The Kauffman Bracket Skein Algebra of the Punctured Torus

by

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ABSTRACT

This dissertation studies the Kauffman bracket skein algebra of the punctured torus.

The first chapter contains the historical background on the Kauffman bracket skein algebra and its applications.

The second chapter contains the multiplication rule for the Kauffman bracket skein algebra of the cylinder over the punctured torus. The explicit formula for the multiplicative rule for the case of the Kauffman bracket skein algebra of the cylinder over torus was found by Frohman and Gelca. In this work, we try to extend their result to the torus with a puncture. The punctured torus has a multiplicative structure of the Kauffman bracket skein algebra that is considerably more complicated than that of the torus, and we ilustrate this with examples for which the crossing number is small.

In Chapter 3, we describe the action of the Kauffman bracket skein algebra on certain vector spaces that arise as relative Kauffman bracket skein modules of tangles in the torus. We analyze several particular cases, then we derive the general formula for the action of the Kauffman bracket skein algebra on the corresponding skein modules by using the geometric properties of the Jones-Wenzl idempotent, which is the main result of the dissertation.

In Chapter 4, we show how the Reshetikhin-Turaev representation of the mapping class group of the punctured torus can be computed from the representation of the Kauffman bracket skein algebra, and based on this we derive explicit formulas for the matrices of the generators of the mapping class group of the punctured torus.

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CHAPTER 1
INTRODUCTION

In 1985 Vaughan Jones introduced a polynomial invariant of knots and links in the three dimensional sphere, as a consequence of his studies of operator algebras. For a knot this is a Laurent polynomial in the variable t, while for a link with an even number of components it is a Laurent polynomial in $t^{\frac{1}{2}}$. It is standard to denote this polynomial invariant by $V_K(t)$, where K is the knot (or link) in question.

The Jones polynomial is computed using a knot (or link) diagram, namely a planar projection of the knot (or link), by the following skein relations:

$$t^{-1}V_{D_+}(t) - tV_{D_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{D_0}(t) \tag{1.1}$$

$$V_0(t) = 1 \tag{1.2}$$

where 0 stands for the trivial knot (or unknot) and D_+, D_- and D_0 are the diagrams of three links identical except in a disk where they look as in Figure 1.1. Jones' discovery

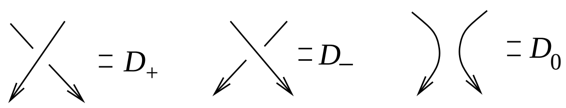

Figure 1.1.

had a huge impact on knot theory. Several other polynomial invariants followed: the Kauffman bracket, the Kauffman polynomial, the HOMFLY polynomial. These proved powerful tools in the classification of knots. As to date, while not a complete invariant, the Jones polynomial distinguishes the unknot from all other knots.

A year after Jones, Louis Kauffman announced the discovery of another polynomial invariant, the Kauffman bracket, denoted by $< K >$. This is an invariant for framed knots and links in the 3-dimensional sphere, meaning that the link components are annuli, not circles. The framing keeps track of how a link component twists about itself. In all that follows we will work with the blackboard framing of all knots and links, meaning that the annuli are always parallel to the plane of the paper. As such, in diagrams we only draw one boundary of the annulus, the other being understood to run parallel to it. In the particular case where the link lies in the cylinder over a surface, we agree that the framing is parallel to the surface.

The Kauffman bracket is computed by the skein relations described in Figure 1.2. The Jones polynomial $V_K(t)$ can be obtained from the Kauffman bracket $f_K(t)$ by replacing variable t by $t^{-\frac{1}{4}}$. More precisely $V_K(t) = f_K(t^{-\frac{1}{4}})$, where $f_K(t) = (-t^3)^{-w(k)} < K >$, $w(K)$ is writhe of K (the number of times K twists about itself), and $< K >$ is bracket polynomial of K. A quick look shows that the Kauffman bracket skein relations are a faster method for computing the Jones polynomial. One should however keep in mind that, unlike the Jones polynomial, the Kauffman bracket keeps track of twistings.

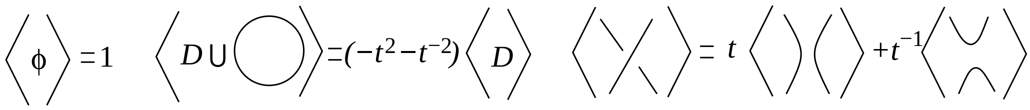

Figure 1.2.

