PROBLEMS IN TOPOLOGY AND OPERATOR THEORY

by

Răzvan Gelca

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
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Thesis supervisors: Professor Raúl Curto
Associate Professor Charles Frohman

ABSTRACT

The topics of my dissertation involve technics and ideas common to both functional analysis and geometry.

The first chapter contains the solution to a problem from the spectral theory of commuting n-tuples of operators. The question is whether one can perturb a Fredholm n-tuple of operators with index zero by compact operators to get an n-tuple with exact Koszul complex. This property is true in the one dimensional case, namely a Fredholm operator with index zero can be compactly perturbed to an invertible one. I prove that there exist Fredholm n-tuples of index zero that cannot be perturbed by compact operators to an n-tuple with exact Koszul complex. I do this by finding an obstruction at the level of the boundary operator from the long exact sequence in cohomology.

The main result of the second chapter is the proof in dimension 2 of a conjecture of Douglas and Paulsen. The conjecture is related to the study of invariant subspaces for multiplication operators on the polydisk, and states that an ideal of polynomials is relatively closed in the Hardy space topology of the unit polydisk if and only if each irreducible component of its variety intersects the polydisk. The conjecture is proved by reducing it to a topological version of the Hilbert Nullstellensatz and using an inequality for polynomials that went unnoticed before. The chapter also includes some Bergman space analogues of this result.

The third chapter gives a positive answer to the question whether the smooth topological quantum field theory of Lickorish, Blanchet, Habeger, Masbaum and Vogel comes from a topological quantum field theory with corners. I give the construction of a topological quantum field theory with corners that satisfies the axioms of K. Walker by describing the basic data and checking its consistency. In addition, I give an axiomatic proof of the Lickorish invariant formula for closed three-manifolds.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the August 1997 graduation.

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To the memory of my father Gheorghe Gelca and grandfather Ioan Drecin

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1.1 Introduction

The spectral theory of linear operators has its roots in the study of differential equations. The spectral decomposition theorem for selfadjoint operators proved by D. Hilbert at the beginning of the century, together with the advances made in quantum physics, caused this area to become a domain of interest in itself. The spectral decomposition theorem produced a comprehensible model for selfadjoint operators, and led to the construction of a functional calculus with measurable functions for these operators. Later, the study of spaces of analytic functions made nonselfadjoint phenomena become attractive to operator theorists; a spectral theory for arbitrary operators, together with an analytic functional calculus was then introduced by F. Riesz and N. Dunford. Fredholm operators appeared first in the study of a class of integral equations done by I. Fredholm, and by now are well understood.

The developments in the study of spaces of functions of more variables made it possible for a multivariable operator theory to emerge. Such a theory already appeared for the selfadjoint case in the works of J. von Neumann, being related to commutative von Neumann algebras. In 1970, J. L. Taylor [37], [38], constructed a spectral theory for commuting n-tuples of operators by using the Koszul complex [28] (a tool that had already been used by the French school in complex analysis). Taylor defined the notion of spectrum, introduced analytic functional calculus, and proved that several important properties valid in the case of one operator are preserved in this more general setting; among these, the fact that the spectrum is compact and nonvoid and that the spectral mapping theorem has a multivariable analogue.

In the late 70's, the notion of Fredholm n-tuples and a definition for the index as the Euler characteristic of the Koszul complex were introduced. Since then a lot of properties that these notions satisfy in the case of a single operator have been shown to be also true for commuting n-tuples. For example the index is continuous ([40], [8]), invariant under compact perturbations ([8], [2]), and additive ([32]).

It is a well known fact that a Fredholm operator of index zero can be perturbed with a compact operator, in fact a finite rank operator, to an invertible one. In [9], R. Curto asked if this remains true in the case of an n-tuple, namely if one can perturb a Fredholm n-tuple of index zero with compact operators to get an n-tuple whose Koszul complex is exact. The purpose of this chapter is to prove the existence of n-tuples for which this property does not hold.

The results from this chapter have appeared in [14] and [15]

We will start by reviewing some facts in multivariable operator theory. Let us discuss first the corresponding notions for a single operator.

Let $\mathcal{X}$ be a Banach space, i. e. a vector space endowed with a norm $\|\cdot\|$ that induces a topology in which $\mathcal{X}$ is complete. If T is a linear operator on $\mathcal{X}$ let $\|T\| := sup_{x\in\mathcal{X}}\|Tx\|/\|x\|$. In case $\|T\|$ is finite, we say that T is a bounded operator, and $\|T\|$ is called the norm of T.

Given an operator T, it is invertible if there exists an operator S with the property that $ST = TS = I$. On the other hand, one says that T is singular if the following complex of Banach spaces is not exact

$$0 \to \mathcal{X} \xrightarrow{T} \mathcal{X} \to 0. \tag{1.1}$$

As a consequence of the open mapping theorem, if T is bounded, then T is invertible if and only if it is not singular.

One defines the spectrum of T to be

$$\sigma(T) := \{z \in \mathbf{C}|\ z - T \text{ is singular}\}.$$

The spectrum is a compact nonvoid set. The number $r := sup\{|z|\ z \in \sigma(T)\}$ is called the spectral radius of T. If f is an analytic function in a neighborhood of $\sigma(T)$ one can define $f(T)$. In particular if $f(z) = \Sigma a_k z^k$ then $f(T) = \Sigma a_k T^k$, and if $f(z) = p(z)/q(z)$ is a rational function having no poles inside $\sigma(T)$, then $f(T) = p(T)q(T)^{-1}$.

The operator T is called Fredholm if the complex (1.1) has finite dimensional quotients. We define $indT := dimkerT - dimcokerT$, which is the Euler characteristic of the complex. The index is continuous and invariant under perturbations with compact operators. Let us recall that an operator is compact if it maps bounded sets from $\mathcal{X}$ into relatively compact sets.

As an example of Fredholm operator we have the unilateral shift $U_+ : l^2 \to l^2$, $U_+(\Sigma_0^\infty a_i e_i) = \Sigma_0^\infty a_i e_{i+1}$. Another example is the operator of multiplication with the variable on the Bergman space of the unit disk (that is the space of analytic, square integrable functions).

The point of view described above makes the following construction of J. L. Taylor [37] natural.

To each n-tuple $T = (T_1, T_2, \cdots, T_n)$ of operators on $\mathcal{X}$ satisfying $T_i T_j = T_j T_i$ for every $1 \leq i, j \leq n$, one attaches a complex of Banach spaces, called the Koszul complex, as follows. Let $\Lambda^p = \Lambda^p[e_1, e_2, \cdots, e_n]$ be the p-forms on $\mathbf{C}^n$. Define the operator $D_T : \mathcal{X} \otimes \Lambda^p \to \mathcal{X} \otimes \Lambda^{p+1}$ by $D_T := T_1 \otimes E_1 + T_2 \otimes E_2 + \cdots + T_n \otimes E_n$, where $E_i \omega := e_i \omega$, $i = 1, \cdots, n$.

The Koszul complex is

$$0 \to \mathcal{X} \bigotimes \Lambda^0 \overset{D_T}{\to} \mathcal{X} \bigotimes \Lambda^1 \overset{D_T}{\to} \cdots \overset{D_T}{\to} \mathcal{X} \bigotimes \Lambda^n \to 0. \tag{1.2}$$

It is easy to check that $D_T^2 = 0$, so this is indeed a complex.

If we consider the algebraic notion of invertibility, namely the existence of the operators $S_1, S_2, \cdots, S_n$ with the property that $S_1T_1 + S_2T_2 + \cdots S_nT_n = 1$, then the lack of an analogue of the open mapping theorem for several variables does not let us relate the invertibility to the Koszul complex. However, because of technical reasons, Taylor developed the whole spectral theory based on this complex. Thus an n-tuple will be called invertible if its Koszul complex is exact.

Let $H^p(T)$ be the cohomology spaces of the complex (1.2). The n-tuple T is invertible if $H^p(T) = 0, 0 \leq p \leq n$, it will be called Fredholm if $dim H^p(T) < \infty, 0 \leq p \leq n$, in which case we define its index to be $ind T := \sum_{p=0}^{n} (-1)^p dim H^p(T)$.

As an example, in the case when $n = 2$ the Koszul complex is

$$0 \to \mathcal{X} \overset{D_T^0}{\to} \mathcal{X} \bigoplus \mathcal{X} \overset{D_T^1}{\to} \mathcal{X} \to 0 \tag{1.3}$$

where $D_T^0(x) = (T_1x, T_2x)$ and $D_T^1(x, y) = T_2x - T_1y$, for any $x, y \in \mathcal{X}$. In this case $H^0(T) = ker D_T^0$, $H^1(T) = ker D_T^1 / ran D_T^0$, and $H^2(T) = \mathcal{X} / ran D_T^1$.

The Taylor spectrum of T, denoted by $\sigma(T)$, is the set of all $z = (z_1, z_2, \cdots, z_n)$ in $\mathbf{C}^n$ such that $z - T = (z_1 - T_1, z_2 - T_2, \cdots, z_n - T_n)$ is not invertible. It is known that $\sigma(T)$ is a compact nonvoid set. For any holomorphic map f on a neighborhood of the spectrum one can define $f(T)$, in particular, if $f(z_1, \cdots, z_n) = \Sigma a_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n}$, then $f(T) = \Sigma a_{i_1, \ldots, i_n} T_1^{i_1} \cdots T_n^{i_n}$. The following result holds (see [38]).

THEOREM 1.1. (Spectral Mapping Theorem) If $f : U \to \mathbf{C}^m$ is holomorphic on a neighborhood U of $\sigma(T)$ then $f(\sigma(T)) = \sigma(f(T))$.

Like every homology theory, the spectral theory for commuting n-tuples has a long exact sequence in cohomology.

THEOREM 1.2. If $T = (T_1, T_2, \cdots, T_n)$ and $T' = (T_1, T_2, \cdots, T_n, S)$ are commuting tuples, then the following sequence is exact:

$$\begin{aligned} 0 \to H^0(T') \to H^0(T) \xrightarrow{\hat{S}} H^0(T) \to H^1(T') \to H^1(T) \to \cdots \\ H^{p-1}(T) \to H^p(T') \to H^p(T) \xrightarrow{\hat{S}} H^p(T) \to \cdots \end{aligned} \tag{1.4}$$

where $\hat{S}$ is the operator induced by $S \otimes 1 : \mathcal{X} \otimes \Lambda^p \to \mathcal{X} \otimes \Lambda^p, 0 \leq p \leq n$.

Let us remark that if T' is invertible, then $\hat{S}$ is an isomorphism at each stage. If T is Fredholm, the long exact sequence in cohomology provides an exact sequence of finite dimensional spaces, consequently the alternated sum of their dimensions is zero. This shows that T' is a Fredholm tuple of index zero. Inductively we see that any n-tuple that contains a Fredholm subtuple is Fredholm of index zero. We will look at this kind of tuples for finding the counterexample to our perturbation problem.

The following result will also be used in the sequel.

THEOREM 1.3. [32] If $T = (T_1, T_2, \cdots, T_n)$ is a Fredholm n-tuple of operators on a Banach space, and if $m = (m_1, m_2, \cdots, m_n) \in \mathbf{N}^n$, then $T^m = (T_1^{m_1}, T_2^{m_2}, \cdots, T_n^{m_n})$ is also Fredholm and

$$ind(T^m) = m_1 m_2 \cdots m_n ind T.$$

1.2 Finite Rank Perturbations

Given a Fredholm operator of index zero T, it is a well known fact that one can add a finite rank operator R to T and make it invertible. The classical construction is to let R be an isomorphism between the kernel and the cokernel of T. Considering the Koszul complex of T, that is the complex (1.1), we see that R induces a finite rank perturbation of this complex to the exact complex

$$0 \to \mathcal{X} \xrightarrow{T+R} \mathcal{X} \to 0. \tag{1.5}$$

Similarly, the Koszul complex of a Fredholm n-tuple of index zero has a finite rank perturbation to an exact complex [8], but usually the new complex is not the Koszul complex of an n-tuple. The purpose of this section is to prove the existence of commuting n-tuples of index zero that do not admit finite rank perturbations to commuting n-tuples with exact Koszul complex.

LEMMA 2.1. Let $(S_1, S_2, \cdots, S_n)$ be an invertible commuting n-tuple and let $f : \mathbf{C}^n \to \mathbf{C}^m$ be a holomorphic function with $f^{-1}(0)$=0. Then $f(S_1, S_2, \cdots, S_n)$ is invertible.

PROOF: Since $f^{-1}(0) \bigcap \sigma(S_1, S_2, \cdots, S_n) = \emptyset$, from the spectral mapping theorem (Theorem 1.1) it follows that 0 is not in the spectrum of $f(S_1, S_2, \cdots, S_n)$, so this m-tuple is invertible.□

LEMMA 2.2. Let $(S_1, S_2, \cdots, S_n)$ be a Fredholm commuting n-tuple, with the property that $ind(S_1, S_2, \cdots, S_n) \neq 0$. Then there exists a sequence of positive integers $\{m_k\}_k$ and $0 \leq p_0 \leq n$ such that $dimH^{p_0}(S_1^{m_k}, S_2, \cdots, S_n) \to \infty$ for $k \to \infty$.

PROOF: By Theorem 1.3, $ind(S_1^m, S_2, \cdots, S_n) = m \cdot ind(S_1, S_2, \cdots, S_n)$, so $\Sigma_{p=0}^n (-1)^p dimH^{p_0}(S_1^m, S_2, \cdots, S_n) \to \infty$ for $k \to \infty$. It follows that $\Sigma_{p=0}^n dimH^{p_0}(S_1^m, S_2, \cdots, S_n) \to \infty$ for $k \to \infty$; so there is a p_0 such that the sequence $dimH^{p_0}(S_1^m, S_2, \cdots, S_n)$ is unbounded, from which the conclusion follows.□

THEOREM 2.3. Let $(T_1, T_2, \cdots, T_n)$ be a Fredholm commuting n-tuple with $ind(T_1, T_2, \cdots, T_n) \neq 0$, and let $p \in \mathbf{C}[z_1, z_2, \cdots, z_n]$ with $p(0) = 0$. Define the operator $T_{n+1} := p(T_1, T_2, \cdots, T_n)$. Then $(T_1, T_2, \cdots, T_n, T_{n+1})$ is Fredholm of index zero, but there do not exist finite rank operators $R_1, R_2, \cdots, R_n, R_{n+1}$ such that $(T_1 + R_1, T_2 + R_2, \cdots, T_{n+1} + R_{n+1})$ is an invertible commuting (n+1)-tuple.

PROOF: Suppose that such finite rank operators exist and let $S_i = T_i +$

R_i, $1 \le i \le n+1$. Applying Lemma 2.1 to the function $\phi : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$, defined by $\phi(z_1, z_2, \cdots, z_n, z_{n+1}) := (z_1, z_2, \cdots, z_n, z_{n+1} - p(z_1, z_2, \cdots, z_n))$ we get that $(S_1, S_2, \cdots, S_n, R)$ must be invertible, where $R := S_{n+1} - p(S_1, S_2, \cdots, S_n)$. Clearly, R is a finite rank operator. By applying again Lemma 2.1 to the function $\psi : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}, \psi(z_1, z_2, \cdots, z_n, z_{n+1}) := (z_1^m, z_2, \cdots, z_n, z_{n+1})$ we get that the $(m+n)$ tuple $(S_1^m, S_2, \cdots, S_n, R)$ is also invertible, for every positive integer m.

Let $\{m_k\}_k$ and p_0 be the numbers obtained by applying Lemma 2.2 to the n-tuple $(S_1, S_2, \cdots, S_n)$, and let $\hat{R} = \hat{R}(m_k, p_0)$ be the operator induced by R on $H^{p_0}(S_1^{m_k}, S_2, \cdots, S_n)$. Because $(S_1^{m_k}, S_2, \cdots, S_n)$ is invertible, from the long exact sequence in cohomology it follows that $\hat{R}$ must be an isomorphism for every m_k.

But this is impossible since $dim H^{p_0}(S_1^{m_k}, S_2, \cdots, S_n) \to \infty$ and $rank(\hat{R}) \le \binom{n}{p_0} \cdot rank(R)$. This completes the proof.$\square$

This result can be generalized as follows.

PROPOSITION 2.4. Let $T = (T_1, T_2, \cdots, T_n)$ be a Fredholm n-tuple with $indT \neq 0$, and let $p \in \mathbf{C}[z_1, z_2, \cdots, z_n]^m$ be a polynomial map satisfying $p(0) = 0$. Then the $(m+n)$-tuple $(T, p(T))$ is Fredholm of index zero, but it cannot be perturbed with finite rank operators to an invertible (m+n)-tuple.

PROOF: Suppose such a perturbation exists. As in the proof of the previous theorem we can deduce that there exists a commuting n-tuple $S_1, S_2, \cdots, S_n)$, that is Fredholm of nonzero index and finite rank operators $F_1, F_2, \cdots, F_m$ such that $(S_1, S_2, \cdots, S_n, F_1, \cdots, F_m)$ is invertible.

As before, from Lemma 2.1 we get that for every k, the $m+n$-tuple $(S_1^k, S_2, \cdots, S_n, F_1, \cdots, F_m)$ is invertible. Choose i to be minimal with the property that $sup_{p,k} dim H_p(S_1^k, S_2, \cdots, S_n, \quad F_1, \cdots, F_i) = \infty$. Such an i exists because of

Lemma 2.2, and it is smaller than m since for $i = m$ all the dimensions are zero.

Let $s := sup_{p,k} dim H_p(S_1^k, S_2, \cdots, S_n, F_1, \cdots, F_i, F_{i+1})$ By Theorem 1.2, for every p we have a short exact sequence

$$H_p(S_1^k, S_2, \cdots, S_n, F_1, \cdots, F_i, F_{i+1}) \to H_p(S_1^k, S_2, \cdots, S_n, F_1, \cdots, F_i) \overset{\hat{F}_{i+1}}{\to}$$
$$\overset{\hat{F}_{i+1}}{\to} H_p(S_1, S_2, \cdots, S_n, F_1, \cdots, F_i) \tag{1.6}$$

from which it follows that $rank \hat{F}_{i+1} \geq dim H_p(S_1^k, S_2, \cdots, S_n, F_1, \cdots, F_i) - s$. But $rank \hat{F}_{i+1} \leq \binom{n+i}{p} rank F_{i+1}$.

Therefore $dim H_p(S_1^k, S_2, \cdots, S_n, F_1, \cdots, F_i) \leq s + \binom{n+i}{p} rank F_{i+1}$, and we get that $sup_{p,k} dim H_p(S_1^k, \ S_2, \cdots, S_n, F_1, \cdots, F_i) < \infty$, a contradiction. This proves the proposition.□

1.3 The Main Example

In what follows we will restrict ourselves to bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$. It is known [8] that the index of an n-tuple is invariant under compact perturbations. We will show the existence of Fredholm pairs of index zero that cannot be perturbed with compact operators to invertible ones. We start with a result about the structure of a Fredholm operator of positive index.

LEMMA 3.1. Let T be a Fredholm operator with $ind T > 0$. Define $\mathcal{H}_n = ker T^n \ominus ker T^{n-1}$. Then $\mathcal{H}_n \neq (0), n \geq 2$. Let $T_n := T|ker T^n$. Write

$$T_n : \mathcal{H}_n \bigoplus ker T^{n-1} \to \mathcal{H}_{n-1} \bigoplus ker T^{n-2}; T_n = \begin{bmatrix} A_n & 0 \\ B_n & C_n \end{bmatrix}. \tag{1.7}$$

Then there exists n_0 such that, for $n > n_0$, A_n is an isomorphism.

PROOF: Suppose that for some n, $\mathcal{H}_n = (0)$. Then $ker T^n = ker T^{n-1}$. Hence $ker T^{n+k} = ker T^n$, for all $k \geq 0$. But this contradicts the fact that $\lim_{n \to \infty} ind T^{n+k} =$

∞.

