Figure 1.** Computation of the loss due to eddy currents.

$$\boldsymbol{H}_{eddy}(P) = \int_\Omega \frac{\mu_0}{4\pi} \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q) = \lim_{\epsilon\to 0} \int_{\Omega\setminus S} \frac{\mu_0}{4\pi} \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q). \tag{6}$$

*Remark:* In the SI system, the left hand side would be called $\boldsymbol{B}_{eddy}$ the effective magnetic flux density, and $\boldsymbol{H}_{eddy}$ is defined according to: $\boldsymbol{B}_{eddy} = \mu_0\, \boldsymbol{H}_{eddy}$. □

We can define:

$$\frac{dR}{dt}(P) = \lim_{\epsilon\to 0} \frac{dR}{dt}(P; S) = \lim_{\epsilon\to 0} \int_{\Omega\setminus S} \frac{\mu_0}{4\pi} \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q) \;\cdot\; \dot{\boldsymbol{M}}(P), \tag{7}$$

and the work done by the eddy current field $\boldsymbol{J}$ in opposing the change in the magnetization in the material is given by:

$$\int_\Omega \frac{dR}{dt}(P)\, dv(P) = \int_\Omega \boldsymbol{H}_{eddy} \cdot \dot{\boldsymbol{M}}\, dv. \tag{8}$$

This is work done by the "system" consisting of the magnetic material on its environment, and hence it is non-positive. Hence, the Rayleigh dissipation function $R_E$ defined by:

$$R_E = -\int_\Omega \boldsymbol{H}_{eddy} \cdot \dot{\boldsymbol{M}}\, dv \geq 0, \tag{9}$$

and it represents the work done by the battery in overcoming the demagnetizing effect of eddy currents.

*Remark:* In the SI system, we have $R_E = -\mu_0 \int_\Omega \boldsymbol{H}_{eddy} \cdot \dot{\boldsymbol{M}}\, dv$ □

Below is the main lemma of this paper.

**Lemma 2.1** *The limit in (6) converges absolutely irrespective of the shape of the open set $S$, which is assumed to contain $P$ and be a subset of $\mathbf{B}_\epsilon(P)$.*

**Proof:** Consider the integral:

$$\mathbf{I} = \int_S \frac{\mu_0}{4\pi} \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q)$$

where $S$ is a small neighborhood $\subset B_\epsilon(P)$ that is a subset of $\Omega$.

The proof of absolute convergence of $\mathbf{I}$ as $\epsilon \to 0$, depends on showing the boundedness of $\boldsymbol{J}$, which is implied by the boundedness of $\boldsymbol{E}$ over all of $\Omega$. As the material is without polarization (that is, $\boldsymbol{P} = \boldsymbol{0}$), we have $\boldsymbol{D} = \epsilon_0\, \boldsymbol{E}$, and the boundedness of $\boldsymbol{E}$ is equivalent to that of $\boldsymbol{D}$. From the Maxwell's equation:

$$\int_V \nabla \cdot \boldsymbol{D}\, dv = \int_{\partial V} \boldsymbol{D} \cdot \boldsymbol{n}\, dS = 0,$$

where $V \subset D$ is any region with a Lipschitz continuous boundary. If $\boldsymbol{D}$ is unbounded or undefined at any point on $\partial V$, then the third equality in the above equation cannot hold. Hence, $\boldsymbol{D}$ must be defined everywhere on the boundary of an arbitrary region $V \subset \Omega$, which implies that $\boldsymbol{D}$ is defined everywhere on $\Omega$. As $\Omega$ is a compact set, $\boldsymbol{D}$ is bounded everywhere on $\Omega$, and there exists $C > 0$ such that $\|\boldsymbol{D}\| \leq C$ on $\Omega$.

The rest of the proof is an adaptation of a method in Leathem [19] (page 15). Let $T$ be a point on the boundary of $S$ that is at the shortest distance from $P$. As the boundary of $S$ is a compact set, there exists at least one such point. Let $\|\boldsymbol{r}_{PT}\| = \eta$. By construction, $\eta < \epsilon$, and:

$$\begin{aligned}
|I| &\leq \frac{\mu_0}{4\pi} \int_S \left| \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3} \right| dv(Q) \\
&\leq \frac{\mu_0}{4\pi} \int_S \frac{\|\boldsymbol{J}(Q)\|}{\|r_{PQ}\|^2}\, dv(Q) \\
&\leq \frac{\mu_0}{4\pi} \int_\eta^\epsilon \frac{\sigma_{\max}\, C}{r^2}\, 4\,\pi\,\epsilon_0\, r^2\, dr \\
&= \frac{\mu_0}{\epsilon_0}\, C\, \sigma_{\max}\, (\epsilon - \eta) \\
&\leq C\, \sigma_{\max}\, Z_0^2\, \epsilon,
\end{aligned}$$