In 1991, Jozef Przytycki introduced the concept of *skein module* of an orientable 3-manifold by generalizing the polynomial knot and link invariants to knots and links in arbitrary manifolds. This was done by imposing skein relations on the formal linear sums of oriented link in 3-manifold. Among these, the most studied were the Kauffman bracket skein modules, mainly due to their relationship to the $SL(2)$-character varieties of the fundamental groups of manifolds.

Skein modules turned out to be closely related to topological quantum field theory, character varieties, and the representation theory of quantum groups. The use of Kauffman bracket skein modules for constructing a topological quantum field theory was done in the early nineties by Lickorish in [12], and Blanchet, Habbegger, Masbaum, and Vogel in [1]. On the other hand Frohman, Bullock, and Sikora related skein modules to $SL(2)$-character varieties of the fundamental groups of 3-manifolds. Frohman, Gelca, Uribe [4], [7], [5] related Kauffman bracket skein modules to quantum mechanics. Let me point out that on cylinders over surfaces the Kauffman bracket skein modules carry a natural algebra structure. Frohman and Gelca [4] gave explicit descriptions of the Kauffman bracket skein algebra of the cylinder over the torus and of the representation of this algebra on the Kauffman bracket skein module of the solid torus. In our work, we will attempt to extend the work of Frohman and Gelca to the case of the torus with one puncture. As such we will study the multiplicative structure of the Kauffman bracket skein algebra of the punctured torus and its representations that arise from the quantum mechanical point of view.

By contrast with the multiplication rule in the Kauffman bracket skein algebra of torus, the multiplication rule in the Kauffman Bracket Skein algebra of a punctured torus is much more complicated, and we were unable to find a general formula. Thus we will restrict ourselves to cases of relatively small crossing numbers. Next we study the representations of this algebra motivated by the associated quantum theory. The algebra will act on skein modules of the solid torus with a distinguished disk on the boundary. Using these representations we will derive the the Reshetikhin-Turaev representation of the mapping class group of a punctured torus.

CHAPTER 2
THE MULTIPLICATIVE STRUCTURE OF THE KAUFFMAN BRACKET SKEIN ALGEBRA OF THE PUNCTURED TORUS

Before we examine the multiplicative structure of the Kauffman bracket skein algebra of a punctured torus, we will review the required background materials.

2.1 Background material

2.1.1 Basic terminology

Throughtout this paper, t will denote a variable or a complex number. By definition, a knot in a 3-dimensional manifold M is an embedding of a circle in M. A link is an embedding of a disjoint union of finitely many circles in M. A framed knot or link in M is defined as the embedding in M of one, respectively several disjoint annluli. We always draw these annuli to be parallel to the plane of the paper, or in case the knot or link is embedded in the cylinder over a surface to be parallel to the surface. We call this framing as blackboard framing. Once this convention is made, for each annulus it suffices to draw just one boundary components, thus framed knots and links can actually be drawn as knots and links.

Definition 1. *Consider the set $\mathcal{L}$ of a isotopy classes of framed links in M, including the empty link (that has no component). Consider $\mathbb{C}[t, t^{-1}]\mathcal{L}$ denote free $\mathbb{C}[t, t^{-1}]$-module with a basis $\mathcal{L}$. Define $S(M)$ to be the smallest submodule of $\mathbb{C}[t, t^{-1}]\mathcal{L}$ contaning all the expressions of the form shown in Figure 2.1, where the links in each expression are identical except inside an embedded ball, where they look as depicted. The* Kauffman bracket skein module *$K_t(M)$ of M is defined to be quotient $\mathbb{C}[t, t^{-1}]\mathcal{L}/S(M)$.*

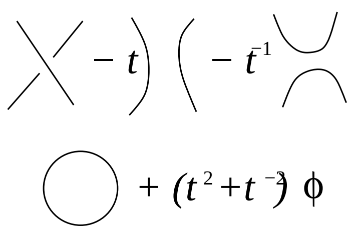

Figure 2.1.

When t is set to be a complex number, the Kauffman bracket skein module becomes a $\mathbb{C}$-vector space. However, by abuse of language, it is still called a module.

If $M = \Sigma \times I$, where Σ is a surface and $I = [0, 1]$, then the Kauffman bracket skein module has a natural multiplication which turns it into an algebra. This multiplication is obtain by gluing two copies of the cylinder to obtain another copy of the cylinder. Explicitly, let α , β be two element in $K_t[\Sigma \times I]$. We can assume α ,β as links in two diffrent cylinders over surface. We glue the 0-end of the cylinder containing α to the 1-end of the cylinder containing β. We get a cylinder containing both α and β. By evaluating these framed links in $K_t(\Sigma \times I)$,we can get $\alpha * \beta$. As such we obtain the *Kauffman bracket skein algebra* of the cylinder over a surface $K_t(\Sigma \times [0, 1])$.