Since $\mathcal{H}_n \perp kerT_{n-1}$, $T|\mathcal{H}_n$ is injective and $T\mathcal{H}_n \bigcap kerT_{n-2} = (0)$. This shows that A_n is injective. But then the sequence $\{dim\mathcal{H}_n\}_n$ is decreasing so it becomes stationary. Let n_0 be such that for $n > n_0$, $dim\mathcal{H}_n = dim\mathcal{H}_{n-1}$. Then for $n > n_0$, A_n is an injective operator between finite dimensional spaces of the same dimension, so it is an isomorphism.□

REMARK. In the case when T is a coisometry, this result provides the Wold decomposition for its adjoint. In this case all the subspaces $\mathcal{H}_n$ are isomorphic, and $\mathcal{H}_0$ is the wandering space of T^*.

LEMMA 3.2. Let T and $\mathcal{H}_n, n{\geq}2$, be as in the statement of the previous lemma. If S is an operator that commutes with T, then for all $n{\geq}1$, $kerT^n$ is an invariant subspace for S. Let $S_n = S|kerT^n$,

$$S_n : \mathcal{H}_n \bigoplus kerT^{n-1} \to \mathcal{H}_n \bigoplus kerT^{n-1}; S_n = \begin{bmatrix} X_n & 0 \\ Y_n & Z_n \end{bmatrix}. \qquad (1.8)$$

Then there is n_0 such that for $n \geq n_0$, X_n is similar to X_{n_0}.

PROOF: The fact that $kerT^n$ is invariant for S follows from commutativity. Let A_n and n_0 be as in Lemma 3.1. Then $ST = TS$ implies $S_{n-1}T_n = T_nS_n, n \geq 2$. Therefore, $X_{n-1}A_n = A_nX_n, n \geq 2$. For $n > n_0$ A_n is an isomorphism hence X_n is similar to X_{n-1}. This proves the lemma.□

LEMMA 3.3. Let (T, S) be an invertible commuting pair. Then for any n, $S|kerT^n$ is an automorphism of $kerT^n$.

PROOF: Applying Lemma 2.1 to (T, S) and $f : \mathbf{C}^2 \to \mathbf{C}^2$, $f(z_1, z_2) = (z_1^n, z_2)$ we get that (T^n, S) is invertible for any n. From the long exact sequence in cohomology it follows that $\hat{S} : H^0(T^n) \to H^0(T^n)$ is an isomorphism. But $H^0(T^n) = kerT^n$, and the lemma is proved.□

THEOREM 3.4. Let T be a Fredholm operator with $indT \neq 0$. Then the pair $(T, 0)$ is Fredholm of index zero and there do not exist compact operators K_1 and K_2 such that $(T + K_1, K_2)$ is an invertible commuting pair.

PROOF: Suppose such K_1 and K_2 exist. Without loss of generality we may assume $indT > 0$, otherwise we take T^* instead of T. We can also assume that $K_1 = 0$, otherwise we can denote $T + K_1$ by T, and let $K_2 := K$.

Consider the spaces $\mathcal{H}_n, n \geq 2$, obtained by applying Lemma 3.1 to T, and let $K_n := K|kerT^n$. By Lemma 3.2,

$$K_n : \mathcal{H}_n \bigoplus kerT^{n-1} \to \mathcal{H}_n \bigoplus kerT^{n-1}; K_n = \begin{bmatrix} X_n & 0 \\ Y_n & Z_n \end{bmatrix}, \tag{1.9}$$

have the property that X_n is similar to X_{n_0} for some n_0 and $n \geq n_0$. Applying Lemma 3.3 we get that the operators $X_n, n \geq 2$ are isomorphisms. If we denote by r the spectral radius of X_{n_0}, then since X_{n_0} is invertible its spectrum contains nonzero elements, so $r > 0$. From the fact that X_n is similar to X_{n_0} for $n \geq n_0$, (so all X_n's have the same spectral radius), it follows that $\|X_n\| \geq r$.

But $\|K|\mathcal{H}_n\| = \|K_n|\mathcal{H}_n\| \geq \|X_n\| \geq r$ for $n \geq n_0$. Because $\mathcal{H}_n \perp \mathcal{H}_m, n \neq m$, and $\mathcal{H}_n \neq (0)$ for any n, it follows that K contains a diagonal that is bounded below in norm, which contradicts the fact that K is compact. Therefore such K_1 and K_2 cannot exist.□

This gives a negative answer to Problem 3 in [9].

PROBLEM. Can a Fredholm pair of index zero be compactly perturbed to an invertible pair?

COROLLARY 3.5. The pair (T, T) can be perturbed by compacts to an invertible commuting pair if and only if T can be perturbed by a compact to an invertible operator.

EXAMPLE: Let $H^2(\mathbf{D})$ be the Hardy space and T_z be the operator of multiplication by the indeterminate z. Then the pair (T_z, T_z) is a Fredholm pair of index zero that cannot be perturbed with compact operators to a pair with exact Koszul complex.

1.4 The Case of n-tuples

In this section we generalize Theorem 3.1 to n-tuples. The obstruction will again be given by the index of one of the operator coordinates of the n-tuple, and it will appear at one end of the Koszul complex. We start with a technical result.

LEMMA 4.1. Let T be such that for any n, $dim ker T^n < \infty$ and $dim ker T^n \to \infty$. If S commutes with T and the sequence $\{dim(ker S \bigcap ker T^n)\}_n$ is bounded, then there exists a sequence of nontrivial orthogonal subspaces $\mathcal{H}_n$ in H such that $P_{\mathcal{H}_n} S|\mathcal{H}_n$ is invertible, and for every m and n, $P_{\mathcal{H}_n} S|\mathcal{H}_n$ is similar to $P_{\mathcal{H}_m} S|\mathcal{H}_m$.

PROOF: Let $\mathcal{K}_n = ker T^n \ominus ker T^{n-1}$. Since $dim ker T^n \to \infty$, the spaces $\mathcal{K}_n$ are nontrivial. Moreover, the operator $P_{\mathcal{K}_n} T|\mathcal{K}_n : \mathcal{K}_n \to \mathcal{K}_{n-1}$ is injective, therefore $dim \mathcal{K}_n \leq dim \mathcal{K}_{n-1}$. This shows that the sequence $dim \mathcal{K}_n$, $n \in \mathbf{N}$ is a decreasing sequence of natural numbers, so it becomes stationary. It follows that there exists a number n_0 such that for $n \geq n_0$, the operator $P_{\mathcal{K}_{n-1}} T|\mathcal{K}_n$ is an isomorphism. Since for every n, $ker T^n \subset ker T^{n+1}$ and the sequence $\{dim(ker S \bigcap ker T_n)\}_n$ is bounded, there exists a number $n_1 > n_0$ such that for $n \geq n_1$, the operator $P_{\mathcal{K}_n} S|\mathcal{K}_n$ is injective, hence invertible. Moreover, the operator $P_{\mathcal{K}_n} T|\mathcal{K}_n$ defines a similarity between $P_{\mathcal{K}_n} S|\mathcal{K}_n$ and $P_{\mathcal{K}_{n+1}} S|\mathcal{K}_{n+1}$ for every $n \geq n_1$. Taking $\mathcal{H}_n = \mathcal{K}_{n+n_1}$, $n \geq 0$, we obtain a sequence of spaces with the desired property.□

THEOREM 4.2. Let $(T_1, T_2, \cdots, T_n)$ be a commuting n-tuple with T_1 Fredholm and $indT_1 \neq 0$. If for each k, $2 \leq k \leq n$, there exists an analytic function of two variables f_k such that

1. $f_k(0, w) = 0$ implies $w = 0$,
2. $f_k(T_1, T_k) = L_k$, L_k compact,

then the n-tuple $(T_1, T_2, \cdots, T_n)$ cannot be perturbed with compact operators to an invertible n-tuple.

PROOF: Suppose that such compacts $K_1, K_2, \cdots, K_n$ exist. Denote $S_i = T_i + K_i$. Then S_1 is Fredholm of nonzero index, we may assume $indS_1 > 0$. We remark that for every k, $2 \leq k \leq n$ the operator $N_k = f_k(S_1, S_k)$ is compact. Consider the analytic function $f : \mathbf{C}^n \to \mathbf{C}^n$, $f(z_1, z_2, \cdots, z_n) = (z_1, f_2(z_1, z_2), \cdots, f_n(z_1, z_n))$. Then $f^{-1}(0) = 0$, and since $(S_1, S_2, \cdots S_n)$ is invertible, from the spectral mapping theorem it follows that $(S_1, N_2, \cdots, N_n)$ is also invertible. Let us show that this is not possible.

Let k be the smallest integer with the property that the sequence $\{dim(kerS_1^m \bigcap kerN_2 \bigcap \cdots \bigcap kerN_k)\}_m$, is bounded. Such a k exists, for by the spectral mapping theorem $(S_1^m, N_2, \cdots, N_n)$ is invertible for every m, hence $kerS_1^m \bigcap kerN_2 \bigcap \cdots \bigcap kerN_n = 0$. Consider the subspace $H_0 = kerN_2 \bigcap \cdots \bigcap N_{k-1}$ (in case $k = 2$ take $H_0 = H$). Since the operators $S_1, N_2, \cdots, N_k$ commute, H_0 is invariant for S_1 and N_k. Moreover, because of the minimality of k, the operators $S_1|H_0$ and $N = N_k|H_0$ satisfy the hypothesis of the previous lemma.

Let $\mathcal{H}_m$ be the spaces obtained by applying the lemma. Since $P_{\mathcal{H}_1}N|\mathcal{H}_1$ is invertible, its spectral radius r is nonzero, so because of the similarity we have $\|P_{\mathcal{H}_m}N|\mathcal{H}_m\| \geq r > 0$ for every m, which contradicts the fact that N is compact.

This proves the theorem.□

COROLLARY 4.3. If T is Fredholm with $indT \neq 0$, and $k_1, k_2, \cdots, k_n$ are positive integers, then the n-tuple $(T^{k_1}, T^{k_2}, \cdots, T^{k_n})$ has index equal to zero, but cannot be perturbed with compact operators to an invertible n-tuple.

EXAMPLE: Let T_z be the operator of multiplication by the indeterminate z, acting on the Hardy space $H^2(\mathbf{D})$. Then the n-tuple $(T_z, T_z, \cdots, T_z)$ cannot be compactly perturbed to an invertible n-tuple.

The following result for triples shows that the obstruction might also be provided by a subtuple.

PROPOSITION 4.4. If the pair (T_1, T_2) is Fredholm of positive index then the triple $(T_1, T_2, 0)$ is Fredholm of index zero and cannot be perturbed with compact operators to an invertible one.

PROOF: Suppose that there exist compact operators K_1, K_2 and K_3 such that the triple (T_1+K_1, T_2+K_2, K_3) is invertible. Let $S_1 = T_1+K_1$ and $S_2 = T_2+K_2$. By Theorem 1.3, $ind(S_1^n, S_2) = n \cdot ind(S_1, S_2) = n$, which shows that $dimH^0(S_1^n, S_2) + dimH^2(S_1^n, S_2) \to \infty$ for $n \to \infty$. So there is a sequence of positive integers $\{n_k\}_k$ such that either $dimH^0(S_1^{n_k}, S_2) \to \infty$ or $dimH^2(S_1^{n_k}, S_2) \to \infty$. Without loss of generality we may assume that $dimH^0(S_1^{n_k}, S_2) \to \infty$. Since $H^0(S_1^n, S_2) = kerS_1^n \cap kerS_2$ and $kerS_1^n \subset kerS_1^{n+1}$ we get that $dim(kerS_1^n \cap kerS_2) \to \infty$ for $n \to \infty$. On the other hand, from the spectral mapping theorem it follows that (S_1^n, S_2, K_3) is invertible for any positive integer n hence $kerS_1^n \cap kerS_2 \cap kerK_3 = 0$. Therefore we can apply Lemma 4.1 to the space $kerS_2$, and to the operators $S_1|kerS_2$ and $K_3|kerS_2$. Using the same idea as in the proof of Theorem 4.2 we contradict the compactness of K_3, which proves the claim.□

EXAMPLE: Let $\mathbf{H}^2(\mathbf{D}^2)$ be the Hardy space on the bidisk, and let T_{z_1} and T_{z_2} be the two shifts defined by $T_{z_1}f(z_1,z_2) := z_1 f(z_1,z_2)$, $T_{z_2}f(z_1,z_2) := z_2 f(z_1,z_2)$, $f \in \mathbf{H}^2(\mathbf{D}^2)$. It is well known that the pair (T_{z_1}, T_{z_2}) is Fredholm of index 1 [5]. Therefore $(T_{z_1}, T_{z_2}, 0)$ is a Fredholm triple of index zero that cannot be perturbed with compact operators to a commuting invertible triple.

It is still not known whether such a result is true in general.

QUESTION: Let $T = (T_1, T_2, \cdots, T_n)$ be a Fredholm n-tuple with $indT \neq 0$, and let $p \in \mathbf{C}[z_1, z_2, \cdots, z_n]^m$ be a polynomial map satisfying $p(0) = 0$. Is it true that the $(m+n)$-tuple $(T, p(T))$ cannot be perturbed with compact operators to a commuting $(m+n)$-tuple whose Koszul complex is exact?

CHAPTER 2
RINGS WITH TOPOLOGIES INDUCED BY SPACES OF FUNCTIONS

2.1 Introduction

In the case of the Hardy space of the unit disk, the invariant subspaces of the operator of multiplication by the independent variable are completely characterized by a well-known theorem of Beurling in terms of inner functions. However, in the case of several variables, this characterization proves to be very difficult.

In the recent years, the theory of Hilbert modules developed by R. G. Douglas and V. Paulsen [11] provided some useful methods to approach this problem. The first jointly invariant subspaces for the multiplication operators studied in this context were the ones that are closures of ideals of polynomials. A surprising result, the Rigidity Theorem [12], shows that unlike the one variable case (in which any two invariant subspaces are unitary equivalent as Hilbert modules), if two subspaces are unitary equivalent as modules, they must coincide. Another result, the characterization of invariant subspaces of finite codimension done by P. Ahern and D. N. Clark (see [1]) proved also to be very natural in this setting. The techniques involved come from commutative algebra and algebraic geometry.

Among the ideals of polynomials in several variables, a special role is played by those that are closed in the relative topology induced on the ring of polynomials by the Hardy space of the polydisk. These ideals can be put in one-to-one correspondence with the invariant subspaces that are their closures in the Hardy space, thus, they can be used to "label" invariant subspaces. Several properties of subspaces can be proved by using their associated ideals.

In what follows we will restrict ourselves to the study of closed ideals. Some results have also been obtained in [12]; let us note in that paper the authors call these ideals contracted. Instead of doing everything the way it is usually done, namely by considering dense rings in Hilbert modules, we will consider Noetherian rings endowed with topologies carrying properties similar to those induced by spaces of functions. The module can be recovered as the topological completion of the ring. Among other things, we prove that if an ideal is closed, then every prime ideal associated to it is closed as well (thus answering a question in [12]), and we prove in dimension two a conjecture of Douglas and Paulsen. Our results will be obtained by using as a major tool the primary decomposition of ideals. The results from this chapter have appeared in [17] and [20].

For a better understanding of the topic let us first discuss the case of the ring of polynomials in one variable with the topology induced by the Hardy space of the unit disk $\mathbf{D}$. Recall that the topology is given by the norm $\|f\|_2 := \sqrt{\Sigma |a_k|^2}$, where $f(z) = \Sigma a_k z^k$. Let us remark that our ring is almost topological, just that multiplication is not continuous, only separately continuous.

Let $I \subset \mathbf{C}[z]$ be an ideal. Then I is generated by some polynomial f. Let $f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$, where $z_1, z_2, \cdots, z_n$ are the (non-necessarily distinct) roots of f. Assume that $z_1, z_2, \cdots, z_r \in \mathbf{D}$, and $z_{r+1}, \cdots, z_n \notin \mathbf{D}$. It is well known that the closure of I is the ideal generated by $(z - z_1)(z - z_2) \cdots (z - z_r)$, and we see that f can be written as the product $g \cdot h$, where $g := (z - z_1)(z - z_2) \cdots (z - z_r)$ generates a closed ideal, and $h = (z - z_{r+1}) \cdots (z - z_n)$ generates a dense ideal. In particular I is closed if and only if all the roots of f are in $\mathbf{D}$ and dense if all the roots lie outside $\mathbf{D}$. Reformulating, I is closed if and only if all the irreducible components of its zero set intersect $\mathbf{D}$, and dense if its zero set is disjoint from $\mathbf{D}$.

If $z_0 \in \mathbf{C}$, the ideal $\mathcal{M}_{z_0}$ generated by $(z - z_0)$ is maximal; it is closed if and only if $z_0 \in \mathbf{D}$. In this case all the powers of this ideal are closed as well. Let us remind that the I-adic topology, determined by an ideal I in a ring $\mathcal{R}$ (see [4]). is the topology characterized by the fact that the closure of a set $A \subset \mathcal{R}$ is $\cap_n(A+I^n)$. Returning to our situation, we see that the $\mathcal{M}_{z_0}$-adic topology is weaker than the Hardy space topology if and only if $z_0 \in \mathbf{D}$.

If we think about $\mathbf{D}$ as a set of distinguished maximal ideals, then we can say that $\mathbf{C}[z]$ endowed with the topology induced by the Hardy space satisfies a topological Hilbert Nullstellensatz, in the sense that an ideal is either dense, or there is a maximal ideal $\mathcal{M} \in \mathbf{D}$ that contains it. Let us now go back to the general setting.

2.2 Hilbert Nullstellensatz for Closed Ideals

Throughout the chapter $\mathcal{R}$ will denote a commutative Noetherian ring with unit, endowed with a topology τ for which addition is continuous and multiplication is separately continuous in each variable.

An example of such a ring is the ring $\mathbf{C}[z_1, z_2, \cdots, z_n]$ of polynomials in n variables with the topology induced by the Hardy space of the polydisk $H^2(\mathbf{D}^n)$, or the Bergman space $L^2_a(\Omega)$ of an open set $\Omega \subset \mathbf{C}^n$. Another example is the ring $\mathcal{O}(\bar{\mathbf{B}})$ of analytic functions in a neighborhood of the closed unit ball $\bar{\mathbf{B}} \subset \mathbf{C}^n$ with the topology induced by $L^2_a(\mathbf{B})$.

Let us remind some basic algebraic facts. For a certain ideal $I \subset \mathcal{R}$ the radical of I is $rad(I) := \{f \in \mathcal{R} | \ \exists n, \ f^n \in I\}$. An ideal P is prime if $f \cdot g \in P$ implies that either $f \in P$ or $g \in P$. An ideal Q is primary if $f \cdot g \in Q$ and $g \notin Q$ implies that $f^n \in Q$ for some power n. If Q is primary then $P := rad(Q)$ is prime, and we say that Q is P-primary. Since $\mathcal{R}$ is Noetherian, every ideal $I \subset \mathcal{R}$ has a (minimal) primary decomposition $I = Q_1 \cap Q_2 \cap \cdots Q_m$ where each Q_i is P_i-primary for some prime ideal P_i. The ideals P_i, $1 \leq i \leq m$ are called the prime ideals associated to I, and we let $\mathcal{P}(I) := \{P_1, P_2, \cdots, P_m\}$. A result that will be used often in the sequel is the Hilbert Nullstellensatz, which states that the set of maximal ideals of $\mathbf{C}[z_1, z_2, \cdots, z_n]$ can be canonically identified with $\mathbf{C}^n$.

As mentioned above, the ring $\mathcal{R}$ is not necessarily a topological ring, and also it is not usually complete in the topology τ. This is where the difficulty lies in developing such a theory. In what follows we study properties of ideals that are closed in the topology τ.

LEMMA 2.1. The radical of a closed ideal is closed.

PROOF: Let I be closed, and $rad(I)$ be its radical. By Proposition 7.14 in [4] there exists an integer k such that $rad(I)^k \subset I$. Let f_n be a sequence of elements in $rad(I)$ converging to some f. We want to prove that $f \in rad(I)$. Since the multiplication is continuous in each variable, for every $g \in rad(I)^{k-1}$, $f_n g \to fg$, hence $fg \in I$, since I is closed. This shows that in particular $fg \in I$ for every $g \in rad(I)^{k-1}$. Repeating the argument we get $f f_n g \to f^2 g$ for any $g \in r(I)^{k-2}$, hence $f^2 g \in I$ for $g \in rad(I)^{k-2}$. Inductively we get $f^r g \in I$, for $g \in r(I)^{k-r}$ and $0 \leq r \leq k$, so $f^k \in I$ which shows that $f \in rad(I)$.$\square$

If Q is P-primary, the lemma above shows that Q closed implies that P is closed. The following result shows that this is true in a more general setting.