where $\sigma_{\max}$ is the maximum value of the conductivity $\sigma$ on $\Omega$, and $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$ is the characteristic impedance of free space. The inequality above shows that irrespective of $S$, we have $\lim_{\epsilon\to 0} |I| = 0$. Now, $\int_{\Omega\setminus S} \left| \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3} \right| dv(Q)$ is finite because $\boldsymbol{J}$ is bounded on $\Omega$, and the denominator never approaches 0. Hence, we have:

$$\frac{\mu_0}{4\pi} \int_\Omega \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q) = \frac{\mu_0}{4\pi} \int_{\Omega\setminus S} \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q) + \frac{\mu_0}{4\pi} \int_S \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q),$$

as both the integrals on the right side exist. At this point, the left hand side seems to be function of $S$. However,

$$\left| \frac{\mu_0}{4\pi} \int_\Omega \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q) - \frac{\mu_0}{4\pi} \int_{\Omega\setminus S} \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q) \right| \leq C\, \sigma_{\max}\, Z_0^2\, \epsilon, \tag{10}$$

which shows that $\boldsymbol{H}_{eddy}(P)$ does not depend on $S$. □

**Corollary 2.1** *Let $P \in \Omega$. The eddy current magnetic field $\mathbf{H}_{eddy}(P)$ does not depend on $\dot{\mathbf{M}}(P)$, and $\frac{\partial R_E}{\partial \dot{\mathbf{M}}}(P) = \mathbf{H}_{eddy}(P)$.*

**Proof:** The claim follows from the fact that $\frac{\mu_0}{4\pi} \int_{\Omega\setminus S} \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q)$ converges absolutely to $\boldsymbol{H}_{eddy}$, and the contribution of $\frac{\mu_0}{4\pi} \int_S \frac{(\boldsymbol{r}_{PQ} \times \boldsymbol{J}(Q))}{\|r_{PQ}\|^3}\, dv(Q)$ to $\boldsymbol{H}_{eddy}$ can be made as small as desired.

As $\boldsymbol{H}_{eddy}(P)$ is independent of $\dot{M}(P)$, we get from (9), the second claim in the corollary. □

*Remark:* In SI units, the corollary above reads $\frac{\partial R_E}{\partial \dot{\boldsymbol{M}}}(P) = \mu_0\, \boldsymbol{H}_{eddy}(P)$. □

## 3. The modified Landau-Lifshitz-Gilbert equation with eddy currents

Brown [8] (pages 31 - 43) showed the existence of a Routhian function $\mathcal{T}$ such that with the Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{G}$ such that:

$$\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{M}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{M}}\right) \cdot \delta \boldsymbol{M} = \frac{\partial \mathcal{R}_G}{\partial \dot{M}} \cdot \delta \boldsymbol{M}, \tag{11}$$

where $\delta \boldsymbol{M} = \boldsymbol{M} \times \mathbf{w}$ is a virtual variation, with $\mathbf{w}$ an arbitrary vector field on $\Omega$, which is $\mathbf{0}$ at the initial and final times, yields the LLG equation (2):

$$\frac{1}{\gamma_0}\dot{\boldsymbol{M}} - \boldsymbol{M} \times \boldsymbol{H}_{eff} = \eta\,(\dot{\boldsymbol{M}} \times \boldsymbol{M}). \tag{12}$$

We consider the dissipation function $\mathcal{R} = \mathcal{R}_G + \mathcal{R}_E$ where $\mathcal{R}_E$ is given by (9), which leads to the Euler-Lagrange equations:

$$\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{M}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{M}}\right) \cdot \delta \boldsymbol{M} = \frac{\partial \mathcal{R}_G}{\partial \dot{M}} \cdot \delta \boldsymbol{M} + \frac{\partial \mathcal{R}_E}{\partial \dot{M}} \cdot \delta \boldsymbol{M}. \tag{13}$$

By Corollary 2.1, we get:

$$\frac{1}{\gamma_0}\dot{\boldsymbol{M}} - \boldsymbol{M} \times \boldsymbol{H}_{eff} = \eta\,(\dot{\boldsymbol{M}} \times \boldsymbol{M}) - \boldsymbol{H}_{eddy} \times \boldsymbol{M}.$$