If M is a 3-dimensional manifold with boundary ∂M, then the topological operation of gluing the cylinder over the boundary to M induces a $K_t(\partial M \times I)$-module structure on $K_t(M)$. Explicitly, if α is an element of $K_t(\partial M \times I)$, and x is an element of $K_t(M)$, glue the 0-end of $\partial M \times I$ the boundary of M. We obtain something that is homeomorphic to M, which now contains both α and x. Define this to be $\alpha \cdot x$. Now project link α onto N. When t is a complex number, we obtain a representation of $K_t(\Sigma \times I)$, where $\Sigma = \partial M$.

In this chapter M will be cylinder over the torus, the cylinder over the torus with one puncture, or the solid torus. Throughout this paper, $\Sigma_{i,j}$ denote surface of genus i with j punctures.

2.1.2 Basic properties of $K_t(\Sigma_{1,0} \times I)$

First, we recall the multiplicative structure to the Kauffman bracket skein module $K_t(\Sigma_{1,0} \times I)$ of the cylinder over the torus as it was given by Frohman and Gelca [4].

The Kauffman bracket skein module $K_t(S^1 \times D)$ of the solid torus is a $K_t(\Sigma_{1,0} \times I)$-module. In addition to this, we also know that $K_t(S^1 \times D)$ has a mulitiplicative structure, being isomorphic to polynomial algebra over $C[t, t_{-1}][\alpha]$, where α is the simple closed curve in the solid torus that generates its fundamental group. We make the convention that α^n consists of n parallel copies of α.

Let T_n be the n-th Chebyshev polynomial defined recursively by $T_0 = x$, $T_{n+1} = T_n \cdot T_1 - T_{n-1}$. For p, q in $\mathbb{Z}$, if p, q are coprime , define $(p, q)_T =(p, q)$ to be the curve of slope q/p on the torus (which defines the first homology class (p, q) of the homology with integer coefficients). We let $(p, q)^k$ denote k parallel copies of the (p, q)-curve on the torus. Then, for general p and q not necessarily coprime, we define $(p, q)_T = T_n(\frac{p}{n}, \frac{q}{n})$, which is an element of $K_t(\Sigma_{1,0} \times I)$ defined by replacing the variable of the Chebyshev polynomial by the $(\frac{p}{n}, \frac{q}{n})$-curve,where n is the greatest common divisor of p and q. For m,n and $gcd(p, q) = gcd(r, s) = 1$, the geometric intersection number (the crossing number) of $T_n(p,q)$ and $T_m(r,s)$ is the absolute value of $mn|^{pq}_{rs}|$,where $|^{pq}_{rs}|$ is the determinant.

The main result about $K_t(\Sigma_{1,0} \times I)$ from [4] is:

Theorem 2.1.1 (the product-to-sum formula)**.** *In $K_t(\Sigma_{1,0} \times I)$ the following relation holds*

$$(p,q)_T * (r,s)_T = t^{|^{pq}_{rs}|}(p+r, q+s)_T + t^{-|^{pq}_{rs}|}(p-r, q-s)_T, \qquad (2.1)$$

for any $p, q, r, s \in \mathbb{Z}$.

2.2 The multiplication structure of $K_t(\Sigma_{1,1} \times I)$

As we saw in theorem 2.2.1 , the multiplicative structure of Kauffman Bracket Skein algebra $K_t(\Sigma_{1,0} \times I)$of torus can be expressed explicitly in the form of the product-to-sum formula. The multiplicative structure of Kauffman Bracket Skein algebra $K_t(\Sigma_{1,1} \times I)$of a pundtured torus dosen't follow the product-to-sum formula as we will see later. For our purpose, we will restrict to the cases with small crossing number 1,2,3,4. To explain multiplicative structure of that case ,it will be sufficient to think about $(p,q)_T * (0,1)_T$ where $p > q$ and $p = 1, 2, 3, 4$. .

2.2.1 Basic material about $K_t(\Sigma_{1,1} \times I)$

Since, every element in $K_t(\Sigma_{1,1} \times I)$ can be described through the (p,q)-curve on the punctured torus $\Sigma_{1,1}$,and multiplicative structure can be explained through the multiplication of curves of the punctured torus $\Sigma_{1,1}$, we give way to describe (p,q)-curve before everything.

At first, we can regard (p,q)-curves as a line segment with two end points $(0,0)$ and (p,q) in $\mathbb{C}$ by viewing $\mathbb{C}$as a covering space of $\Sigma_{1,0}$ .This way of viewing curve can be applied to $\Sigma_{1,1}$ also. The description of some elements in $K_t(\Sigma_{1,1} \times I)$ as curves on a functured torus $\Sigma_{1,1}$ is given in the following example.