THEOREM 2.2. If an ideal I is closed, then every prime ideal $P \in \mathcal{P}(I)$ is closed.

PROOF: If I is closed and $f \in \mathcal{R}$, then the ideal $(I : f) := \{g \in \mathcal{R}, gf \in I\}$

is closed. Indeed, if $g_n \in (I : f)$ and $g_n \to g$ then $g_n f \to gf$, so $gf \in I$ which shows that $g \in (I : f)$.

From Lemma 2.1 we get that $rad((I : f))$ is closed for every f. By Theorem 4.5. in [4] every prime ideal associated to I is of this form hence it is closed.$\square$

This gives a positive answer to a question raised in [12]. The converse of this theorem is not always true; for example, if we endow $\mathbf{C}$ with the topology induced by $A(\mathbf{D})$, the Dirichlet algebra of the unit disk, by choosing $z_0 \in \partial\mathbf{D}$ we get that the ideal generated by $(z - z_0)$ is closed, but the one generated by $(z - z_0)^2$ is not.

REMARK. Since the closure of an ideal is an ideal, a maximal ideal is either closed or dense.

Given an ideal $J \subset \mathcal{R}$, the J-adic topology on $\mathcal{R}$ is the topology determined by the powers of J, so in this topology the closure of a set $A \subset \mathcal{R}$ is $\cap_n (A + J^n)$. For more details the reader can consult [4].

Let

$$\mathcal{C} := \{\mathcal{M} \subset \mathcal{R}, \mathcal{M} \text{ maximal ideal and the } \mathcal{M}\text{-adic topology is weaker than } \tau\}.$$

We see that $\mathcal{C}$ consists of those maximal ideals $\mathcal{M}$ for which $\mathcal{M}^n$ is dense for every integer n. As an example, if $\mathcal{R} = \mathbf{C}[z_1, z_2, \cdots, z_n]$, and τ is induced by the Hardy space of the polydisk, then $\mathcal{C}$ coincides with the polydisk, when making the usual identification between points and maximal ideals via the Hilbert Nullstellensatz. The following result is a slightly modified version of Theorem 2.7 in [12].

THEOREM 2.3. If an ideal I has the property that for every prime $P \in \mathcal{P}(I)$ there exists $\mathcal{M} \in \mathcal{C}$ with $P \subset \mathcal{M}$, then I is closed.

PROOF: Let $P_1, P_2, \cdots, P_m$ be the prime ideals associated to I, $P_i \subset \mathcal{M}_i$,

$\mathcal{M}_i \in \mathcal{C}$. If $J = \mathcal{M}_1\mathcal{M}_2 \cdots \mathcal{M}_m$ then the J-adic topology is weaker than τ, so it suffices to prove that I is closed in the J-adic topology. Without loss of generality we may assume that $I = (0)$, in which case the ideals P_i are the primes associated to (0), so by Proposition 4.7 in [4] they contain all zero divisors. Now let us suppose that (0) is not closed. By Krull's Theorem ([3, Theorem 10.7]) there exists $f \in J$ such that $1 + f$ is a zero divisor. It follows that there exists i such that $1 + f \in P_i$, so the unit can be written as a sum of an element in J and one in P_i. But this is impossible since both J and P_i are contained in $\mathcal{M}_i$. This proves the theorem.□

DEFINITION. The pair $(\mathcal{R}, \tau)$ is said to satisfy the (topological) Hilbert Nullstellensatz if every ideal $I \subset \mathcal{R}$ is either dense, or there exists $\mathcal{M} \in \mathcal{C}$ with $I \subset \mathcal{M}$.

Let us remark that if $(\mathcal{R}, \tau)$ satisfies Hilbert's Nullstellensatz then any closed ideal is contained in a maximal closed ideal, which motivates the terminology. By Krull's Theorem, $(\mathcal{R}, \tau)$ satisfies Hilbert's Nullstellensatz for every J-adic topology τ. The ring $\mathbf{C}[z]$ with the topology induced by $H^2(\mathbf{D})$ or $L^2_a(\mathbf{D})$ also satisfies this property. In [33], a class of strongly pseudoconvex domains Ω for which $\mathcal{O}(\bar{\Omega})$ with the topology induced by $L^2_a(\Omega)$ satisfies the topological Hilbert Nullstellensatz has been exhibited.

LEMMA 2.4. If I and J are two dense ideals in $\mathcal{R}$ then $I \cdot J$ is dense.

PROOF: Let $f_n \to 1$, $n \to \infty$, $f_n \in I$. If $g \in J$ then $f_n g \to g$ which shows that $I \cdot J$ is dense in J, hence dense in $\mathcal{R}$.□

THEOREM 2.5. If $(\mathcal{R}, \tau)$ satisfies Hilbert's Nullstellensatz then an ideal $I \subset \mathcal{R}$ is closed if and only if every prime associated to I is closed. Moreover, the closure of an ideal in $\mathcal{R}$ is equal to the intersection of its primary components that are contained in closed maximal ideals.

PROOF: If I is closed then every prime associated to I is closed by Theorem 2.2. Conversely, if a prime associated to I is closed, then it is not dense, so it is included in an $\mathcal{M} \in \mathcal{C}$. The fact that I is closed now follows from Theorem 2.3.

For the second part, let $\bar{I}$ be the closure of I in $\mathcal{R}$, and let $I = \cap_{i=1}^{m} Q_i$ be a (minimal) primary decomposition of I, such that $Q_1, Q_2, \cdots, Q_r$ are included in maximal ideals that are in $\mathcal{C}$, hence closed, and $Q_{r+1}, Q_{r+2}, \cdots, Q_m$ are not, so they are dense. Then from the first part of the proof we get $\bar{I} \subset \cap_{i=1}^{n} Q_i$.

On the other hand, by Lemma 2.4 the ideal $Q_{r+1}Q_{r+2}\cdots Q_m$ is dense in $\mathcal{R}$, hence $Q_{r+1}Q_{r+2}\cdots Q_m(Q_1 \cap Q_2 \cap \cdots \cap Q_r)$ is dense in $Q_1 \cap Q_2 \cap \cdots \cap Q_r$, hence I is dense in $Q_1 \cap Q_2 \cap \cdots \cap Q_r$, so $\bar{I} = Q_1 \cap Q_2 \cap \cdots \cap Q_r$.$\square$

This result shows that in a ring that satisfies the topological Hilbert Nullstellensatz, the closed ideals can be easily classified. In particular, in the case when $\mathcal{R}$ is the ring of polynomials and $\tau$is induced by the Hardy space we see that the following conjecture of R. G. Douglas and V. Paulsen ([12], [31]) is equivalent to the fact that the topological Hilbert Nullstellensatz is satisfied.

CONJECTURE. Let $\mathcal{R} = \mathbf{C}[z_1, z_2, \cdots, z_n]$ be endowed with the topology induced by $H^2(\mathbf{D}^n)$. Then an ideal I is closed if and only if every irreducible component of the zero set of I intersects $\mathbf{D}^n$.

2.3 The Case of the Bidisk

In this section we prove the above mentioned conjecture for the case of two variables. Let us denote by $\mathbf{T}^2$ the 2-dimensional torus $\{(z_1, z_2) \in \mathbf{C}^2 | \; |z_i| = 1, i = 1, 2\}$.

LEMMA 3.1. If $\alpha \in \mathbf{C}$, $|\alpha| \geq 1$ and $1/2 < r < 1$ then for any z with $|z| = 1$ we have $|(z - \alpha)/(rz - \alpha)| \leq 2$.

PROOF: The result follows from $|z-\alpha| \leq |z-\alpha/r|$. The last inequality is obvious since in the triangle formed by the points z, α and α/r the angle at α is obtuse.$\square$

LEMMA 3.2. Let $p(z) = a_n(z-z_1)(z-z_2)\cdots(z-z_n)$ be such that $|z_i| \geq 1$. If $1/2 < r < 1$ then for any z with $|z| = 1$, $|p(z)/p(rz)| \leq 2^n$.

PROOF: The result follows by applying the previous lemma to each of the factors in the decomposition of p.$\square$

PROPOSITION 3.3. Let $p \in \mathbf{C}[z_1, z_2, \cdots, z_n]$ be a polynomial having no zeros inside $\mathbf{D}^n$. Then $pH^2(\mathbf{D}^n)$ is dense in $H^2(\mathbf{D}^n)$.

PROOF: For a fixed k and r, $1/2 < r < 1$, the polynomial $p(rz_1, \cdots, rz_{k-1}, z_k, \cdots, z_n)$ has no zeros in $\mathbf{D}^n$, so if we consider z_k as the independent variable, by Lemma 3.2 we get

$$|p(rz_1, \cdots, rz_{k-1}, z_k, \cdots, z_n)/p(rz_1, \cdots, rz_k, \cdots, z_n)| \leq 2^{deg_k p} \tag{2.1}$$

for all $(z_1, z_2, \cdots, z_n) \in \mathbf{D}^n$, and $1/2 < r < 1$, where $deg_k p$ is the degree of p in z_k.

Multiplying these inequalities for $1 \leq k \leq n$ we get

$|p(rz_1, z_2, \cdots, z_n)/p(rz_1 rz_2, \cdots, rz_n)| \leq 2^{n \cdot degp}$, $\forall (z_1, z_2, \cdots, z_n) \in \mathbf{D}^n, 1/2 < r < 1$, where $degp$ is the total degree of p. By continuity, the same inequality holds on $\mathbf{T}^n$.

Let $f_r(z) := p(z)/p(rz)$, $z = (z_1, z_2, \cdots, z_n) \in \mathbf{D}^n$. Since p has no zeros in $\mathbf{D}^n$ we see that $f_r \in pH^2(\mathbf{D}^n)$. If we show that $f_r \to 1$ for $r \to 1$ in the L^2-norm, then we are done.

The set $A := V(p) \cap \mathbf{T}^n$, where $V(p)$ is the zero set of p, has measure zero on the torus, and $f_r \to 1$ uniformly on compact subsets of $\mathbf{T}^n \backslash A$. For $\epsilon > 0$, choose W a neighborhood of A on the torus, with measure smaller than $\epsilon/(2(2^m+1)^2)$. Also choose r_0, $1/2 < r_0 < 1$, such that for $r > r_0$, $\|f_r - 1\|^2_{2,\mathbf{T}^n \backslash W} < \epsilon/2$. It follows that for $r > r_0$, $\|f_r - 1\|_2^2 < \|f_r - 1\|_{2,\mathbf{T}^n \setminus W} + \|f_r - 1\|_{2,W} < (2^m + 1)^2 \epsilon / (2(2^m + 1)^2) + \epsilon/2 = \epsilon$, which proves the assertion.$\Box$

REMARK. We see that the only nontrivial situations where this result applies are those when the zero set of the polynomial touches the boundary of $\mathbf{D}^n$. Here are some examples of such polynomials: $z_1 z_2 - 1$, $z_1 + z_2 - 2$, $2z_1 z_2 + z_1 + z_2 + 2$.

THEOREM 3.4. The ring $\mathbf{C}[z_1, z_2]$, with the topology induced by the Hardy space, satisfies the topological Hilbert Nullstellensatz.

PROOF: Let us first prove the property for prime ideals. Using the classical Hilbert Nullstellensatz we see that the only maximal ideals in $\mathcal{C}$ are those corresponding to points in $\mathbf{D}^2$. Moreover, the other maximal ideals are dense. So by Theorem 2.3 we only have to show that if P is prime and $V(P) \cap \mathbf{D}^2 = \emptyset$, where $V(P)$ is the zero set of P, then P is dense in $\mathbf{C}[z_1, z_2]$. Standard results in dimension theory (see [4]) show that P is either maximal or principal. Indeed the maximal length of a chain of nonzero prime ideals containing P is 2. If we have $P_0 \subset P$ then P is maximal and the result follows easily. If $P \subset P_1$ then P must be principal since if P is generated by $g_1, g_2, \cdots, g_k$ we can take g_1 to be irreducible (using the fact that P is prime), and then the ideal generated by g_1 is included in P, so it must coincide with P. The density in the second case follows from Proposition 3.3.

If I is an arbitrary ideal having no zeros in $\mathbf{D}^2$, let us show that it is dense. If $P_1, P_2, \cdots, P_m$ are the primes associated to it then from what has been established above it follows that each P_i is dense. By Proposition 7.14 in [4] there exists an integer k such that $(P_1 \cdot P_2 \cdot \ldots \cdot P_n)^k \subset I$. It follows from Lemma 2.4 that I itself is dense in $\mathbf{C}[z_1, z_2]$, which proves the theorem.$\Box$

As a direct consequence of this result and Theorem 2.5 we get

COROLLARY 3.5. Let $\mathbf{C}[z_1, z_2]$ be endowed with the topology induced by the Hardy space. Then an ideal is closed if and only if each of the irreducible components of its zero set intersects $\mathbf{D}^2$.

COROLLARY 3.6. If I is a *principal* ideal in $\mathbf{C}[z_1, z_2, \cdots, z_n]$ and we endow this ring with the topology induced by the Hardy space, then I is closed if and only if each algebraic component of its zero set intersects $\mathbf{D}^n$.

2.4 The Case of Reinhardt Domains

In the sequel we are going to describe another situation in which the topological Hilbert Nullstellensatz is true. Following [10] we introduce the pseudoconvex domains in $\mathbf{C}^2$

$$\Omega_{p,q} := \{z = (z_1, z_2) \in \mathbf{C}^2 \ \mid \ |z_1|^p + |z_2|^q < 1\}, \ (1 \leq p, q < \infty).$$

A study of the Bergman spaces of these domains has been done in [10]. We will prove that $\mathbf{C}[z_1, z_2]$ with the topology induced by the L^2-norm on $\Omega_{p,q}$ satisfies the topological Hilbert Nullstellensatz. This will be done in two stages. First, we prove that $\mathcal{C}$ coincides with the set of maximal ideals coming from points in $\Omega_{p,q}$, and the other maximal ideals are dense. Then, we prove a density result analogous to Proposition 3.3, and conclude that the Nullstellensatz holds. We start with some technical results.

LEMMA 4.1. If $0 \leq a \leq 1$ then the series

$$\sum_{m,n \geq 0} \binom{m+n}{m} a^m (1-a)^n \tag{2.2}$$

diverges.

PROOF: By symmetry, we can assume $a < 1$. We have

$$\sum_{m,n\geq 0}\begin{pmatrix} m+n \\ m \end{pmatrix} a^m(1-a)^n = \sum_{p\geq 0}\sum_{q\leq p}\begin{pmatrix} p \\ q \end{pmatrix} a^q(1-a)^{p-q} =$$

$$= \sum_{p\geq 0}(1-a)^p\sum_{q\leq p}\begin{pmatrix} p \\ q \end{pmatrix} a^q(1-a)^{p-q} =$$

$$= \sum_{p\geq 0}(1-a)^p\sum_{q\leq p}\begin{pmatrix} p \\ q \end{pmatrix} (\frac{a}{1-a})^q = \sum_{p\geq 0}(1-a)^p\cdot\frac{1}{(1-a)^p} = \infty.\Box$$

COROLLARY 4.2. If $0\leq a\leq 1$ then the series

$$\sum_{m,n>0}\begin{pmatrix} 2(m+n)-2 \\ 2m-1 \end{pmatrix} a^{2m-1}(1-a)^{2n-1} \tag{2.3}$$

diverges.

Let B denote the beta function, defined by $B(r,s) := \Gamma(r)\Gamma(s)/\Gamma(r+s)$, $r,s>0$.

By [6],

$$B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}dt,\ (r,s>0).$$

LEMMA 4.3. If $(z_1,z_2)\in\partial\Omega_{p,q}$ then the series

$$\sum_{r_1,r_2\geq 0}\frac{|z_1|^{2r_1}|z_2|^{2r_2}}{B(\frac{2r_1+2}{p},\frac{2r_2+2}{q}+2)}$$

diverges.

PROOF: Since $B(r,s)$ is a decreasing function in r and s, by taking the subseries corresponding to indices with the property that $pm\leq r_1\leq pm+1$ and $qn\leq r_2\leq qn+1$ we get the inequality

$$\sum_{r_1,r_2\geq 0}\frac{|z_1|^{2r_1}|z_2|^{2r_2}}{B(\frac{2r_1+2}{p},\frac{2r_2+2}{q}+2)} \geq |z_1|^{p+2}|z_2|^{q+2}\sum_{m,n\geq 0}\frac{|z_1|^{(2m-1)p}|z_2|^{(2n-1)q}}{B(2m,2n)}. \tag{2.4}$$

Since $|z_1|^q = 1 - |z_2|^p$ the sum of the latter series is equal to

$$\sum_{m,n\geq 0} (|z_1|^p)^{2m-1}(1-|z_1|p)^{2n-1}/B(2m,2n).$$

Applying Corollary 4.2 for $a = |z_1|^p$, and using the fact that

$$B(2m,2n) = (2m-1)!(2n-1)!/(2m+2n-1)! < 1/\binom{2(m+n)-2}{2m-1}$$

(since $\Gamma(k) = (k-1)!$ for k a positive integer) we get the desired result.□

Recall that $\mathcal{C}$ is the set of maximal ideals $\mathcal{M}$ with the property that the $\mathcal{M}$-adic topology is weaker than the Bergman space topology, and that by the classical Hilbert Nullstellensatz it can be identified with a subset of $\mathbf{C}^2$.

PROPOSITION 4.4. If we endow the ring $\mathbf{C}[z_1, z_2]$ with the topology induced by $L^2(\Omega_{p,q})$, then $\mathcal{C} = \Omega_{p,q}$, and if $\mathcal{M} \notin \mathcal{C}$ then $\mathcal{M}$ is dense.

PROOF: By Theorem 4.9 c) and Example 5.2 in [10], an ideal $\mathcal{M} = (z_1 - w_1, z_2 - w_2)$ is dense in $\mathbf{C}[z_1, z_2]$ with the L^2-norm if and only if

$$\frac{2(\pi)^2}{p} \sum_{r_1,r_2\geq 0} (r_2+1)\frac{|w_1|^{2r_1}|w_2|^{2r_2}}{B(\frac{2r_1+2}{p}, \frac{2r_2+2}{q}+1)}. \tag{2.5}$$

diverges.

By Lemma 4.3 this series diverges if $(w_1, w_2) \in \partial\Omega_{p,q}$, so if $w_1, w_2 \notin \Omega_{p,q}$ the ideal $\mathcal{M}$ is dense.

It is not difficult to check that $\mathcal{M}^m$ is closed whenever $(w_1, w_2) \in \Omega_{p,q}$ and $m \in \mathbf{N}$. Indeed, if we consider a polydisk centered in (w_1, w_2) contained in $\Omega_{p,q}$, the topology τ induced by the L^2-norm on this polydisk is weaker than the one induced by the L^2-norm on $\Omega_{p,q}$, and the ideals $\mathcal{M}^n$ are all closed in this topology, so they are also closed in τ. It follows that $\mathcal{C} = \Omega_{p,q}$, and if $\mathcal{M} \notin \mathcal{C}$ then $\mathcal{M}$ is dense.□

The following result is analogous to Proposition 3.3.

PROPOSITION 4.5. Let Ω be a bounded, complete Reinhardt domain and let $p \in \mathbf{C}[z_1, z_2, \cdots, z_n]$ be a polynomial having no zeros in Ω. Then $pL^2_a(\Omega)$ is dense in $L^2_a(\Omega)$.