Therefore, modified LLG equation is given by:

$$\dot{\boldsymbol{M}} = \gamma_0\,\boldsymbol{M} \times \boldsymbol{H}_{total} - \gamma_0\,\eta\,\boldsymbol{M} \times \dot{\boldsymbol{M}} = \gamma_0\,\boldsymbol{M} \times (\boldsymbol{H}_{total} - \eta\,\dot{\boldsymbol{M}}), \tag{14}$$

where $\eta > 0$ and $\gamma_0 = \frac{e}{m\,c} < 0$. The total magnetic field is given by:

$$\begin{aligned}\boldsymbol{H}_{total} &= \boldsymbol{H}_{eff} + \boldsymbol{H}_{eddy}\\ &= \boldsymbol{H}_0 + \boldsymbol{H}_{demag} + \boldsymbol{H}_{anis} + \boldsymbol{H}_{exch} + \boldsymbol{H}_{me} + \boldsymbol{H}_{eddy}\\ &= \boldsymbol{H} + \boldsymbol{H}_{anis} + \boldsymbol{H}_{exch} + \boldsymbol{H}_{me},\end{aligned}$$

as the magnetic field $\boldsymbol{H}$ in Maxwell's equations is given by:

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_{demag} + \boldsymbol{H}_{eddy}.$$

From (14),

$$\dot{\boldsymbol{M}} \cdot (\boldsymbol{H}_{total} - \eta\,\dot{\boldsymbol{M}}) = \mathbf{0},$$

which implies:

$$\dot{\boldsymbol{M}} \cdot \boldsymbol{H}_{eff} = \eta\dot{\boldsymbol{M}} \cdot \dot{\boldsymbol{M}} - \dot{\boldsymbol{M}} \cdot \boldsymbol{H}_{eddy}. \tag{15}$$

Now,

$$\frac{dG}{dt}(\boldsymbol{M}, \boldsymbol{H}_0)dt = \int_\Omega \frac{\partial G}{\partial \boldsymbol{M}}\dot{\boldsymbol{M}} + \frac{\partial G}{\partial \boldsymbol{H}_0}\dot{\boldsymbol{H}}_0\,dv = \int_\Omega (-\boldsymbol{H}_{eff} \cdot \dot{\boldsymbol{M}} - \boldsymbol{M} \cdot \dot{\boldsymbol{H}}_0)\,dv. \tag{16}$$

Using the reciprocity relation for magnetostatics (page 37, [8]),

$$\int_\Omega \boldsymbol{M} \cdot \dot{\boldsymbol{H}}_0\,dv = \int_\Omega \dot{\boldsymbol{M}} \cdot \boldsymbol{H}_0\,dv,$$

we get,

$$\frac{dG}{dt}(\boldsymbol{M}, \boldsymbol{H}_0)dt = \int_\Omega (-\boldsymbol{H}_{eff} \cdot \dot{\boldsymbol{M}} - \dot{\boldsymbol{M}} \cdot \boldsymbol{H}_0)\,dv \tag{17}$$

$$= \int_\Omega (-\eta\,\dot{\boldsymbol{M}} \cdot \dot{\boldsymbol{M}} + \dot{\boldsymbol{M}} \cdot \boldsymbol{H}_{eddy} - \dot{\boldsymbol{M}} \cdot \boldsymbol{H}_0)\,dv. \tag{18}$$

Each of the three terms on the right hand side is non-negative. The second term is negative as it is work done by the magnetic material on its environment, while the term $\int_\Omega \dot{\boldsymbol{M}} \cdot \boldsymbol{H}_0\,dv$ is positive as it is the work done by the battery on changing the magnetization of the material. Hence, Equation (14) describes a physically valid, irreversible phenomenon.

## 4. Conclusion

In this paper, we studied the modified Landau-Lifshitz-Gilbert (LLG) equation for of a conducting, magnetic body. The modified LLG equations include the magnetic field due to eddy currents in the total effective magnetic field. We derived an expression for the magnetic field due to eddy current losses and show that it is well defined. We showed that the work done by the eddy currents in opposing the change of magnetization is a Rayleigh type dissipation function, and derive the modified LLG equations using the calculus of variations. Finally, we showed that the modified LLG equations lead to a decrease in the Gibbs energy. This implies that the LLG equations describes a dynamic process proceeding spontaneously forward in time.

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