Example 2.2.1. *The simplest skeins are shown in Figure 2.2.*

we disregard the orientation of curve on the punctured torus, so (p,q)-curve and $(-p,-q)$-curve denote same curve. In general,(p,q)-curve on the punctured torus can be described as follow. Assume p, q are coprime, and $p > q$. A punctured torus is a quotient space of a unit square with a hole, described as follow. As we can see through the

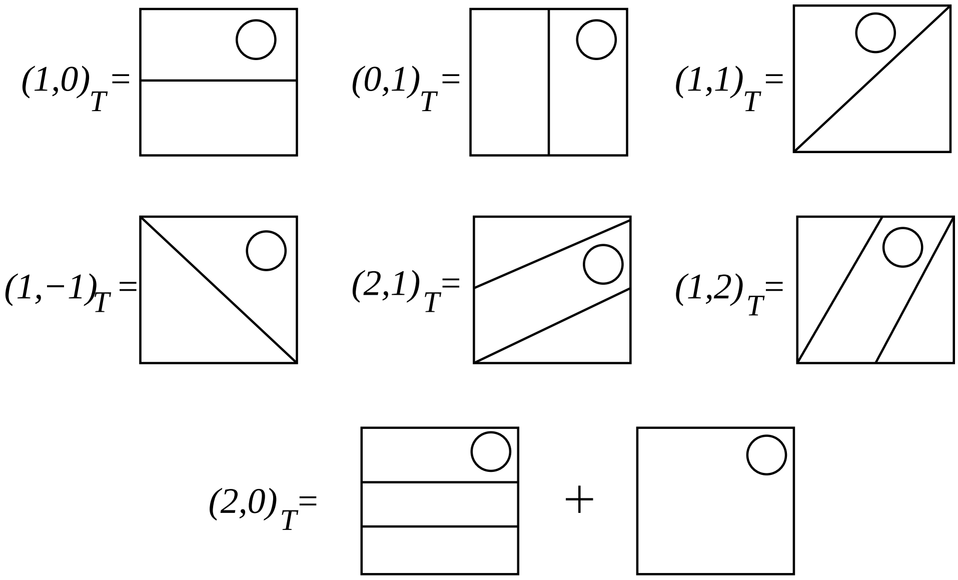

Figure 2.2.

picture, we gave coordinate for all edges of unit square as follow. Left and right vertical edges have coordinate $(0, \frac{i}{p})$, $(1, \frac{i}{q})$ and upper and lower edges have coordinate $(\frac{j}{q}, 1)$ and$(\frac{j}{q}, 0)$,respectively , where $0 \le i \le p$ and $0 \le j \le q$. At fist, draw a line from $(0, 0)$ to $(1, \frac{q}{p})$with slope $\frac{q}{p}$),next, draw a line from$(0, \frac{q}{p})$to $(\frac{x}{q}, 1)$with slope $\frac{q}{p}$.Thirdly, draw another line from $(\frac{x}{q}, 0)$ to $(1, y)$ with slope $\frac{q}{p}$. By keeping continuing this process untill we have last point as $(1, 1)$ we can get (p, q)-curve on the punctured torus.

Now we will give the basic results about $K_t(\Sigma_{1,1} \times I)$

Lemma 2.2.1. *The following formula holds*

$$(1, 0)_T * (0, 1)_T = t(1, 1)_T + t^{-1}(1, -1)^T.$$

Proof. The computation is shown in Figure 2.3.

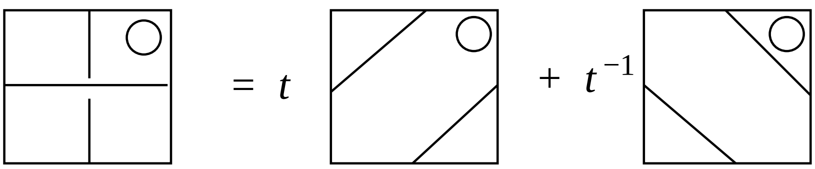

Figure 2.3.

□

For the next result, let us introduce the element η defined in Figure 2.4.

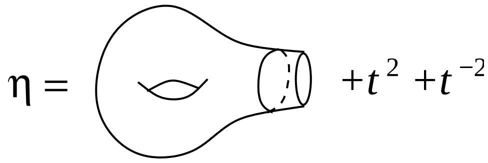

Figure 2.4.