PROOF: Since the Reinhardt domain is complete, for any r, $1/2 < r < 1$, $p(rz)$ has no zeros inside Ω. The proof of Proposition 3.3 applies mutatis mutandis to show that $f_r(z) := p(z)/p(rz)$ is a family of functions in $L^2_a(\Omega)$ that is uniformly bounded. Since this family converges uniformly on compacts to 1, it follows that it converges in $L^2_a(\Omega)$ to 1, which proves the density.$\Box$

THEOREM 4.6 The ring $\mathbf{C}[z_1, z_2]$ with the topology induced by $L^2_a(\Omega_{p,q})$ satisfies the topological Hilbert Nullstellensatz. In particular $\mathbf{C}[z_1, z_2]$ with the topology induced by $L^2_a(\mathbf{B})$ satisfies the topological Hilbert Nullstellensatz.

PROOF: The proof is similar to that of Theorem 3.4; the second part of the statement follows from the fact that $\mathbf{B} = \Omega_{2,2}$.$\Box$

We now prove an analogous for the ring of germs of analytic functions in the neighborhood of the unit ball. Let us denote by $\mathbf{B}^2$ the open unit ball in $\mathbf{C}^2$ and by $\mathcal{O}(\overline{\mathbf{B}^2})$ the ring of germs of analytic functions defined in a neighborhood of $\overline{\mathbf{B}^2}$. We want to prove the topological Hilbert Nullstellensatz for $\mathcal{O}(\overline{\mathbf{B}^2})$ with the topology induced by the Bergman space. For $k \geq 1$, denote by S^k the unit sphere in $\mathbf{R}^{k+1}$ and by $\mathbf{D}$ the unit disk in the plane. We will need the following technical result.

LEMMA 4.7. Let $f \in \mathcal{O}(\overline{\mathbf{B}^2})$, f not identically equal to zero. Then there exists a transformation $\rho : \mathbf{C}^2 \to \mathbf{C}^2$ such that $f \circ \rho(0, e^{i\alpha}) \neq 0$ for all $\alpha \in [0, 2\pi)$.

PROOF: For $S^3 = \partial \mathbf{B}^2$ consider the Hopf fibration

$$S^1 \to S^3 \xrightarrow{\pi} S^2$$

where we recall that the projection π is given by the equivalence relation $(a, b) \sim (\lambda a, \lambda b)$ for $|\lambda| = 1$. If we denote by $V(f)$ the zero set of f, then $V(f) \cap S^3$ has dimension at most one, hence $\pi(V(f)\cap S^3)$ has dimension at most one. It follows that there exists $x \in S^2\backslash\pi(V(f)\cap S^3)$. Let $(a,b)\in \mathbf{C}^2$, $|a|^2+|b|^2=1$ with the property that $\pi(a,b)=x$. Then $f(ae^{i\alpha},be^{i\alpha})\neq 0$ for all $\alpha\in[0,2\pi)$. If we choose $\rho = \begin{pmatrix} \bar{b} & a \\ -\bar{a} & b \end{pmatrix}$ then $f\circ\rho$ satisfies the desired property.$\square$

PROPOSITION 4.8. Let $f\in\mathcal{O}(\overline{\mathbf{B}^2})$ be an analytic function having no zeros in $\mathbf{B}^2$. Then the space $fL^2_a(\mathbf{B}^2)$ is dense in $L^2_a(\mathbf{B}^2)$.

PROOF: By Lemma 4.7 we may assume that if $(w_1,w_2)\in\overline{\mathbf{B}^2}$ and $w_1=0$ then $f(w_1,w_2)\neq 0$. Let us show that there exists $C>0$ such that for every $1/2<r<1$ and $(z_1,z_2)\in\overline{\mathbf{B}^2}$, $|f(z_1,z_2)/f(rz_1,z_2)|<C$.

Fix $(w_1,w_2)\in\overline{\mathbf{B}^2}$, and assume that $w_1\neq 0$. Note that $f(z,w_2)$ is not identically zero as a function of z_1, otherwise $(0,w_2)$ would be a zero for f lying inside of $\mathbf{B}^2$. Thus there exists $a>0$ such that $w_1\in a\mathbf{D}$ and $f(\cdot,w_2)$ is defined in a neighborhood of $\overline{a\mathbf{D}}$ and has no zeros on $aS^1=\{z \mid |z|=a\}$. It follows that $f(\cdot,w_2)$ has a finite number of zeros in $a\mathbf{D}$, say m, counting multiplicities. By Rouché's and has no zeros on $aS^1=\{z\mid |z|=a\}$. It follows that $f(\cdot,w_2)$ has a finite number of zeros in $a\mathbf{D}$, say m, counting multiplicities. By Rouché's Theorem there is a compact neighborhood K of w_2 such that $\overline{a\mathbf{D}}\times K$ is contained in the domain of f, and for every $z_2\in K$, $f(\cdot,z_2)$ has exactly m zeros in $a\mathbf{D}$, counting multiplicities, and no zero on aS^1. Thus on $\overline{a\mathbf{D}}\times K$ we can write $f(z_1,z_2)=p_{z_2}(z_1)g_{z_2}(z_1)$ where for each z_2, $p_{z_2}(z_1)$ is a polynomial of degree m and $g_{z_2}(z_1)$ is an analytic function having no zeros in $\overline{a\mathbf{D}}$. Another application of Rouché's Theorem and the maximum modulus principle shows that g_{z_2} depends continuously on z_2.

It follows that for $1/2\le r\le 1$ the family $\{g_{z_2}(rz_1)\}_r$ is bounded away from zero, hence

$$C_1 = sup_{1/2 \leq r \leq 1} sup_{\overline{a\mathbf{D}} \times K} |g_{z_2}(z_1)/g_{z_2}(rz_1)| < \infty.$$

By Proposition 3.2 for $1/2 \leq r \leq 1$ and $(z_1, z_2) \in \overline{a\mathbf{D}} \times K \cap \overline{\mathbf{B}^2}$

$$|p_{z_2}(z_1)/p_{z_2}(rz_1)| \leq 2^m.$$

Thus there exists a neighborhood U of (w_1, w_2) and a constant $C_2 > 0$ such that for $1/2 < r < 1$ and $(z_1, z_2) \in U \cap \overline{\mathbf{B}^2}$

$$|f(z_1, z_2)/f(rz_1, z_2)| < C_2.$$

If $w_1 = 0$ then $f(z_1, z_2) \neq 0$ in a neighborhood of (w_1, w_2), thus a similar inequality holds there. From the compactness of $\overline{\mathbf{B}^2}$ it follows that there exists a constant $C > 0$ such that for $1/2 < r < 1$ and $(z_1, z_2) \in \overline{\mathbf{B}^2}$, $|f(z_1, z_2)/f(rz_1, z_2)| < C$.

As in the proof of Proposition 4.5, the family $h_r(z_1, z_2) = f(z_1, z_2)/f(rz_1, z_2)$ is in $fL^2_a(\mathbf{B}^2)$ and tends to 1 as $r \to 1$, so the conclusion follows.□

THEOREM 4.9. The ring $\mathcal{O}(\overline{\mathbf{B}^2})$ with the topology induced by $L^2_a(\mathbf{B}^2)$ satisfies the topological Hilbert Nullstellensatz.

PROOF: The ring $\mathcal{O}(\overline{\mathbf{B}^2})$ is Noetherian [36], and has dimension 2. Indeed, if there existed distinct prime ideals $P_0 \subset P_1 \subset P_2 \subset P_3$, by localizing at a maximal ideal $\mathcal{M} \supset P_3$ we would get a chain of four distinct prime ideals in the local ring $\mathcal{O}_{\mathcal{M}}$, which would contradict the fact that the latter ring has dimension 2. So the proof of Theorem 3.4. applies to give the desired conclusion.□

COROLLARY 4.10. Let $\mathcal{O}(\overline{\mathbf{B}^2})$ be endowed with the topology induced by the Bergman space. Then an ideal is closed if and only if each irreducible component of its zero set intersects the unit ball.

2.5 Ideals of Finite Codimension

Now let us suppose that $\mathcal{R}$ is also a k-algebra, where k is an algebraically closed field, and that the scalar multiplication is *continuous* (not just separately continuous). By the classical Hilbert Nullstellensatz ([3, Corollary 5.24]), $\mathcal{R}/\mathcal{M} \cong k$ for every maximal ideal $\mathcal{M}$. Let us also assume that the family $\mathcal{C}$ defined right before the statement of Theorem 2.3. consists of all closed maximal ideals. Thus in this case a maximal ideal $\mathcal{M}$ is either dense, or the $\mathcal{M}$-adic topology is weaker then the topology of $\mathcal{R}$.

EXAMPLE. If we consider $\mathbf{C}[z_1, z_2, \cdots, z_n]$ with the topology induced by $H^2(\mathbf{D}^n)$ then the condition above is satisfied. In this case we have $\mathcal{C} = \mathbf{D}^n$.

In a similar way as we proved Theorem 2.3 we can establish the following result.

LEMMA 5.1. Given an ideal I whose associated prime ideals are maximal, I is closed if and only if every maximal ideal belonging to I is closed.

LEMMA 5.2. An ideal $I \subset \mathcal{R}$ has finite codimension in $\mathcal{R}$ if and only if every prime ideal belonging to I is maximal.

PROOF: Let $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$, Q_i $\mathcal{M}_i$-primary. By Proposition 7.14 in [4] there exists an integer n such that $\mathcal{M}_i^n \subset Q_i$, so $(\mathcal{M}_1\mathcal{M}_2 \cdots \mathcal{M}_m)^n \subset I$. Since $dim\mathcal{R}/(\mathcal{M}_1\mathcal{M}_2 \cdots \mathcal{M}_m)^n < \infty$, I has finite codimension.

For the converse, let P be a prime ideal belonging to I. Then P has finite codimension as well, so $\mathcal{R}/P$ is an integral domain that is finite over k, and since k is algebraically closed we must have $\mathcal{R}/P \cong k$, therefore P is maximal.□

Let $\widetilde{\mathcal{R}}$ be the closure of $\mathcal{R}$ in the topology τ. Since multiplication is only separately continuous, $\widetilde{\mathcal{R}}$ is no longer a ring, but it is an $\mathcal{R}$-module. Each element

$x \in \mathcal{R}$ induces a continuous multiplication morphism T_x on $\widetilde{\mathcal{R}}$. We shall denote by $\widetilde{I}$ the closure in $\widetilde{\mathcal{R}}$ of an ideal I in $\mathcal{R}$, to avoid confusion with $\bar{I}$, the closure of I in $\mathcal{R}$. Clearly $\widetilde{I}$ is a closed submodule of $\widetilde{\mathcal{R}}$. Also, if $Y \subset \widetilde{\mathcal{R}}$ is a closed submodule, then $Y \cap \mathcal{R}$ is an ideal that is closed in $\mathcal{R}$.

DEFINITION. (see [4], page 58) A submodule $Y \subset \widetilde{\mathcal{R}}$ is called primary in $\widetilde{\mathcal{R}}$ if $Y \neq \widetilde{\mathcal{R}}$ and every zero-divisor in $\widetilde{\mathcal{R}}/Y$ is nilpotent.(An element $x \in \mathcal{R}$ is called a *zero-divisor* if the morphism induced by T_x on $\widetilde{\mathcal{R}}/Y$ has nontrivial kernel, and *nilpotent* if this morphism is nilpotent).

REMARK. If $Y \subset \widetilde{\mathcal{R}}$ is primary then $(Y : \widetilde{\mathcal{R}}) := \{x \in \mathcal{R} \mid T_x\widetilde{\mathcal{R}} \subset Y\}$ is primary, so $P := r((Y : \widetilde{\mathcal{R}}))$ is prime. We say that Y is P-primary. Moreover, $(Y : \widetilde{\mathcal{R}}) = Y \cap \mathcal{R}$ so $Y \cap \mathcal{R}$ is also P-primary.

Although every ideal in $\mathcal{R}$ has a primary decomposition, this is not true in general for the submodules of $\widetilde{\mathcal{R}}$. For example in the case of the module $H^2(\mathbf{D}$ the zero sets of primary submodules consist of a single point, thus a closed submodule whose zero set inside the unit disk is infinite does not have a primary decomposition. Such a submodule can arise from a Blaschke product. The next result shows that closed submodules of $\widetilde{\mathcal{R}}$ of finite codimension admit primary decompositions.

THEOREM 5.3. There is a one-to-one correspondence between ideals in $\mathcal{R}$ whose associated prime ideals are maximal and closed in $\mathcal{R}$, and closed submodules in $\widetilde{\mathcal{R}}$ of finite codimension, given by the maps $I \to \widetilde{I}$ and $Y \to Y \cap \mathcal{R}$. Moreover, if $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ is a (minimal) primary decomposition for I, then $\widetilde{I} = \widetilde{Q}_1 \cap \widetilde{Q}_2 \cap \cdots \cap \widetilde{Q}_m$ is a (minimal) primary decomposition for $\widetilde{I}$.

PROOF: By Lemmas 5.1 and 5.2 we have to show that the maps indicated above establish a one-to-one correspondence between ideals of finite codimension that are closed in the topology of $\mathcal{R}$ and closed submodules of finite codimension in $\widetilde{\mathcal{R}}$.

If Y is a closed submodule of $\widetilde{\mathcal{R}}$ of finite codimension then $\mathcal{R}/(Y \cap \mathcal{R}) \cong \widetilde{\mathcal{R}}/Y$ since the canonical map $\mathcal{R} \to \widetilde{\mathcal{R}}/Y$ is surjective, $\mathcal{R}$ being dense in $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{R}}/Y$ being finite dimensional, and the kernel of this map is $Y \cap \mathcal{R}$. On the other hand $\widetilde{\mathcal{R}}/(\widetilde{Y \cap \mathcal{R}}) \cong \mathcal{R}/(Y \cap \mathcal{R})$, hence $Y = (\widetilde{Y \cap \mathcal{R}})$. Also for every ideal $I \subset \mathcal{R}$ that is closed in the topology of $\mathcal{R}$, $\widetilde{I} \cap \mathcal{R} = I$, so the two maps are inverses of one another, and the one-to-one correspondence is proved.

Let $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ be a primary decomposition of I. Then $\widetilde{I} \subset \widetilde{Q}_1 \cap \widetilde{Q}_2 \cap \cdots \cap \widetilde{Q}_m$, and since $\widetilde{I} \cap \mathcal{R} = \widetilde{Q}_1 \cap \widetilde{Q}_2 \cap \cdots \cap \widetilde{Q}_m \cap \mathcal{R} = Q_1 \cap Q_2 \cap \cdots Q_m = I$, by the first part of the proof the two must be equal.

In the commutative diagram below the horizontal arrows are isomorphisms

$$\begin{array}{ccc} \mathcal{R}/Q_i & \xrightarrow{\cong} & \widetilde{\mathcal{R}}/\widetilde{Q}_i \\ {\scriptstyle T_x}\downarrow & & \downarrow{\scriptstyle T_x} \\ \mathcal{R}/Q_i & \xrightarrow{\cong} & \widetilde{\mathcal{R}}/\widetilde{Q}_i \end{array}$$

so the fact that Q_i is a primary ideal (hence a primary $\mathcal{R}$-module as well) implies that $\widetilde{Q}_i$ is a primary submodule of $\widetilde{\mathcal{R}}$.

If the primary decomposition of I is minimal let us show that the corresponding decomposition for $\widetilde{I}$ is also minimal. Suppose that there exists j such that $\widetilde{I} = \widetilde{Q}_1 \cap \cdots \widetilde{Q}_{j-1} \cap \cap \widetilde{Q}_{j+1} \cap \cdots \cap \widetilde{Q}_m$. Then $I = \widetilde{I} \cap \mathcal{R} = Q_1 \cap \cdots \cap Q_{j-1} \cap Q_{j+1} \cap \cdots \cap Q_m$, which contradicts the minimality of the primary decomposition of I. The proof the theorem is complete.$\square$

From the previous proof it follows that the associated primes of I and $\widetilde{I}$ coincide. Corollary 4.11 in [4] shows that in this case the minimal primary decomposition is unique.

REMARK. In the case when the topology on $\mathcal{R}$ comes from a norm, the first part of the theorem is contained in [8], Corollary 2.8.

EXAMPLES. 1. For the case $\mathcal{R} = \mathbf{C}[z_1, z_2, \cdots, z_n]$ and $\widetilde{\mathcal{R}} = H^2(\mathbf{D}^n)$ Theorem 5.3 already appears in the work of Ahern and Clark [1]. The primary closed $\mathbf{C}[z_1, z_2, \cdots, z_n]$-submodules of finite codimension of $H^2(\mathbf{D}^n)$ are those closed submodules Y for which there exists a point $\lambda \in \mathbf{D}^n$ and a number $m \in \mathbf{N}$ such that Y contains the space of functions f that satisfy $(\partial^m/\partial z_1^m \partial^m/\partial z_2^m \cdots \partial^m/\partial z_n^m f)(\lambda) = 0$.

2. If $\mathbf{B}$ is the unit ball in $\mathbf{C}^n$, $\mathcal{R} = \mathcal{O}(\bar{\mathbf{B}})$ and $\widetilde{\mathcal{R}} = L_a^2(\mathbf{B})$ then $\mathcal{C} = \mathbf{B}$, and if $\mathcal{M}$ is a maximal ideal corresponding to a point in $\partial\mathbf{B}$ then $\mathcal{M}$ is dense in $L_a^2(\mathbf{B})$ (see [17]). This shows that the conditions listed at the beginning of this section are satisfied, so Theorem 5.3 holds. The primary closed $\mathcal{O}(\bar{\mathbf{B}})$-modules of finite codimension have the same description as in the previous example.

CHAPTER 3
TOPOLOGICAL QUANTUM FIELD THEORY WITH CORNERS BASED ON THE KAUFFMAN BRACKET

3.1 Introduction

In 1984 V.F.R. Jones [23] discovered a polynomial invariant for knots in three dimensional space. The definition of this invariant was purely combinatorial and topologists started to look for a geometric explanation of its existence. The first major progress was made by E. Witten [44] who described a construction of this invariant by making use of the Feynman path integral from quantum field theory. As he pointed out, the Jones polynomial is related to a new set of topological three manifold invariants. However, his approach lacks a rigorous mathematical foundation, since the construction uses an integral over the space of all connections.

M.F. Atiyah noted [3] that if the path integral existed, it had to satisfy a certain number of axioms. According to Atiyah, a topological quantum field theory (TQFT) consists of a functor from the category of surfaces to that of finite dimensional vector spaces, and a partition function that associates to each three manifold a vector in the vector space of its boundary. This theory is multiplicative with respect to disjoint union, and the invariants of cobordisms multiply like matrices under composition of cobordisms. Moreover, it has no dynamics, i.e. the invariant of the mapping cylinder of a surface is the identity matrix.

A first example of a TQFT that satisfies Atiyah's axioms and is related to the Jones polynomial was produced by Reshetikhin and Turaev in [34], and makes use of the representation theory of Hopf algebras. Then, an alternative construction based on geometric techniques has been worked out by Kohno in [27]. A combinatorial approach based on skein spaces associated to the Kauffman bracket [25], another polynomial invariant closely related to the Jones polynomial, was exhibited by Lickorish in [30] and [29] and by Blanchet, Habegger, Masbaum, and Vogel in [5]. All these theories are smooth, in the sense that manifolds can be glued only along closed surfaces in their boundary, and as a consequence, the axioms are not sufficient to enable the computation of invariants from the ones of very simple manifolds.

In an attempt to give a more axiomatic approach to such a theory, and also to make it easier to handle, K. Walker described in [41] a system of axioms for a TQFT in which one allows gluings along surfaces with boundary, a so called TQFT with corners. He also described the minimal amount of initial information (basic data) that one needs to know in order to be able to recover the whole theory from axioms. He based his theory on the decompositions of surfaces into disks, annuli and pairs of pants, and along with the mapping class group of a surface he considered the groupoid of transformations of these decompositions.

Following partial work from [41], in [13] and [18] we exhibited a TQFT with corners associated to the Reshitikhin-Turaev theory. We mention that in this construction one encounters a sign problem at the level of the groupoid of transformations of decompositions. The presence of this sign problem was due to the fact [21] that the theory was based on the Jones polynomial, whose skein relation is defined for oriented links.