Lemma 2.2.2. *The following formula holds*

$$(2,1)_T * (0,1)_T = t^2(2,2)_T + t^{-2}(2,0)_T + t^2 + t^{-2} + \eta.$$

Proof. We proceed as in Figure 2.5. This is further equal to

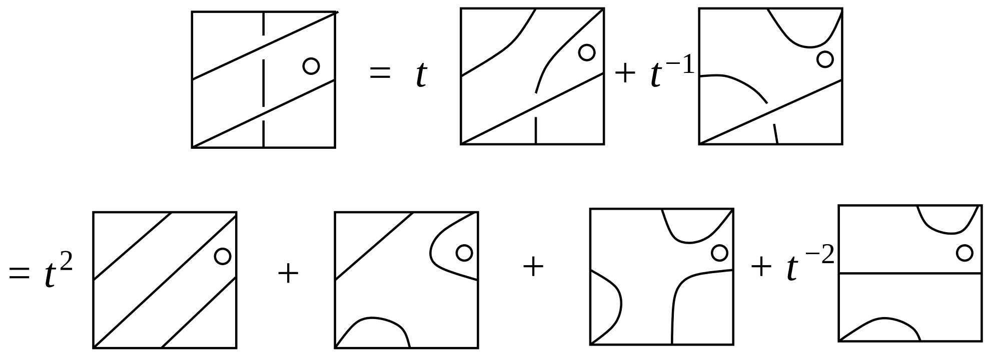

Figure 2.5.

$formula$

□

This lemma shows the difference between the multiplication rule in $K_t(\Sigma_{1,1} \times I)$ and that in $K_t(\Sigma_{1,0} \times I)$, in particular we see that the product-to-sum formula does not extend to this situation.

Lemma 2.2.3. *The following formula holds*

$$(2,0)_T * (0,1)_T = t^2(2,1)_T + t^{-2}(2,-1)_T.$$

Proof. Using the definition of $(2,0)_T$, we can write

$$(2,0)_T * (0,1)_T = [(1,0)_T * (1,0)_T - 2] * (0,1)_T$$
$$= [t(1,0)_T * (1,1)_T] + [t^{-1}(1,0)_T * (1,-1)_T] - 2(0,1)_T$$
$$= t^2(2,1)_T + (0,1)_T + t^{-2}(2,-1)_T + (0,1)_T - 2(0,1)_T$$
$$= t^2(2,1)_T + t^{-2}(2,-1)_T$$

as desired. □

Using these two results we will exhibit a formula for the multiplication of two skeins with algebraic intersection number equal to 2.

Let L denote the longitude and M the meridian curves in $\Sigma_{1,0}$ respectively $\Sigma_{1,1}$. Let $h_L, h_M, and, h_R : \Sigma_{1,i} \to \Sigma_{1,i}$ be the maps definded by $h_L(e^{i\theta}, e^{i\phi}) = (e^{i(\theta+\phi)}, e^{i\phi})$, $h_M(e^{i\theta}, e^{i\phi}) = (e^{i\theta}, e^{i(\theta+\phi)})$, $h_R(e^{i\theta}, e^{i\phi}) = (e^{i\phi}, e^{i\theta})$, repectively, where $0 \le \theta, \phi \le 2\pi$. As such h_L and h_M are the longitudinal respectively meridinal twist (or Dehn twist). We know that $h_L|M = h_M|L = identity$ in $\Sigma_{1,i}$ where $i = 0, 1$.

Now we consider homology group with integer coefficients $H_1(\Sigma_{1,i}, \mathbb{Z})$, which is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. We know that L and M determine the basis elements $(1,0)$ and $(0,1)$ respectively.

Let $h_{L_*}, h_{M_*}, h_{R_*} : H_1(\Sigma_{1,i}) \to H_1(\Sigma_{1,i})$ be the induced homology maps. Under this basis, the matrix representations for $h_{L_*}, h_{M_*}, and, h_{R_*}$ are as follows:

$$h_{L_*} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, h_{M_*} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, h_{R_*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The images of a (p,q)-curve on the torus under these maps and their inverses are

$$h_{L_*}(p,q) = (p, p+q), \quad h_{M_*}(p,q) = (p+q, q), \quad h_{R_*}(p,q) = (q,p),$$
$$h^{-1}_{L_*}(p,q) = (p, q-p), \quad h^{-1}_{M_*}(p,q) = (p-q, q), \quad h^{-1}_{R_*}(p,q) = (q,p).$$

Lemma 2.2.4. *Let (p,q) , (r,s) be the $(p,q)-curve$, $(r,s)-curve$ in $\Sigma_{1,i}$,respectively and $i = 0,1$. Suppose $|^{pq}_{rs}|$=$m(>0)$ and $gcd(r,s) = 1$, then there exist $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{Z})$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} m \\ k \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,where $k = 1, 2, ..., m-1$.*