In this chapter we describe the construction of a TQFT with corners that underlies the smooth TQFT of Lickorish [30], [29]. It is based on the skein theory of the Kauffman bracket. Note that since the Kauffman bracket is defined for unoriented links, we will not encounter any sign problem this time. The main elements involved in our construction are the Jones-Wenzl idempotents [43], which appeared in the work of Jones on the index of subfactors. They are the analogues of the irreducible representations of irreducible representations of the quantum deformations of $sl(2, C)$ (see [34]). Regarding the computations, we make the observation that in our case they will be done either in the skein space of the plane, or in that of the disk with points on the boundary, although the spaces associated to closed surfaces are skein spaces of handlebodies [29], [35].

In Section 2 we review the definitions from [41]. Section 3 starts with a review of facts about skein spaces and then proceeds with the description of the basic data. In Section 4 we prove that the basic data gives rise to a well defined TQFT. As a main device involved in the proof we exhibit a tensor contraction formula. In the fifth section we generalize to surfaces with boundary a well known formula for the invariant of the product of a closed surface with a circle. Next we show that the invariants of 3-manifolds with boundary have a distinguished vector component which satisfies the Kauffman bracket skein relation. As a consequence, we compute the invariant of the complement of a regular neighborhood of a link, and explain how the invariants of closed manifolds arise when doing surgery on such links.

The results from this section have appeared in [16], citesaptesprezece, [19], [21].

3.2 Facts About TQFT's With Corners

A TQFT with corners is one that allows gluings of 3-manifolds along surfaces in their boundary that themselves have boundary. In order to be able to understand such a theory we must first briefly describe its objects, the extended surfaces and 3-manifolds. For an extensive discussion we recommend [41]. The adjective "extended" comes from the way the projective ambiguity of the invariants is resolved, which is done, as usually, via an extension of the mapping class group. All surfaces and 3-manifolds throughout the paper are supposed to be piecewise linear, compact and orientable.

In order to fulfill the needs of a TQFT with corners, the concept of extended surface will involve slightly more structure than the usual Lagrangian space, namely the decomposition into disks, annuli and pairs of pants (shortly DAP-decomposition).

DEFINITION. A DAP-decomposition of a surface Σ consists of

- a collection of disjoint simple closed curves in the interior of Σ that cut Σ into elementary surfaces: disks, annuli, and pairs of pants, and an ordering of these elementary surfaces;

- a numbering of the boundary components of each elementary surface Σ_0 by 1 if Σ_0 is a disk, 1 and 2 if Σ_0 is an annulus, and 1,2 and 3 if Σ_0 is a pair of pants;

- a parametrization of each boundary component C of Σ_0 by $S^1 = \{z| \ |z| = 1\}$ (the parametrization being compatible with the orientation of Σ_0 under the convention "first out");

- fixed disjoint embedded arcs in Σ_0 joining $e^{i\epsilon}$ (where $\epsilon > 0$ is small) on the j-th boundary component to $e^{-i\epsilon}$ on the $j+1$-st (modulo the number of boundary components of Σ_0). These arcs will be called seams.

An example of a DAP-decomposition is shown in Fig 3.1. Two decompositions are considered identical if they coincide up to isotopy. We also make the convention that whenever we talk about the decomposition curves we also include the boundary components of the surface as well.

DEFINITION. An extended surface (abbreviated e-surface) is a pair (Σ, D)

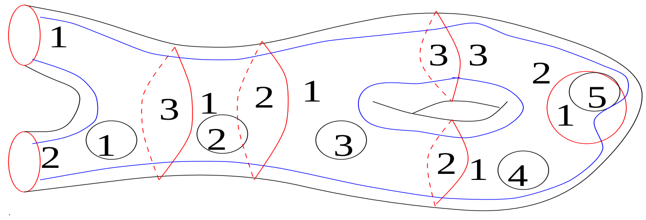

Figure 3.1: DAP-decomposition

where Σ is a surface and D is a DAP-decomposition of Σ.

Let us note that in the case of smooth TQFT's one is only interested in the Lagrangian subspace spanned by the decomposition curves of D in $H_1(\Sigma)$. In our case, we will be interested in the decomposition itself, since we can always arrange the gluing to be along a collection of elementary subsurfaces in the boundary of the 3-manifold. We emphasize that the DAP-decomposition plays the same role as the basis plays for a vector space.

If we change the orientation of a surface, the DAP-decomposition should be changed by reversing all orientations and subsequently by permuting the numbers 2 and 3 in the pairs of pants.

In what follows, we will call a move any transformation of one DAP-decomposition into another. By using Cerf theory [7] one can show that any move can be written as a composition of the elementary moves described in Fig. 3.2 and their inverses, together with the permutation map P that changes the order of elementary surfaces. In the sequel T_1 will be called a twist, R rotation, the maps A and D contractions of annuli, respectively disks, and their inverses expansions of annuli and disks.

DEFINITION. An extended morphism (shortly e-morphism) is a map between two e-surfaces $(f, n) : (\Sigma_1, D_1) \to (\Sigma_2, D_2)$ where f is a homeomorphism and n is an integer.

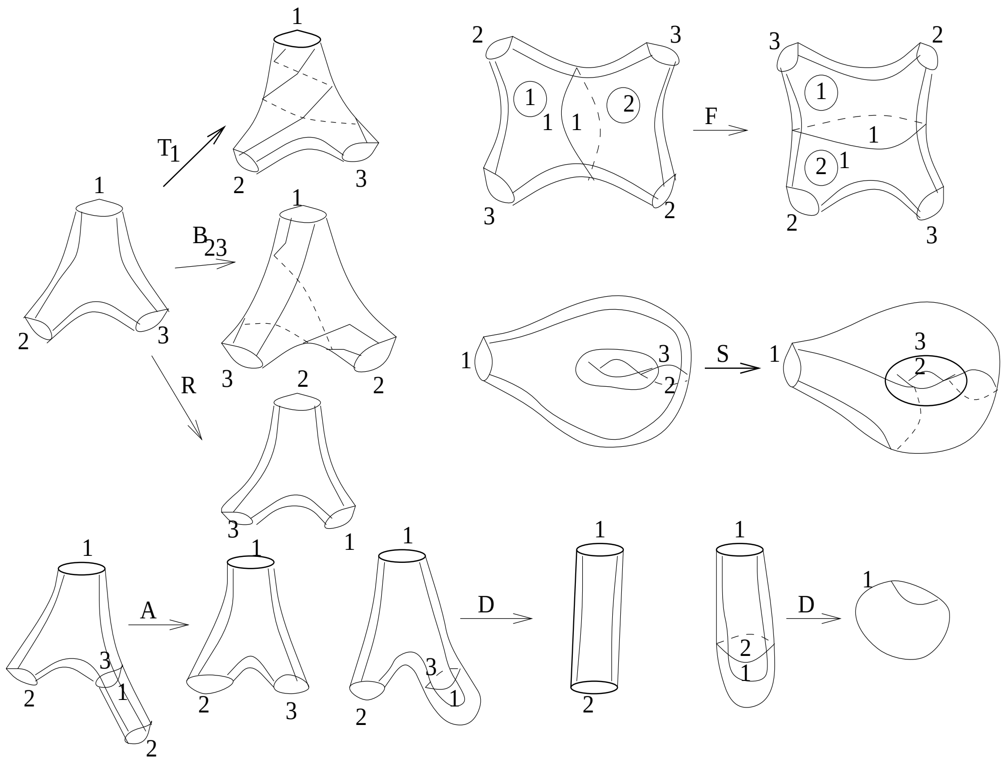

Figure 3.2: Elementary moves

Note that such an e-morphism can be written as a composition of a homeomorphism $(f, 0) : ((\Sigma_1, D_1) \to (\Sigma_2, f(D_1))$, a move $(\Sigma_2, f(D_1)) \to (\Sigma_2, D_2)$ and the morphism $(0, n) : (\Sigma_2, D_2) \to (\Sigma_2, D_2)$. Note also that the moves from Fig 2.2 have the associated homeomorphism equal to the identity.

The set of e-morphisms is given a groupoid structure by means of the following composition law. For $(f_1, n_1) : (\Sigma_1, D_1) \to (\Sigma_2, D_2)$ and $(f_2, n_2) : (\Sigma_2, D_2) \to (\Sigma_3, D_3)$ let

$$(f_2, n_2)(f_1, n_1) := (f_2 f_1, n_1 + n_2 - \sigma((f_2 f_1)_* L_1, (f_2)_* L_2, L_3)$$

where σ is Wall's nonadditivity function [42] and $L_i \subset H_1(\Sigma_i)$ is the subspace generated by the decomposition curves of D_i, $i = 1, 2, 3$.

Let us now review some facts about extended 3-manifolds.

DEFINITION. The triple (M, D, n) is called an extended 3-manifold (e-3-manifold) if M is a 3-manifold, D is a DAP-decomposition of ∂M and $n \in \mathbf{Z}$.

The boundary operator, disjoint union and mapping cylinder are defined in the canonical way, namely $\partial(M, D, n) = (\partial M, D)$, $(M_1, D_1, n_1) \sqcup (M_2, D_2, n_2) = (M_1 \sqcup M_2, D_1 \sqcup D_2, n_1 + n_2)$ and for $(f, n) : (\Sigma_1, D_1) \to (\Sigma_2, D_2)$, $I_{(f,n)} = (I_f, D, n)$, with the only modification that in I_f we identify the boundary components of $-\Sigma_1$ with those of Σ_2 that they get mapped onto, thus $\partial I_f = -\Sigma_1 \cup \Sigma_2$ and $D = D_1 \cup D_2$. More complicated is the gluing of e-3-manifolds, which is done as follows.

DEFINITION. Let (M, D, n) be an e-3-manifold and (Σ_1, D_1) and (Σ_2, D_2) be two disjoint surfaces in its boundary. Let $(f, m) : (\Sigma_1, D_1) \to (\Sigma_2, D_2)$ be an e-morphism. Define the gluing of (M, D, n) by (f, m) to be

$$(M, D, n)_{(f,m)} := (M_f, D', m + n - \sigma(K, L_1 \oplus L_2, \Delta^-))$$

where M_f is the gluing of M by f, D' is the image of D under this gluing, σ is Wall's nonadditivity function, K is the subspace of $H_1(\partial M)$ spanned by the kernel of $H_1(\Sigma_1 \cup \Sigma_2) \to H_1(M)$, $\partial\Sigma_1$ and $\partial\Sigma_2$, L_i are the subspaces of $H_1(\Sigma_i)$ generated by the decomposition curves of D_i and $\Delta^- = \{(x, -f_*(x)), x \in H_1(\Sigma_1)\}$.

For a geometric explanation of this definition see [41].

In order to define a TQFT we also need a finite set of labels $\mathcal{L}$, with a distinguished element $0 \in \mathcal{L}$. Consider the category of labeled extended surfaces (le-surfaces) whose objects are e-surfaces with the boundary components labeled by elements in $\mathcal{L}$ (le-surfaces), and whose morphisms are the e-morphisms that preserve labeling (called labeled extended morphism and abbreviated le-morphisms).

An le-surface is thus a triple (Σ, D, l), where l is a labeling function.

Following [Wa] we define a TQFT with label set $\mathcal{L}$ to consist out of

-a functor V from the category of le-surfaces to that of finite dimensional vector spaces, called modular functor,

-a partition function Z that associates to each 3-manifold a vector in the vector space of its boundary.

The two should satisfy the following axioms:

(2.1) (disjoint union)$V(\Sigma_1 \sqcup \Sigma_2, D_1 \sqcup D_2, l_1 \sqcup l_2) = V(\Sigma_1, D_1, l_1) \bigotimes V(\Sigma_2, D_2, l_2)$;

(2.2) (gluing for V) Let (Σ, D) be an le-surface, C, C' two subsets of boundary components of (Σ), and $g : C \to C'$ the homeomorphism which is the parametrization reflecting map. Let Σ_g be the gluing of Σ by g, and D_g the DAP-decomposition induced by D. Then, for a certain labeling l of $\partial\Sigma$ we have

$$V(\Sigma_g, D_g, l) = \bigoplus_{x \in \mathcal{L}(C)} V(\Sigma, D, (l, x, x))$$

where the sum is over all labelings of C and C' by x.

(2.3) (duality) $V(\Sigma, D, l)^* = V(-\Sigma, -D, l)$ and the identifications $V(\Sigma, D, l) = V(-\Sigma, -D, l)^*$ and $V(-\Sigma, -D, l) = V(\Sigma, D, l)^*$ are mutually adjoint. Moreover, the following conditions should be satisfied

-if (f, n) is an le-morphism between to le-surfaces, then $V(\bar{f}, -n)$ is the adjoint inverse of $V(f, n)$, where we denote by $\bar{f}$ the homeomorphism induced between the surfaces with reversed orientation.

-if $\alpha_1 \otimes \alpha_2 \in V(\Sigma_1, D_1, l_1) \bigotimes V(\Sigma_2, D_2, l_2)$ and $\beta_1 \otimes \beta_2 \in V(-\Sigma_1, -D_1, l_1) \bigotimes V(-\Sigma_2, -D_2, l_2)$ then $< \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2 >=< \alpha_1, \beta_1 >< \alpha_2, \beta_2 >$,

-there exists a function $S : \mathcal{L} \to \mathbf{C}^*$ such that with the notations from axiom (2.2) if $\oplus_x \alpha_x \in \bigoplus_{x \in \mathcal{L}(C)} V(\Sigma, D, (l, x, x))$ and $\oplus_x \beta_x \in \bigoplus_{x \in \mathcal{L}(C)} V(-\Sigma, -D, (l, x, x))$ then the pairing on the glued surface is given by $< \oplus_x \alpha_x, \oplus_x \beta_x >= \sum_x S(x) <$

$\alpha_x, \beta_x >$, where $(x = (x_1, x_2, \cdots, x_n)$ and $S(x) = S(x_1)S(x_2)\cdots S(x_n)$;

(2.4) (empty surface) $V(\emptyset) = \mathbf{C}$;

(2.5) (disk) If $\mathbf{D}$ is a disk $V(\mathbf{D}, m) = \mathbf{C}$ if $m = 0$ and 0 otherwise;

(2.6) (annulus) If A is an annulus then $V(A, (m, n)) = \mathbf{C}$ if $m = n$ and 0 otherwise;

(2.7) (disjoint union for Z) $Z((M_1, D_1, n_1) \sqcup (M_2, D_2, n_2)) = Z(M_1, D_1, n_1) \otimes Z(M_2, D_2, n_2)$;

(2.8) (naturality) Let $(f, 0) : (M_1, D, n) \to (M_2, f(D), n)$. Then

$V(f|\partial(M_1, D, n))Z(M_1, D, n) = Z(M_2, f(D), n)$.

(2.9) (gluing for Z) Let (Σ_1, D_1), $(\Sigma_2, D_2) \subset \partial(M, D, m)$ be disjoint, and let $(f, n) : (\Sigma_1, D_1) \to (\Sigma_2, D_2)$. Then by (2.2)

$V(\partial(M, D, m)) = \bigoplus_{l_1, l_2} V(\Sigma_1, D_1, l_1) \otimes V(\Sigma_2, D_2, l_2) \otimes V(\partial(M, D, m) \backslash ((\Sigma_1, D_1) \cup (\Sigma_2, D_2), l_1 \cup l_2)$

hence $Z(M, D, m) = \bigoplus_{l_1, l_2} \sum_j \alpha_{l_1}^{(j)} \otimes \beta_{l_2}^{(j)} \otimes \gamma_{l_1, l_2}^{(j)}$. The axiom states that

$Z((M, D, m)_{(f,n)}) = \oplus_l \sum_j < V(f, n)\alpha_l^{(j)}, \beta_l^{(j)} > \gamma_{ll}^{(j)}$,

-where l runs through all labelings of $\partial\Sigma_1$;

(2.10) (mapping cylinder axiom) For $(id, 0) : (\Sigma, D) \to (\Sigma, D)$ we have

$$Z(I_{(id,0)}) = \oplus_{l \in \mathcal{L}(\partial\Sigma)} id_l$$

where id_l is the identity matrix in $V(\Sigma, D, l) \otimes V(\Sigma, D, l)^*$.

3.3 The Basic Data

In order to construct a TQFT with corners one needs to specify a certain amount of information, called basic data, from which the modular functor and partition function can be recovered via the axioms. Note that the partition function is completely determined by the modular functor, so we only need to know that latter. Moreover, the modular functor is determined by the vector spaces associated to le-disks, annuli and pairs of pants, and by the linear maps associated to le-morphisms. An important observation is that the matrix of a morphism $V(f,0)$, where $(f,0) : (\Sigma_1, D) \to (\Sigma_2, f(D))$, is the identity matrix, so one only needs to know the values of the functor for moves, hence for the elementary moves described in Fig. 3.2. Of course we also need to know its value for the map $C = (id, 1)$.

The possibility of relating our theory to the Kauffman bracket depends on the choice of basic data. Our construction has been inspired by [30]. We will review the notions we need from that paper and then proceed with our definitions.

Let Σ be a surface with a collection of $2n$ points on its boundary ($n \geq 0$). A link diagram in Σ is an immersed compact 1-manifold L in Σ with the property that $L \cap \partial\Sigma = \partial L$, ∂L consists of the $2n$ distinguished point on $\partial\Sigma$, the singular points of L are in the interior of Σ and are transverse double points, and for each such point the "under" and "over" information is recorded.

Let $A \in \mathbf{C}$ be fixed. The skein vector space of Σ, denoted by $\mathcal{S}(\Sigma)$, is defined to be the complex vector space spanned by all link diagrams factored by the following two relations:

a). $L\cup$(trivial closed curve)$= -(A^2 + A^{-2})L$,

b). $L_1 = AL_2 + A^{-1}L_3$,

where L_1, L_2 and L_3 are any three diagrams that coincide except in a small disk, where they look like in Fig. 3.3.

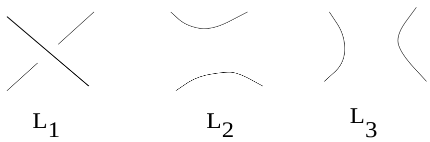

Figure 3.3: Diagrams for crossings

For simplicity, from now on, whenever in a diagram we have an integer, say k, written next to a strand we will actually mean that we have k parallel strands there. Also rectangles (coupons) inserted in diagrams will stand for elements of the skein space of the rectangle inserted there.

Three examples are useful to consider. The first one is the skein space of the plane, which is the same as the one of the sphere, and it is well known that it is isomorphic to $\mathbf{C}$.

The second example is that of an annulus A with no points on the boundary. It is also a well known fact that $\mathcal{S}(A)$ is isomorphic to the ring of polynomials $\mathbf{C}[\alpha]$, (if endowed with the multiplication defined by the gluing of annuli). The independent variable α is the diagram with one strand parallel to the boundary of the annulus. Recall from [30] that every link diagram L in the plane determines a map

$$< \cdot, \cdot, \cdots, \cdot >_L : \mathcal{S}(A) \times : \mathcal{S}(A) \times \cdots : \mathcal{S}(A) \to \mathcal{S}(\mathbf{R}^2)$$

obtained by first expanding each component of L to an annulus via the blackboard framing and then homeomorphically mapping A onto it.

The third example is the skein space of a disk with $2n$ points on the boundary. If the disk is viewed as a rectangle with n points on one side and n on the opposite, then we can define a multiplication rule on the skein space by juxtaposing rectangles, obtaining the Temperley-Lieb algebra TL_n. Recall that TL_n is generated by the elements $1, e_1, e_2, \cdots, e_{n-1}$, where e_i is described in Fig. 3.4.

There exists a map from TL_n to $\mathcal{S}(\mathbf{R}^2)$ obtained by closing the elements in TL_n by n parallel arcs. This map plays the rôle of a quantum trace. It splits in a canonical way as $TL_n \to \mathcal{S}(A) \to \mathcal{S}(\mathbf{R}^2)$ by first closing the elements in an annulus and then including them in a plane.

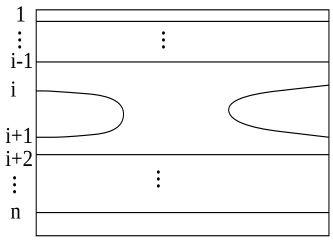

Figure 3.4: Basis for TL_n

At this moment we recall the definition of the Jones-Wenzl idempotents [43]. They are of great importance for our construction, since they mimic the behavior of the finite dimensional irreducible representations from the Reshetikhin-Turaev theory [34]. For this let $r > 1$ be an integer (which will be the level of our TQFT). Let $A = e^{i\pi/(2r)}$. Recall that for each n one denotes by $[n]$ the quantized integer $(A^{2n} - A^{-2n})/(A^2 - A^{-2})$.

The Jones-Wenzl idempotents are the unique elements $f^{(n)} \in TL_n$, $0 \leq n \leq r-1$, that satisfy the following properties:

1) $f^{(n)} e_i = 0 = e_i f^{(n)}$, for $0 \leq i \leq n-1$,

2) $(f^{(n)} - 1)$ belongs to the algebra generated by $e_1, e_2, \cdots, e_{n-1}$,

3) $f^{(n)}$ is an idempotent,

4) $\Delta_n = (-1)^n [n+1]$

where Δ_n is the image of $f^{(n)}$ through the map $TL_n \to \mathcal{S}(\mathbf{R}^2)$.

In the sequel we will have to work with the square root of Δ_n so we make the notation $d_n = i^n \sqrt{[n+1]}$, thus $\Delta_n = d_n^2$.

Following [30], in a diagram we will always denote $f^{(n)}$ by an empty coupon (see Fig. 3.5).

The image of $f^{(n)}$ through the map $TL_n \to \mathcal{S}(A)$ will be denoted by $S_n(\alpha)$ We

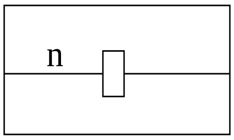

Figure 3.5: Jones-Wenzl idempotent

will also need the element $\omega \in \mathcal{S}(A)$, $\omega = \sum_{n=0}^{r-2} d_n^2 S_n(\alpha)$. Given a link diagram L in the plane, whenever we label one of its components by ω we actually mean that we inserted ω in the way described in the definition of $< \cdot, \cdot, \cdots, \cdot >_L$. Note that one can perform handle slides (also called second Kirby moves [26]), over components labeled by ω without changing the value of the diagram (see [30]).

Now we can define the basic data for a TQFT in level r, where r, as said, is an integer greater than 1. Let $\mathcal{L} = \{0, 1, \cdots, r-2\}$. Make the notation $X = (i\sqrt{2r})/(A^2 - A^{-2})$, that is $X^2 = \sum d_n^4 =< \omega >_U$, where U is the unknot with zero framing.

Notice that by gluing two disks along the boundary we get a pairing map $\mathcal{S}(\mathbf{D}, 2n) \times \mathcal{S}(\mathbf{D}, 2n) \to \mathcal{S}(S^2) = \mathbf{C}$, hence we can view $\mathcal{S}(\mathbf{D}, 2n)$ as a set of functionals acting on the skein space of the exterior. In what follows, whenever we mention the skein space of a disk, we will always mean the skein space as a set of functionals in this way. For example this will enable us to get rid of the diagrams that have a strand labeled by $r-1$ (see also [30], [24]). The point of view is similar to that of factoring by the bad part of a representation (the one of quantum trace 0) in the Reshetikhin-Turaev setting.

To a disk with boundary labeled by 0 we associate the vector space V_0 which is the skein space of a disk with no points on the boundary. Of course for any other label a we put $V_a = 0$. It is obvious that $V_0 = \mathbf{C}$. We let β_0 be the empty diagram.

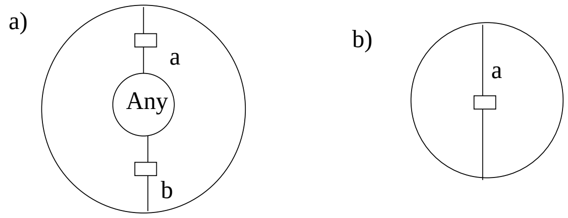

Figure 3.6: Spaces associated to annuli

To an annulus with boundary components labeled by a and b we associate the vector space V_{ab} which is the subspace of $\mathcal{S}(\mathbf{D}, a+b)$ spanned by all diagrams of the form indicated in Fig. 3.6. a), where in the smaller disk can be inserted any diagram from $\mathcal{S}(\mathbf{D}, a+b)$. The first condition in the definition of the Jones-Wenzl idempotents implies that $V_{ab} = 0$ if $a \neq b$ and V_{aa} is one dimensional and is spanned by the diagram from Fig. 3.6. b). We will denote by β_{aa} this diagram multiplied by $1/d_a$, where we recall that $d_a = i^a\sqrt{[a+1]}$. The element β_{aa} has the property that paired with itself on the outside gives 1.

To a pair of pants with boundary components labeled by a, b, and c we put into correspondence the space V_{abc}, which is the space spanned by all diagrams of the form described in Fig. 3.7. a), where in the inside disk we allow any diagram from $\mathcal{S}(\mathbf{D}, a+b+c)$.

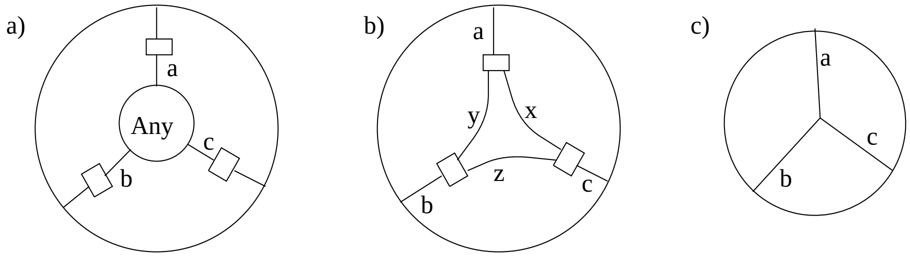

Figure 3.7: Spaces associated to pairs of pants

The reader will notice that there is some ambiguity in this definition. To make it rigorous, we have to mark a point on the circle, from which all points are counted. We will keep this in mind although we will no longer mention it.

The results from [24] and [30] show that V_{abc} can either be one dimensional or it is equal to zero. The triple (a, b, c) is said to be admissible if $V_{abc} \neq 0$. This is exactly the case when $a + b + c$ is even, $a + b + c \leq 2(r - 2)$ and $a \leq b + c$, $b \leq a + c$, $c \leq a + b$. In this case the space V_{abc} is spanned by the triad introduced by Kauffman [24] which is described in Fig. 3.7. b). Here the numbers x, y, z satisfy $a = x + y$, $b = y + z$, $c = z + x$.

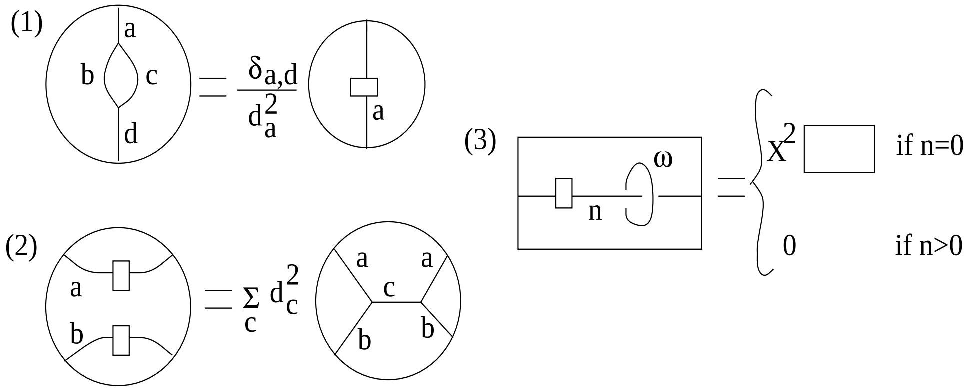

Figure 3.8: The identities of Lickorish

In [29] it is shown that if we pair the diagram from Fig. 3.7. b) with the one corresponding to V_{acb} on the outside we get the complex number $\theta(x + y, y + z, z + x) = (\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!)/\ (\Delta_{y+z-1}!\Delta_{z+x-1}!\Delta_{x+y-1}!)$, where $\Delta_n = \Delta_1\Delta_2\cdots\Delta_n$ and $\Delta_{-1} = 1$. Thus if we denote by β_{abc} the product of this diagram with $(d_{x+y+z}!d_{x-1}!d_{y-1}!d_{z-1}!)^{-1}(d_{y+z-1}!d_{z+x-1}!d_{x+y-1}!) = 1/\sqrt{\theta(a, b, c)}$ (with the same convention for factorials), then β_{abc} paired on the outside with β_{acb} will give 1.

In diagrams, whenever we have a β_{abc} we make the notation from Fig. 3.7. c). This notation is different from the one with a dot in the middle from [30] , in the sense that we have a different normalization! We prefer this notation because it will simplify diagrams in the future, so whenever in a diagram we have a trivalent vertex, we consider that we have a β inserted there. In particular, a diagram that looks like the Greek letter θ will be equal to 1 in $\mathcal{S}(\mathbf{R}^2)$. The elements β_{abc} are the analogues of the quantum Clebsch-Gordan coefficients.

In the sequel we will need the three identities described in Fig. 3.8, whose proofs can be found in [30]. Here δ_{ad} is the Kronecker symbol.

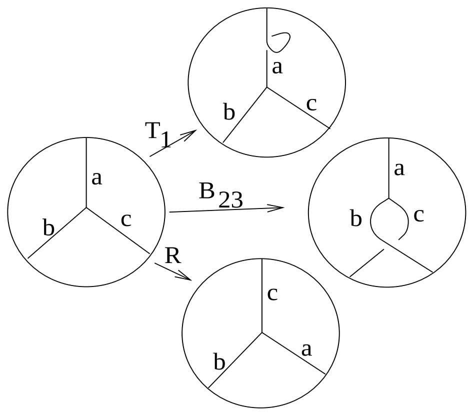

Figure 3.9: Morphisms for pairs of pants

Let us define the dual spaces. It is natural to let the dual of V_0 to be V_0, that of V_{aa} to be V_{aa}, and that of V_{abc} to be V_{acb}. However the pairings will look peculiar. This is due to the fact that we want the mapping cylinder to be satisfied. So we let $<,>: V_0 \times V_0 \to \mathbf{C}$ be defined by $<\beta_0, \beta_0>= 1$, $<,>: V_{aa} \times V_{aa} \to \mathbf{C}$ be defined by $<\beta_{aa}, \beta_{aa}>= X/d_a^2$, and $<,>: V_{abc} \times V_{acb} \to \mathbf{C}$ be defined by

$< \beta_{abc}, \beta_{acb} >= X^2/(d_a d_b d_c)$.

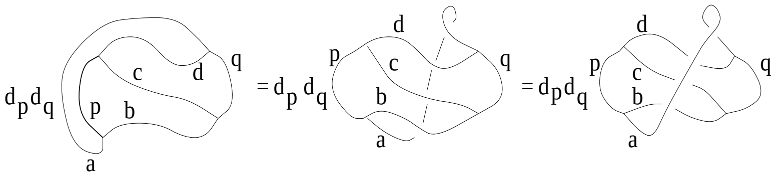

Figure 3.10: Fusion matrix

Before we define the morphisms associated to the elementary moves we make the convention that for any e-morphism f we will denote $V(f)$ also by f.

The morphisms corresponding to the three elementary moves on a pair of pants are described in Fig. 3.9. Further, we let $F : \bigoplus_p V_{pab} \otimes V_{pcd} \to \bigoplus_p V_{qda} \otimes V_{qbc}$ be defined by $F\beta_{pab} \otimes \beta_{pcd} = \sum_q \{^{bcp}_{adq}\}\beta_{qda} \otimes \beta_{qbc}$ The coefficients c_{abcdpq} being given by any of the three equal diagrams from Fig. 3.10. Note that $c_{abcdpq} = d_p^{-1} d_q \{^{bcp}_{adq}\}$ where $\{^{bcp}_{adq}\}$ are the 6j-symbols.

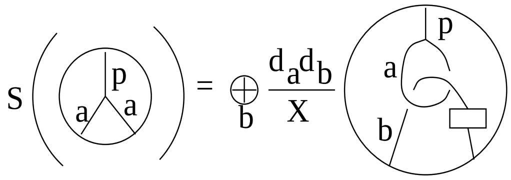

Figure 3.11: S-matrix

Also the map $S : \bigoplus V_{paa} \to \bigoplus V_{pbb}$ is described in Fig. 3.11.

The maps A, D and P are given by relations of the form $A(x \otimes \beta_{aa}) = x$, $D(\beta_{aa0} \otimes \beta_0) = \beta_{aa}$ and $P(x \otimes y) = y \otimes x)$. The map C is the multiplication by the value of the diagram described in Fig. 3.12. a). Note that Lemma 4 in [30] implies that the inverse of C is the multiplication by the diagram from Fig. 3.12.

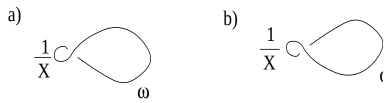

Figure 3.12: Framing adjusting morphism

b). Finally, $S(a) = d_a^2/X$, $a \in \mathcal{L}$.

REMARK. The reader should note that the crossings from all these diagrams are negative. We make this choice because, returning to the analogy with vector spaces, all the maps we defined behave like changes of basis rather than like morphisms.

3.4 The Compatibility Conditions

In order for the basic data to give rise to a well defined TQFT, it has to satisfy certain conditions. A list of such conditions has been exhibited in [41], by making use of techniques of Cerf theory similar to those from [22]. The first group of relations, the so called Moore-Seiberg equations, are the conditions that have to be satisfied in order for the modular functor to exist. They are as follows:

1. at the level of a pair of pants:

a) $T_1B_{23} = B_{23}T_1$, $T_2B_{23} = B_{23}T_3$, $T_3B_{23} = B_{23}T_2$, where $T_2 = RT_1R^{-1}$ and $T_3 = R^{-1}T_1R$,

b) $B_{23}^2 = T_1T_2^{-1}T_3^{-1}$,

c) $R^3 = 1$,

d) $RB_{23}R^2B_{23}RB_{23}R^2 = 1$,

2. relations defining inverses:

a) $P^{(12)}F^2 = 1$,

b) $T_3^{-1}B_{23}^{-1}S^2 = 1$,

3. relations coming from "codimension 2 singularities":

a) $P^{(13)}R^{(2)}F^{(12)}R^{(2)}F^{(23)}R^{(2)}F^{(12)}R^{(2)}F^{(23)}R^{(2)}F^{(12)} = 1$,

b) $T_3^{(1)}FB_{23}^{(1)}FB_{23}^{(1)}FB_{23}^{(1)} = 1$,

c) $C^{-1}B_{23}^{-1}T_3^{-2}ST_3^{-1}ST_3^{-1}S = 1$,

d) $R^{(1)}(R^{(2)})^{-1}FS^{(1)}FB_{23}^{(2)}B_{23}^{(1)} = FS^{(2)}T_3^{(2)}(T_1^{(2)})^{-1}B_{23}^{(2)}F$,

4. relations involving annuli and disks:

a) $F(\beta_p^{mn} \otimes \beta_p^{p0}) = \beta_m^{0m} \otimes \beta_m^{np}$,

b) $A_2^{(12)}D_3^{(13)} = D_2D_3^{(13)}$,

c) $A^{(12)}A^{(23)} = A^{(23)}A^{(12)}$,

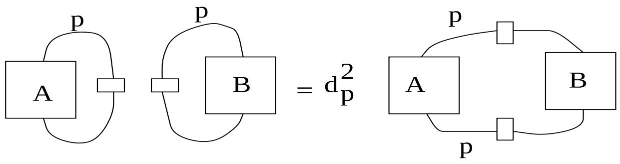

Figure 3.13: First recombination formula

5. relations coming from duality:

-for any elementary move f, one must have $f^+ = \bar{f}$, where f^+ is the adjoint of $f$with respect to the pairing, and $\bar{f}$ is the morphism induced by f on the surface with reversed orientation,

6. relations expressing the compatibility between the pairing, and moves A and D:

a) $< \beta_m^m, \beta_m^m >= S(m)^{-1}$

b) $< \beta_m^{m0}, \beta_m^{m0} >= S(0)^{-1}S(m)^{-1}$.

In addition one also has to consider two conditions that guarantee that the partition function is well defined.

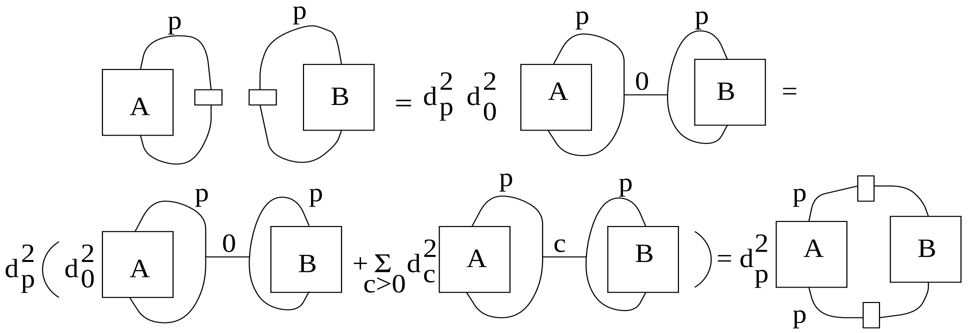

Figure 3.14: Proof of Lemma 4.1

7. a) $S(m) = S_{0m}$ where $[S_{xy}]_{x,y}$ is the matrix of move S on the torus (which can be thought as the punctured torus capped with a disk),

b) $F(\beta_0^{mm} \otimes \beta_0^{nn}) = \bigoplus S(m)^{-1}S(n)^{-1}id_{pmn}$ where id_{pmn} is the identity matrix in $(V_{pmn})^* \bigotimes V_{pmn}$.

In all these relations, the superscripts in parenthesis indicate the index of the elementary surface(s) on which the map acts, and the subscripts indicate the number of the boundary component.

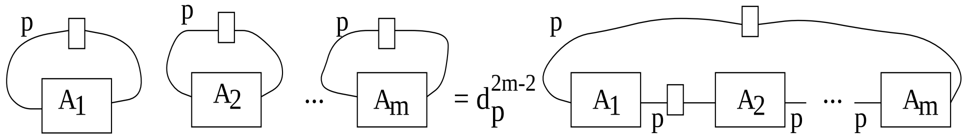

Figure 3.15: Second recombination formula

We will prove that our basic data satisfies these relations. For the proof we will need a contraction formula similar to the tensor contraction formula that one encounters in the case of TQFT's based on representations of Hopf algebras (see [39], [13], [18]).

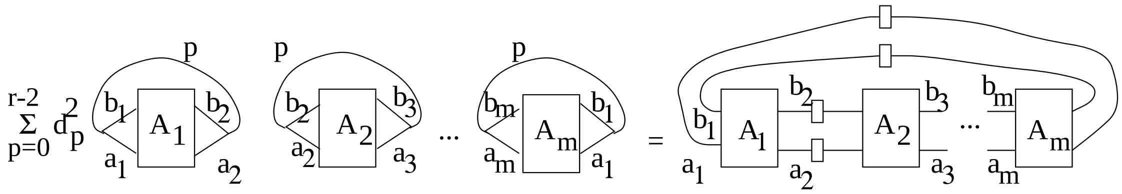

Figure 3.16: Bessel type formula

LEMMA 4.1. For any $A, B \in TL_p$ the equality from Fig. 3.13 holds.

PROOF: The proof is contained in Fig. 3.14. In this chain of equalities the first one is trivial, the second one holds because the sum that appears in the third term is trivial (by the first property of Jones-Wenzl idempotents, since such an idempotent lies on the strand labeled by c; more explanations about this phenomenon can be found in [L1] and [R]), and the last equality follows from identity (2) in Fig. 3.8.□

LEMMA 4.2. If $A_1, A_2, \cdots, A_m \in TL_p$ then the identity from Fig. 3.15 holds.

PROOF: Follows by induction from Lemma 4.1.□.