Proof. If $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$=$m$, then we know that the crossing number of curves (p,q) and (r,s) is m. Suppose $|r|>|s|$, then, by division algorithm, there are i and j in $\mathbb{Z}$ such that $r=s\cdot i+j$ where $0\leqq j<|s|$. So $h_{L_*}^{-i}(r,s)=(j,s)$. Suppose $|r|<|s|$, then there are t and u in $\mathbb{Z}$ such that $s=r\cdot t+u$ with $0\leqq u<|r|$. This time $h_{M_*}^{-t}(r,s)=(r,u)$. By repeating these alternately, we can reach the curves $(0,1)$ or $(1,0)$. If it is $(1,0)$, then we can switch this into $(0,1)$ using $h_{R_*}(1,0)=(0,1)$. This implies that there exist 2×2 matrix $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ in $SL_2(\mathbb{Z})$ such that $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$, where p_1,q_1 belong to $\mathbb{Z}$. We also know that $\begin{vmatrix} p_1 & q_1 \\ 0 & 1 \end{vmatrix}$=$m$. This implies that $p_1=m$. If $|q_1|>m$, then $q_1=m\cdot l+k$ where $0\leqq k<m$. Now $h_{M_*}^{-l}\cdot\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} m \\ k \end{pmatrix}$ and

$$h_{M_*}^{-l}\cdot\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}=h_{M_*}^{-l}\cdot\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. Then this matrix belongs to $SL_2(\mathbb{Z})$. □

Theorem 2.2.5. *If* $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$ $=\pm 2$, *then*

$$(p,q)_T*(r,s)_T=t^{|{pq \atop rs}|}(p+r,q+s)_T+t^{-|{pq \atop rs}|}(p-r,q-s)_T+\rho$$

where $\rho=\begin{cases} \eta & \textit{if gcd(p,q)=gcd(r,s)=1;} \\ 0 & \textit{otherwise.} \end{cases}$.

Proof. It suffices to check the case $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$=2. Then we know that the curves (p,q) and (r,s) cross in two points. By above proposition, we know that there exist a homeomorphism $f:\Sigma_{1,1}\to\Sigma_{1,1}$ such that the induced isomorphism $f_*=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(Z)$ satisifies $f_*(p,q)=(p_0,q_0)$ and $f_*(r,s)=(0,d)$ where $d=gcd(r,s)$. Also we know that $\begin{vmatrix} p_0 & q_0 \\ 0 & d \end{vmatrix}=2$, and $(p_0,q_0)*(0,1)=t^{p_0}(p_0,q_0+1)_T+t^{-p_0}(p_0,q_0-1)_T+\rho$ where $p_0=1,2,or q_0=0$. In particular, if $gcd(p,q) = gcd(r,s) = 1$, then $d = 1$, $p_0 = 2$, and , $q_0 = 1$. This implies that $f_*(p,q)_T = (2,1)_T, f_*(r,s)_T = (0,1)_T$. So

$$(p,q)_T * (r,s)_T = f_*^{-1}[(2,1)_T * (0,1)_T] = f_*^{-1}[t^2(2,2)_T + t^{-2}(2,0)_T + \eta].$$

Therefore

$$(p,q)_T * (r,s)_T = t^2(p+r, q+s)_T + t^{-2}(p-r, q-s)_T + \eta.$$

If $gcd(r,s) = d, d \neq 1$, then $d = 2, p_0 = 1$ and $q_0 = 0$. Hence

$$(1,q_0)_T * (0,2)_T = t^2(1,q_0+2)_T + t^{-2}(1,q_0-2)_T = t^2(1,2)_T + t^{-2}(1,-2)_T.$$

We obtain $(p,q)_T * (r,s)_T = t^2(p+r,q+s)_T + t^{-2}(p-r,q-s)_T$. □

In particular, if $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \pm 1$, the product-to-sum fomula from [4] holds.

Let us consider a few other situations.