THEOREM 4.1. Suppose that $A_i \in \mathcal{S}(\mathbf{D}, a_i + b_i + a_{i+1} + b_{i+1})$, $i = 1, 2, \cdots, m$, where a_i and b_i are integers with $a_{m+1} = a_1$ and $b_{m+1} = b_1$. Then the identity described in Fig. 3.16 holds.

PROOF: By Lemma 4.2, the left hand side is equal to the expression described in Fig. 3.17. a).

On the other hand, if $p \neq q$, by using the identity (2) from Fig. 3.8, we get the chain of equalities from Fig. 3.17, where the last one follows from the fact that on the strand labeled by c there is a Jones-Wenzl idempotent and using the first property of these idempotents.

As a consequence of this fact we get that our expression is equal to the one

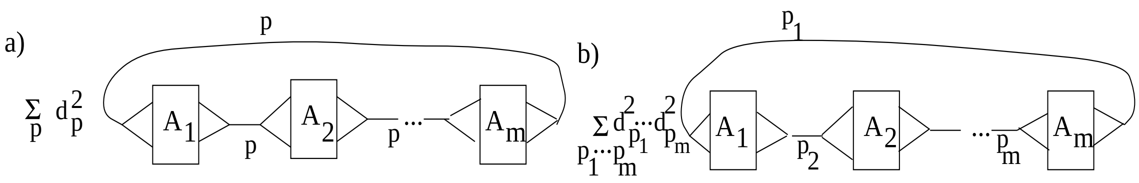

Figure 3.17: Proof of Theorem 4.1.

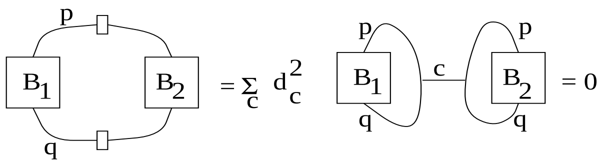

Figure 3.18: Bottle neck type identity

from Fig. 3.16. b), and then by applying the identity (2) from Fig. 3.6 several times we get the desired result.□.

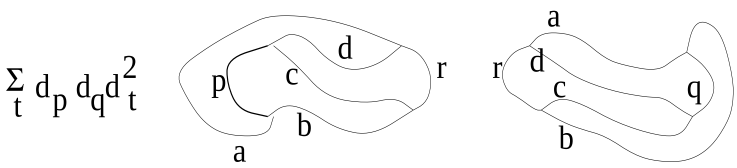

Figure 3.19: Proof of 2. a), first part

We can proceed with proving the compatibility conditions. The proofs are similar to the ones in [13] and [18], but one should note that they are simpler. First, the relations on a pair of pants are obviously satisfied. This can be seen at first glance for 1.a) and 1.c), then 1.d) is the third Reidemeister move, and 1.b) is equivalent to 1.c) (see [13] or Chap. VI in [39]).

For the proof of 2. a) we write $FPF\beta_{pab} \otimes \beta_{pcd} = \sum_q c_{abcdpq}\beta_{qab} \otimes \beta_{qcd}$. Since we have a matrix multiplication here we see that the coefficient c_{abcdpq} is given by the diagram from Fig. 3.19.

By using Theorem 4.1 this becomes the expression from Fig. 3.20. Using identity (1) from Fig. 3.8. wee see that this is equal to δ_{pq} multiplied by the Greek letter θ diagram, therefore is equal to δ_{pq} and the identity is proved.

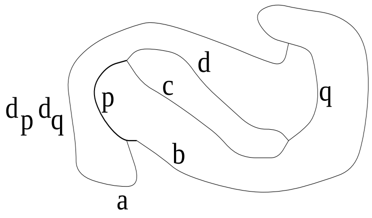

Figure 3.20: Proof of 2. a), second part

For 2. b) we have that $T_3^{-1}B_{23}^{-1}S^2\beta_{paa}$ is equal to the first term in Fig. 3.21. We get the chain of equalities from this figure by pulling first the strand labeled by ω down and using the identity (2) from Fig. 3.8, and then using identity (3) from Fig. 3.8. The last term is equal to β_{aa}.

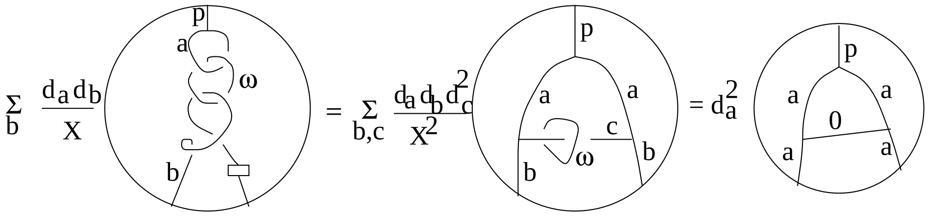

Figure 3.21: Proof of 2. b)

Now we describe the proof of the pentagon identity. We are interested in computing the coefficient of $\beta_{sde}\otimes\beta_{sct}\otimes\beta_{rab}$ in $F^{(12)}R^{(2)}F^{(23)}R^{(2)}F^{(12)}R^{(2)}F^{(23)}R^{(2)}F^{(12)}$

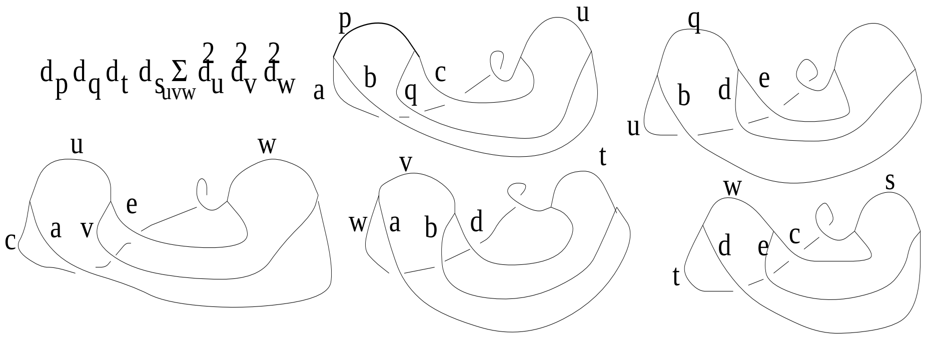

Figure 3.22: Coefficient for pentagon

$\beta_{pab} \otimes \beta_{pqc} \otimes \beta_{qde}$. Again, by using the formula for matrix multiplication we get that this coefficient is described in Fig. 3.22.

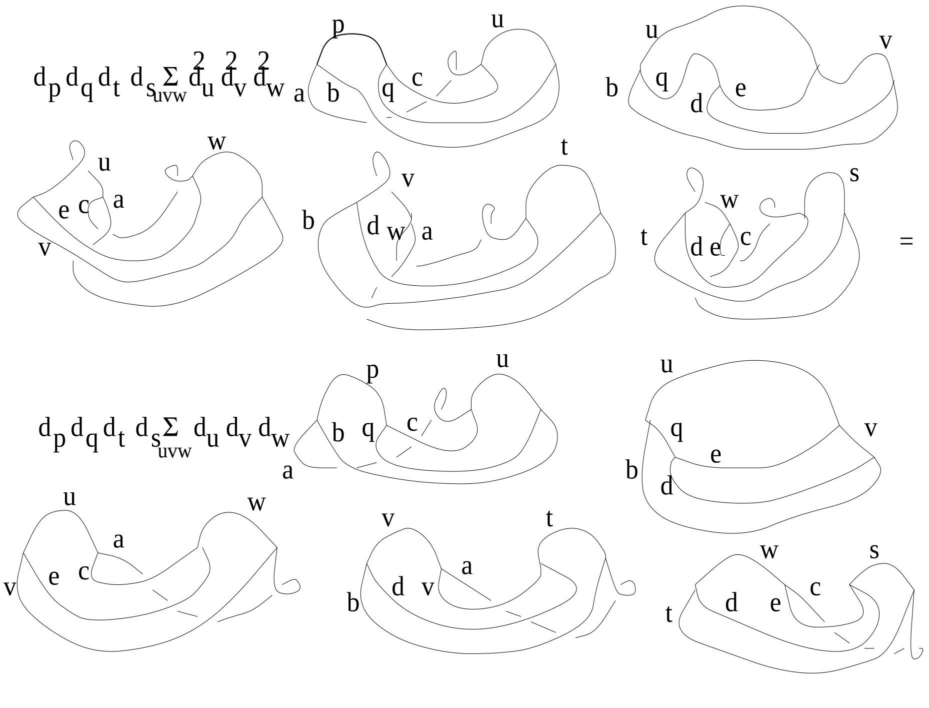

Figure 3.23: Proof of pentagon, first part

By doing a flip in the third, fourth and fifth factor we get the first term from

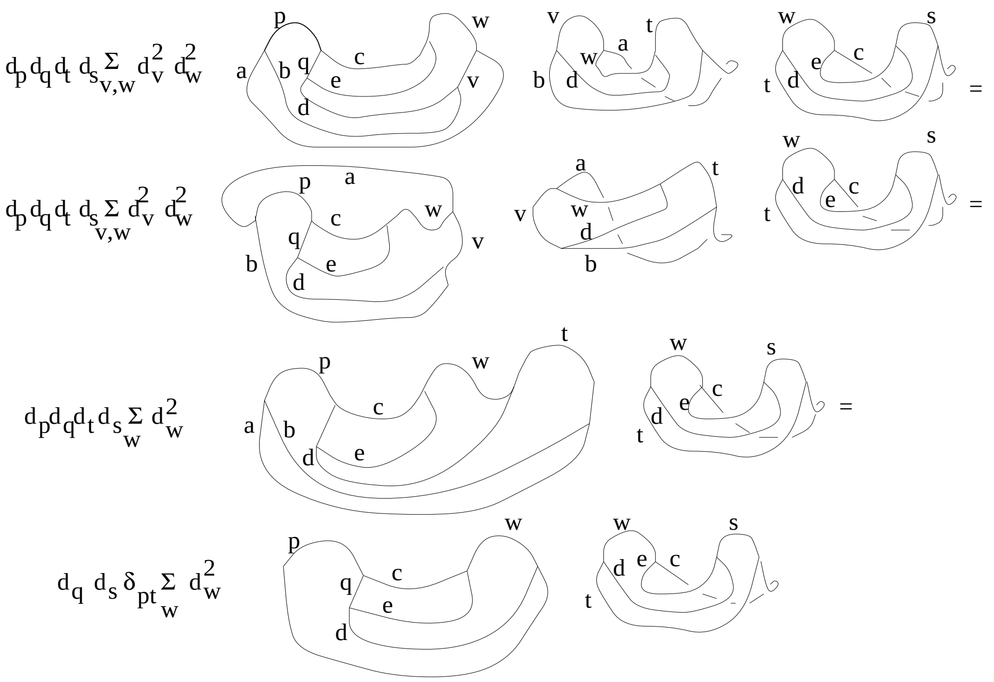

Figure 3.24: Proof of pentagon, second part

the equality shown in Fig. 3.23, which is further transformed into the second by applying three times 1. b)). Apply Theorem 4.1 to contract with respect to u, then continue like in Fig. 3.24, namely pull the strand of a over, then apply Theorem 4.1 for the sum over v and then use for the last equality formula (1) in Fig. 3.8. Finally, if we use Theorem 4.1 once more and then formula (1) in Fig. 3.8, we get $\delta_{pt}\delta_{qs}$ times a diagram of the form of letter θ. Hence the final answer is $\delta_{pt}\delta_{qs}$ and the identity is proved.

In order for the F-triangle to hold we have to show that the coefficient of $\beta_{qab} \otimes \beta_{qcd}$ in $T_3^{(1)}FB_{23}^{(1)}FB_{23}^{(1)}FB_{23}^{(1)}\beta_{pab}\beta_{pcd}$ is δ_{pq}. The coefficient is given in Fig. 3.25.

We transform the second factor as shown in Fig. 3.26 by first doing two flips

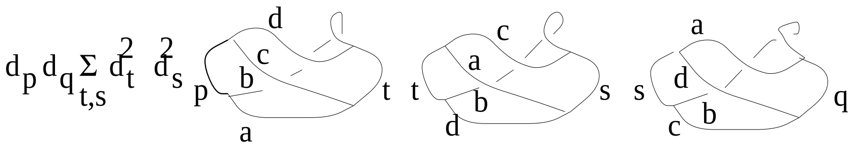

Figure 3.25: Coefficient for F-triangle

and then using 1. b) twice. Then contract the product via Theorem 4.1 to get the first term from the equality from Fig. 3.27, then transform it into the second by using again 1. b). As before, this is equal to δ_{pq}.

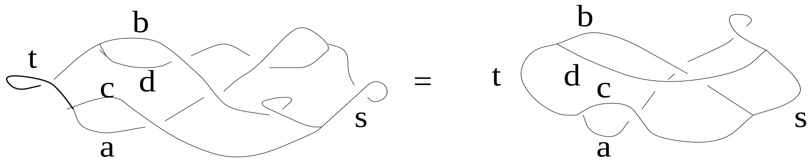

Figure 3.26: Middle factor for F-triangle

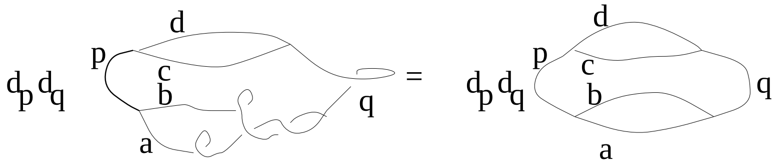

Figure 3.27: Proof of F-triangle

In the case of the S-triangle, it is not hard to see that $C^{-1}B_{23}^{-1}T_3^{-2}ST_3^{-1}ST_3^{-1}S\beta_{aa}$ is equal to the expression from Fig. 3.28. Lemma 3 in [L1] enables us to do Kirby moves over components labeled by ω, so we get the first term from Fig. 3.29, which is equal to the second one by Lemma 4 in [30]. From here we continue like in the proof of 2. b).

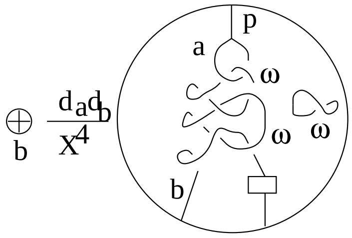

Figure 3.28: Coefficient for S-triangle

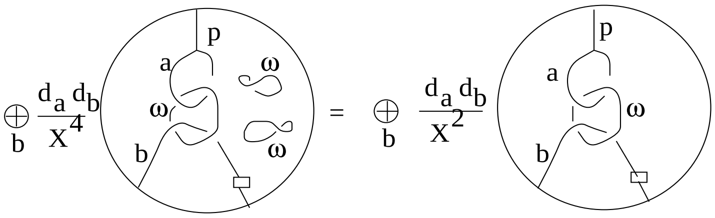

Figure 3.29: Proof of S-triangle

Let us prove 3. d). We have to show that the coefficient of $\beta_{qdc} \otimes \beta_{qda}$ in $FS^{(2)}T_3^{(2)}(T_1^{(2)})^{-1}B_{23}^{(2)}F\ \beta_{pab} \otimes \beta_{pbc}$ is the same as the coefficient of this vector in $R^{(1)}(R^{(2)})^{-1}FS^{(1)}FB_{23}^{(1)}B_{23}^{(2)}\beta_{pab} \otimes \beta_{pbc}$. For the first one we have the sequence of equalities from Fig. 3.30, where the second equality is obtained by contracting via Theorem 4.1. For the second one we have the equalities from Fig. 3.31, where at the first step we used a combination of a flip and 1.b) and at the second step we contracted. By moving strands around the reader can convince himself that the two are equal.

The groups of relations 4, 5, and 6 are straightforward. Also, we see that the function S has been chosen such that 7.a) holds. Let's prove 7.b). Here is the place where we see why we normalized the pairing the way we did. We have to prove that $d_m^2 d_n^2 X^{-2} F\beta_{0mm} \otimes \beta_{0nn} = \oplus_p d_m d_n d_p X^{-2} \beta_{pnm} \otimes \beta_{pmn}$. We see in Fig. 3.32 that this

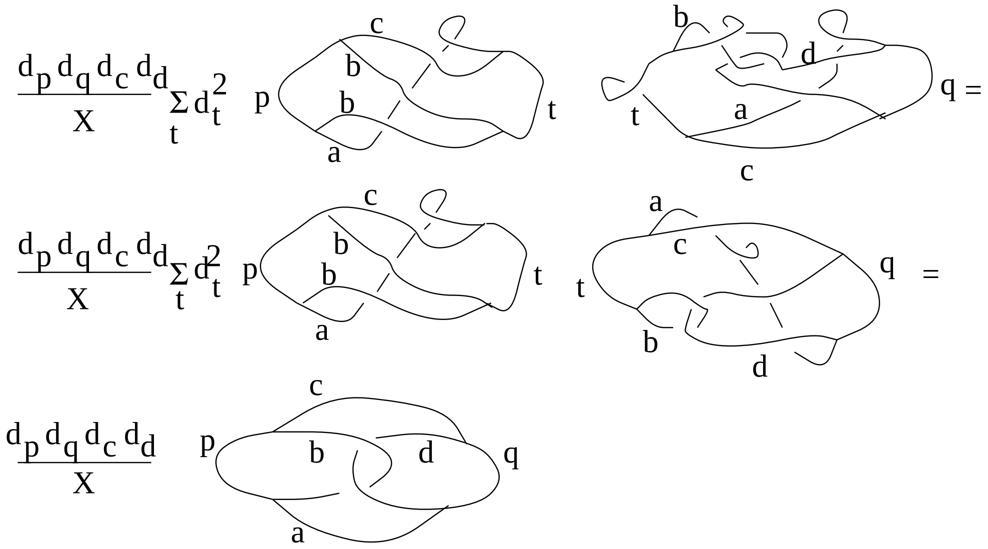

Figure 3.30: Left hand side for FSF-identity

is true.

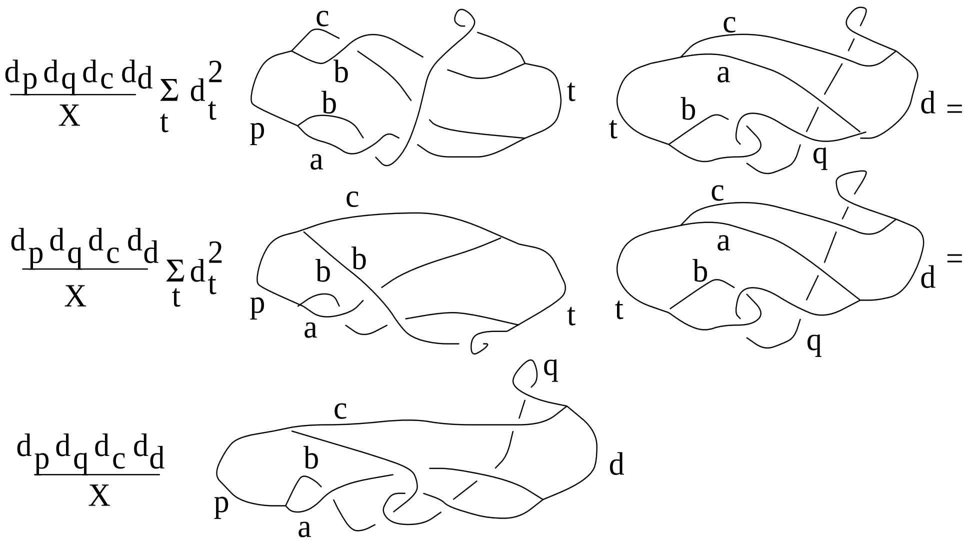

Figure 3.31: Right hand side for FSF-identity

3.5 Properties of Invariants of 3-manifolds

We begin this section with the generalization of Theorem 8 from [30] (see also Proposition 10.1 in [5]) to surfaces with boundary.