Proposition 2.2.6. *The following formula holds*

$$(3,0)_T * (0,1)_T = t^3(3,1)_T + t^{-3}(3,1)_T.$$

Proof. By applying the above result, and using the fact that $(3,0)_T = (1,0)_T^3 - 3(1,0)_T$, we obtain

$$\begin{aligned}
&(3,0)_T * (0,1)_T = [(1,0)_T^3 - 3(1,0)_T] * (0,1)_T = [(1,0)_T^3 * (0,1)_T] - 3[(1,0)_T * (0,1)_T] \\
&= ((1,0)_T * [t(1,0)_T * (1,1)_T] + [t^{-1}(1,0)_T * (1,-1)_T]) - [3(1,0)_T * (0,1)_T] \\
&= t^2[t(3,1)_T + t^{-1}(1,1)_T] + 2t(1,1)_T + 2t^{-1}(1,1)_T + t^{-2}[t(1,1)_T + t^{-1}(3,-1)_T] \\
&\quad -[3(1,0)_T * (0,1)_T] = t^3(3,1)_T + t^{-3}(3,-1)_T,
\end{aligned}$$

as desired. □

Proposition 2.2.7. *The following formula holds*

$$(3,1)_T * (0,1)_T = t^3(3,2)_T + t^{-3}(3,0)_T + t^{-1}(1,0)_T\eta$$

Proof. Write $(3,1)_T = t^{-1}(1,0)_T * (2,1)_T - t^{-2}(1,1)_T$. Applying Theorem 2.2.5, we can write

$$
\begin{aligned}
&(3,1)_T * (0,1)_T = [t^{-1}(1,0)_T * (2,1)_T - t^{-2}(1,1)_T] * (0,1)_T \\
&= t^{-1}(1,0)_T[t^2(2,2)_T + t^{-2}(2,0)_T + \eta] - t^{-2}[t(1,2)_T + t^{-1}(1,0)_T] \\
&= t[t^2(3,2)_T + t^{-2}(1,2)_T] + t^{-3}[(3,0)_T + (1,0)_T] + t^{-1}(1,0)_T\eta - t^{-1}(1,2)_T \\
&+t^{-3}(1,0)_T = t^3(3,2)_T + t^{-3}(3,0)_T + t^{-1}(1,0)_T\eta.
\end{aligned}
$$

□

Next we use the reflection homeomorphism $f\Sigma_{1,i} \to \Sigma_{1,i}$ defined by $f(x,y) = (x,-y)$. The induced mapf_* can be represented as $f_* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ We will use this to prove following result.

Proposition 2.2.8. *The following formula holds*

$$(3,2)_T * (0,1)_T = t^3(3,3)_T + t^{-3}(3,1)_T + t^{-1}(1,1)_T\eta.$$

Proof. We have

$$
\begin{aligned}
&(3,2)_T * (0,1)_T = h_{L_*}(f_*(3,1)_T * (0,1)) \\
&= h_{L*}(f_*(t^3(3,2)_T + t^{-3}(3,0)_T + t_T^{-1}(1,0)_T\eta)) \\
&= h_{L*}(t^{-3}(3,-2)_T + t^3(3,0)_T + t(1,0)_T) = t^{-3}(3,1)_T + t^3(3,3)_T + t(1,1)_T\eta.
\end{aligned}
$$

Here, when we applied f_*, the coefficient changed to its reciprocal due to the change of sign in the intersection number. As a matter of fact, we can also use the method of Proposition 2.2.9 to derive this result. □

Now we investigate the case of the crossing number equal to 4.

Proposition 2.2.9. *The following formula holds*

$$(4,0)_T * (0,1)_T = t^4(4,1)_T + t^{-4}(4,-1)_T$$

Proof. Note that $(4,0)_T = (1,0)_T^4 - 4(1,0)_T^2 + 2$, so

$$(4,0)_T * (0,1)_T = [(1,0)_T^4 * (0,1)_T] - 4[(1,0)_T^2 * (0,1)_T] + 2(0,1)_T$$

Now we know that

$$(1,0)_T^4 * (0,1)_T = (1,0)_T^3[(1,0)_T * (0,1)_T] = (1,0)_T^2[t(1,0)_T * (1,1)_T$$
$$+t^{-1}(1,0)_T * (1,-1)_T] = (1,0)_T^2[t^2(2,1)_T + (0,-1)_T + t^{-2}(2,-1)_T + (0,1)_T]$$
$$= (1,0)_T[t^3(3,1)_T + t(-1,-1)_T + 2t(1,1)_T + 2t^{-1}(1,-1)_T + t^{-1}(-1,1)_T$$
$$+t^{-3}(3,-1)_T].$$

This is further equal to

$$t^3(1,0)_T * (3,1)_T + t(1,0)_T * (-1,-1)_T + 2t(1,0)_T * (1,1)_T$$
$$+2t^{-1}(1,0)_T * (1,-1)_T + t^{-1}(1,0)_T * (-1,1)_T + t^{-3}(1,0)_T * (3,-1)_T$$
$$= t^4(4,1)_T + t^2(-2,-1)_T + (0,1)_T + t^2(2,1)_T + 2t^2(2,1)_T + 2(0,1)_T$$
$$+2t^{-2}(2,-1)_T + 2(0,1)_T + (0,1)_T$$
$$+t^{-2}(2,-1)_T + t^{-4}(4,-1)_T + t^{-2}(-2,1)_T = t^4(4,1)_T + t^{-4}(4,-1)_T$$
$$+4t^2(2,1)_T + 4t^{-2}(2,-1)_T + 6(0,1)_T$$