PROPOSITION 5.1. Let Σ be a surface of genus g with n boundary components, and let D be the DAP-decomposition of $\Sigma \times S^1$ whose decomposition circles are the components of $\partial\Sigma \times \{1\}$ and whose seams are of the form $\{x\} \times S^1$, with $x \in \partial\Sigma$. Then

$$Z(\Sigma \times S^1, D, 0) = \sum_{j_1, j_2, \cdots, j_n} c_{j_1, j_2, \cdots, j_n} \beta_{j_1 j_1} \otimes \beta_{j_2 j_2} \otimes \cdots \otimes \beta_{j_n j_n}$$

where $j_1, j_2, \cdots, j_n$ run over all labelings of $\partial\Sigma$ and $c_{j_1, j_2, \cdots, j_n}$ is the number of ways of labeling the diagram in Fig. 3.33 with integers i_k such that at each node we have an admissible triple.

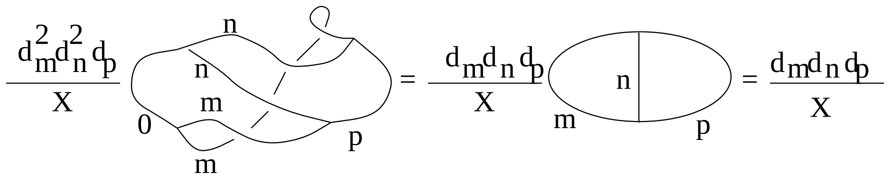

Figure 3.32: Compatibility for partition function

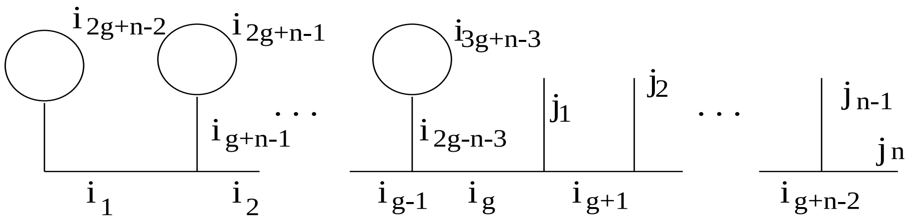

Figure 3.33: Diagram for Proposition 5.1.

PROOF: Consider on Σ a DAP-decomposition D_0 with decomposition curves as shown in Fig. 3.34. Put on $\Sigma \times I$ the DAP-decomposition D' that coincides with D_0 on $\Sigma \times \{1\}$, with $-D_0$ on $-\Sigma \times \{0\}$, and on $\partial\Sigma \times I$ there are no extra decomposition circles, and the seams are vertical (i.e. of the form $\{x\} \times I$).

It follows that $(\Sigma \times I, D', 0)$ is the mapping cylinder of $(id, 0)$ (with vertical annuli no longer contracted like in the definition of the mapping cylinder from Section 2). The mapping cylinder axiom implies that

$$Z(\Sigma \times I, D', 0) = \bigotimes_{j_1, j_2, \cdots, j_n} id_{j_1, j_2, \cdots, j_n} \beta_{j_1 j_1} \otimes \beta_{j_2 j_2} \otimes \cdots \otimes \beta_{j_n j_n}$$

where $id_{j_1, j_2, \cdots, j_n}$ is the identity endomorphism on $V(\Sigma, D_0, (j_1, j_2, \cdots, j_n))$.

If we glue the ends of $\Sigma \times I$ via the identity map we get the e-3-manifold from the statement. The gluing axiom implies that in the formula above the identity matrices get replaced by their traces. Therefore

$$Z(\Sigma \times S^1, D, 0) = \bigotimes_{j_1, j_2, \cdots, j_n} dim V(\Sigma, D_0, (j_1, j_2, \cdots, j_n)) \beta_{j_1 j_1} \otimes \beta_{j_2 j_2} \otimes \cdots \otimes \beta_{j_n j_n}$$

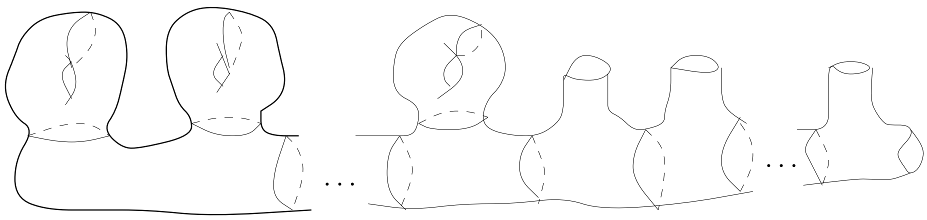

Figure 3.34: Surface for Proposition 5.1.

On the other hand the gluing axiom for V implies that

$dim V(\Sigma, D_0, (j_1, j_2, \cdots, j_n)) = c_{j_1, j_2, \cdots, j_n}$, which proves the proposition.$\square$

The following result shows that the Kauffman bracket not only determines our TQFT, but also can be recovered from it. It is an analogue of Theorem 1.1 in [21] which showed the presence of the skein relation of the Jones polynomial in the context of the Reshetikhin-Turaev TQFT. Before we state the theorem we have to introduce some notation.

Let us assume that the three e-manifolds $(M_1, D_1, 0)$, $(M_2, D_2, 0)$ and $(M_3, D_3, 0)$ are obtained by gluing to the same e-manifold the genus 2 e-handlebodies from Fig. 3.35 respectively, where the gluing occurs along the "exterior" punctured spheres. Note that the three handlebodies have the same structure on the "exterior" spheres, so they produce the same change of framing (if any) when gluing.

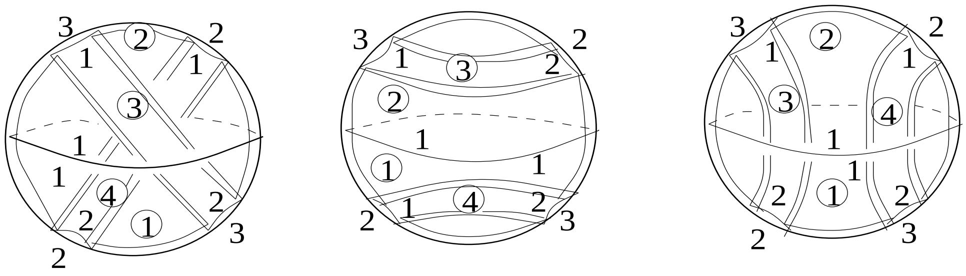

Figure 3.35: Extended genus 2 handlebodies

The "interior" annuli of the handlebodies are part of the boundaries of our 3-manifolds. The gluing axiom implies that $V(\partial M_i, D_i)$ splits as a direct sum $V_i \bigoplus V_i'$, where V_i is the subspace corresponding to the labeling of the ends of the annuli by 1. Moreover, the gluing axiom for Z implies that $Z(M_i, D_i, 0)$ also splits as $v_i \oplus v_i'$ where $v_i \in V_i$ and $v_i' \in V_i'$. On the other hand the spaces V_1, V_2 and V_3 are canonically isomorphic. Indeed, they have a common part, to which the vector spaces corresponding to the two annuli with ends labeled by 1 are attached via the map $x \to x \otimes \beta_{11} \otimes \beta_{11}$. Thus v_1, v_2 and v_3 can be thought as lying in the same vector space. With this convention in mind, the following result holds.

THEOREM 5.1. The vectors v_1, v_2, and v_3 satisfy the Kauffman bracket skein relation

$$v_1 = Av_2 + A^{-1}v_3.$$

Proof: By the gluing axiom for Z we see that it suffices to prove the theorem in the case where M_1, M_2 and M_3 coincide with the three handlebodies (i.e. when the manifold to which they get glued is empty).

The first e-manifold is obtained by first taking the mapping cylinder of the homeomorphism on a pair of pants that takes the "right leg" over the "left leg" as shown in Fig. 3.36 (it should be distinguished from a move in the sense that it really maps one seam into the other), then composing it with the move $B_{23}^{(1)}$, and finally by expanding two annuli via moves of type A^{-1}.

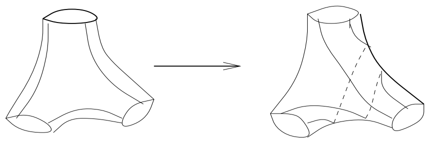

Figure 3.36: Map on a pair of pants

We get

$$v_1 = B_{23}\hat{\beta}_{011} \otimes \beta_{011} \otimes \beta_{11} \otimes \beta_{11} + B_{23}\hat{\beta}_{211} \otimes \beta_{211} \otimes \beta_{11} \otimes \beta_{11}$$

where for $x \in V_{abc}$ we denote by $\hat{x}$ the vector in $(V_{abc})^*$ with the property that $< x, \hat{x} >= 1$. By the definition of the pairing $\hat{\beta}_{011} = d_1^2X^{-2}\beta_{011}$ and $\hat{\beta}_{211} = d_1^2d_2X^{-2}\beta_{211}$. The computation of $B_{23}\beta_{011}$ and $B_{23}\beta_{211}$ is described in Fig. 3.37.

Hence

$$v_1 = -A^3d_1^2X^{-2}\beta_{011} \otimes \beta_{011} \otimes \beta_{11} \otimes \beta_{11} + A^{-1}d_1^2d_2X^{-2}\beta_{211} \otimes \beta_{211} \otimes \beta_{11} \otimes \beta_{11}.$$

The second manifold can be obtained by gluing along a disk the mapping cylinders of two annuli. The mapping cylinder of an annulus has the invariant

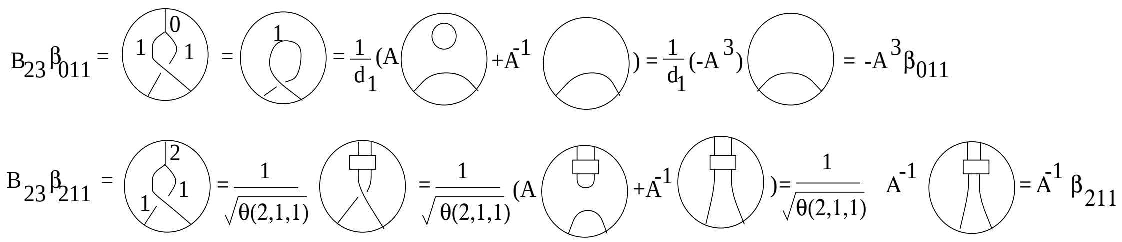

Figure 3.37: Skein computation

$\oplus_a \hat{\beta}_{aa} \otimes \beta_{aa} = \oplus_a d_a^2 X^{-1} \beta_{aa} \otimes \beta_{aa}$, so after expanding a disk and gluing the two copies together we get $\oplus_{a,b} d_a^2 d_b^2 X^{-2} \beta_{0aa} \otimes \beta_{0bb} \otimes \beta_{aa} \otimes \beta_{bb}$. But we are only interested in the component of the invariant for which $a = b = 1$, hence $v_2 = d_1^4 X_{-2} \beta_{011} \otimes \beta_{011} \otimes \beta_{11} \otimes \beta_{11}$.

Finally, the third e-manifold is the mapping cylinder of the identity with two expanded annuli, hence

$$v_3 = -d_1^2 X^{-2} \beta_{011} \otimes \beta_{011} \otimes \beta_{11} \otimes \beta_{11} + d_1^2 d_2 X^{-2} \beta_{211} \otimes \beta_{211} \otimes \beta_{11} \otimes \beta_{11}.$$

The conclusion follows by noting that the diagram that gives the value of $d_1^2 = \Delta_1$ is the unknot, hence $d_1^2 = -A^2 - A^{-2}$.$\square$

As a consequence of the theorem we will compute the formula for the invariant of the complement of a regular neighborhood of a link.

PROPOSITION 5.2. Let L be a framed link with k components, and M be the complement of a regular neighborhood of L. Consider on ∂M the DAP-decomposition D whose decomposition curves are the meridinal circles of L (one for each component) and whose seams are parallel to the framing (see Fig. 3.38.a)). Then

$$Z(M, D, 0) = \frac{1}{X} \sum_{n_1 n_2 \cdots n_k} < S_{n_1}(\alpha), S_{n_2}(\alpha), \cdots, S_{n_k}(\alpha) >_L \beta_{n_1 n_1} \otimes \beta_{n_2 n_2} \otimes \cdots \beta_{n_k n_k}$$

where the sum is over all labels, and $< \cdot, \cdot, \cdots, \cdot >_L$ is the link invariant defined in

Section 3.

PROOF: We assume that L is given by a diagram in the plane with the blackboard framing. When L is the unknot the invariant can be obtained from Proposition 5.1 applied to the case where Σ is a disk, so in this situation $Z(M,D,0) = 1/X\Sigma_n d_n^2\beta_{nn}$ and the formula holds. By taking the connected sum of k copies of the complement of the unknot, and using the gluing axiom for Z we see that the formula also holds for the trivial link with k components. Let us prove it in the general case. Put $Z(M,D,0) = 1/X\sum_{n_1n_2\cdots n_k} c_{n_1n_2\cdots n_k}\beta_{n_1n_1}\otimes\beta_{n_2n_2}\otimes\cdots\beta_{n_kn_k}$. We want to prove that

$$c_{n_1n_2\cdots n_k} =< S_{n_1}(\alpha), S_{n_2}(\alpha), \cdots, S_{n_k}(\alpha) >_L . \tag{3.1}$$

Since by Theorem 5.1, $c_{11\cdots 1}$ and $< S_1(\alpha), S_1(\alpha), \cdots, S_1(\alpha) >_L$ satisfy both the Kauffman bracket skein relation, the equality holds when all indices are equal to 1. If some of the indices are equal to 0, the corresponding link components can be neglected (by erasing them in the case of the link, and by gluing inside solid tori in the trivial way in the case of the 3-manifold). Therefore the equality holds if $n_i = 0, 1$, $i = 1, 2, \cdots, k$.

For a tuple $\mathbf{n} = (n_1, n_2, \cdots, n_k)$ let $\mu(\mathbf{n}) = max\{n_i | i = 1, 2, \cdots, k\}$ and $\nu(\mathbf{n}) = card\{i|\ n_i = \mu(\mathbf{n})\}$. We will prove (1) by induction on $(\mu(\mathbf{n}), \nu(\mathbf{n}))$, where the pairs are ordered lexicographically. Suppose that the property is true for all links and all tuples $\mathbf{n}'$ with $(\mu(\mathbf{n}'), \nu(\mathbf{n}')) < (\mu(\mathbf{n}), \nu(\mathbf{n}))$ and let us prove it for $(\mu(\mathbf{n}), \nu(\mathbf{n}))$.

Let M_0 be the product of a pair of pants with a circle. Put on M_0 a DAP-decomposition D_0 as described in Proposition 5.1. Then

$$Z(M_0, D_0, 0) = \sum_{mnp} \delta_{mnp}\beta_{mm}\otimes\beta_{nn}\otimes\beta_{pp}$$

where $\delta_{mnp} = 1$ if (m, n, p) is admissible and 0 otherwise.

Assume that in the tuple $\mathbf{n} = (n_1, n_2, \cdots, n_k)$, $n_k = \mu(\mathbf{n})$. Glue M_0 to M

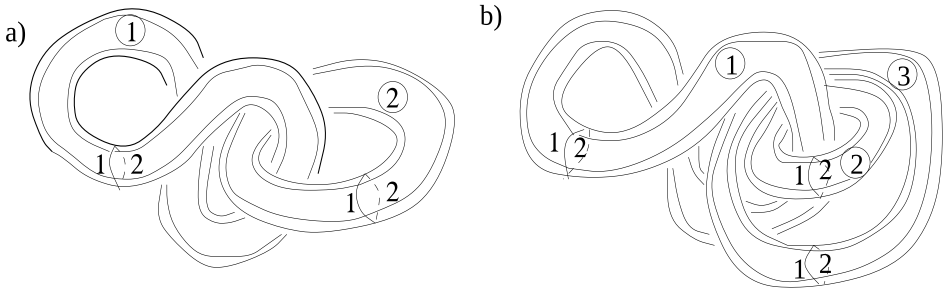

Figure 3.38: Link complements

along the k-th torus of M such that in the gluing process the DAP-decompositions of the two tori overlap. We get an e-3-manifold $(M_1, D_1, 0)$ that is nothing but the manifold associated to the link L' obtained from L by doubling the last component (see Fig. 3.38.b)).

Let $Z(M_1, D_1, 0) = 1/X\Sigma d_{m_1m_2\cdots m_k,m_{k+1}}\beta_{m_1m_1} \otimes \beta_{m_2m_2} \otimes \cdots \beta_{m_{k+1}m_{k+1}}$. The gluing axiom, together with relation 6.a) from Section 3 imply that $d_{m_1,m_2,\cdots,m_{k+1}} = \sum_p \delta_{m_km_{k+1}p}c_{m_1,m_2,\cdots,m_{k-1},p}$. In particular

$$d_{n_1,n_2,\cdots,n_{k-1},n_k-1,1} = c_{n_1,n_2,\cdots,n_k-2} + c_{n_1,n_2,\cdots,n_k}.$$

Applying the induction hypothesis we get

$$c_{n_1n_2\cdots n_k} =< S_{n_1}(\alpha), \cdots, S_{n_{k-1}}(\alpha), S_{n_k-1}(\alpha), \alpha >_{L'} - < S_{n_1}(\alpha, \cdots, S_{n_{k-1}}(\alpha), S_{n_k-2}(\alpha) >_L .$$

But $< S_{n_1}(\alpha, \cdots, S_{n_{k-1}}(\alpha), S_{n_k-1}(\alpha), \alpha >_{L'}=< S_{n_1}(\alpha, \cdots, S_{n_{k-1}}(\alpha), \alpha S_{n_k-1}(\alpha) >_L$ and since $S_{n_k}(\alpha) = \alpha S_{n_k-1}(\alpha) - S_{n_k-2}(\alpha)$ (see [30]), we obtain the equality in (1) and the proposition is proved.$\square$

REMARK. As an easy consequence of this result one can give a short proof of the formula for the colored Jones polynomials of cable knots.

COROLLARY. If M is a closed 3-manifold obtained by performing surgery on the framed link L with k components, then

$$Z(M,0) = X^{-k-1}C^{-\sigma} < \omega, \omega, \cdots, \omega >_L$$

where Σ is the signature of the linking matrix of L.

PROOF: We may assume that L is given by a link diagram in the plane and its framing is the blackboard framing. Let $(M_1, D_1, 0)$ be the e-3-manifold associated to L as in the statement of Proposition 5.2. Consider the e-manifold $(M_2, D_2, 0)$ where M_2 is the solid torus and $D_2$is described in Fig. 3.39. Applying Proposition 5.2 to the unknot we see that the invariant of this e-manifold is $1/X\Sigma_n d_n^2 \beta_{nn}$.

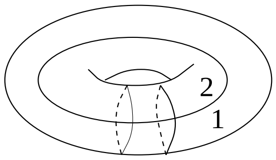

Figure 3.39: Extended solid torus

If we glue k copies of this manifold to M_1 such that the DAP-decompositions overlap we get M. In the gluing process the framing changes by $-\sigma(L_1, L_2, L_3)$ (see Section 1) where L_1 is the kernel of $H_1(\partial M_1) \to H_1(M_1)$, L_2 is the Lagrangian space spanned in $H_1(\partial M)$ by the meridinal circles of the link, and L_3 is the one spanned by the curves that give the framing. It is a standard result in knot theory that $-\sigma(L_1, L_2, L_3) = \sigma$, the linking matrix of L. Using the gluing axiom for Z we get

$$Z(M,\sigma) = X^{-k-1} \sum_{n_1, n_2, \cdots, n_k} d_{n_1}^2 d_{n_2}^2 \cdots d_{n_k}^2 < S_{n_1}(\alpha), S_{n_2}(\alpha), \cdots, S_{n_k}(\alpha) >_L =$$

$$= X^{-k-1} < \omega, \omega, \cdots, \omega >_L$$

hence

$$Z(M,0) = X^{-k-1}C^{-\sigma} < \omega, \omega, \cdots, \omega >_L . \square$$

We make the remark that this gives the invariants of 3-manifolds as normalized in [30].

A similar argument, based again on the skein relation for the invariants of three manifolds from Theorem 5.1, can be used to prove the formula for the quantum invariant of three manifolds with boundary. The TQFT with corners can also be used for the proof of the formulas of Rozansky for the invariants of Seifert fibered spaces and for a more direct approach to the theory of Turaev-Viro modules in the context of cyclic covers of complements of knots. We also consider that the above ideas can be followed to construct a universal TQFT.

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