Also

$$4(1,0)_T^2 * (0,1)_T = 4[t(1,0)_T * (1,1)_T + t^{-1}(1,0)_T * (1,-1)_T]$$
$$= 4[t^2(2,1)_T + (0,-1)_T + t^{-2}(2,-1)_T + (0,1)_T]$$

This implies that $(4,0)_T * (0,1)_T = t^4(4,1)_T + t^{-4}(4,-1)_T$, as desired. □

Next three results proposition show some other aspects of the multiplication of $K_t(\Sigma_{1,1} \times I)$. Let $S_n(x)$ be the Chebyshev polynomial of second kind defined by $S_0 = 1$, $S_1 = x$, and $S_{n+1} = xS_n - S_{n-1}$.

Proposition 2.2.10. *The following formula holds,*

$$(4,1)_T * (0,1)_T = t^4(4,2)_T + t^{-4}(4,0)_T + [t^{-2}(2,0)_S + t^(0,0)_S]\eta$$

where $(p,q)_S = S_n(\frac{p}{n}, \frac{q}{n})$ *with* $n = gcd(p,q)$.

Proof. Write $(4,1)_T = t^{-1}[(1,0)_T * (3,1)_T] - t^{-2}(2,1)_T$.This implies that

$$
\begin{aligned}
&(4,1)_T * (0,1)_T = t^{-1}(1,0)_T[(3,1)_T * (0,1)_T] - t^{-2}[(2,1)_T * (0,1)_T] \\
&= t^{-1}(1,0)_T[t^3(3,2)_T + t^{-3}(3,0)_T + t^{-1}(1,0)_T\eta] - t^{-2}[t^2(2,2)_T + t^{-2}(2,0)_T + \eta] \\
&= t^2[(1,0)_T * (3,2)_T] + t^{-4}[(1,0)_T * (3,0)_T] + t^{-2}[(1,0)_T * (1,0)_T\eta] - (2,2)_T \\
&-t^{-4}(2,0)_T - t^{-2}\eta \\
&= t^2[t^2(4,2)_T + t^{-2}(2,2)_T + \eta] + t^{-4}[(4,0)_T + (2,0)_T] + t^{-2}(1,0)^2\eta \\
&-(2,2)_T - t^{-4}(2,0)_T - t^{-2}\eta \\
&= t^4(4,2)_T + (2,2)_T + t^2\eta + t^{-4}(4,0)_T + t^{-4}(2,0)_T + t^{-2}(2,0)_T + t^{-2}(1,0)^2\eta \\
&-(2,2)_T - t^{-4}(2,0)_T - t^{-2}\eta \\
&= t^4(4,2)_T + t^{-4}(4,0)_T + t^{-2}(1,0)^2\eta + t^2\eta - t^{-2}\eta \\
&= t^4(4,2)_T + t^{-4}(4,0)_T + [t^{-2}(2,0)_S + t^2(0,0)_S]\eta
\end{aligned}
$$

This is the result what we wanted. □

Proposition 2.2.11. *The following formula holds*

$$(4,2)_T * (0,1)_T = t^4(4,3)_T + t^{-4}(4,1)_T + (2,1)_S\eta$$

Proof. We have

$$
\begin{aligned}
&(4,2)_T * (0,1)_T = [(2,1)_T * (2,1)_T - 2] * (0,1)_T \\
&= (2,1)_T * [t^2(2,2)_T + t^{-2}(2,0)_T + \eta] - 2(0,1)_T \\
&= t^2(2,1)_T[(1,1)_T^2 - 2] + t^{-2}(2,1)_T[(1,0)_T - 2] + (2,1)_T\eta - 2(0,1)_T \\
&= t^2[(2,1)_T * (1,1)_T] * (1,1)_T + 2t^2(2,1)_T + t^{-2}[(2,1)_T * (0,1)_T](1,1)_T \\
&-2t^{-2}(2,1)_T + (2,1)_T\eta - 2(0,1)_T \\
&= t^2[t(3,2)_T + t^{-1}(1,0)_T] * (1,1)_T - 2t^2(2,1)_T + t^{-2}[t^{-1}(3,1)_T + t(1,1)_T] * (1,0)_T \\
&-2t^{-2}(2,1)_T + (2,1)_T\eta - 2(0,1)_T \\
&= t^4(4,3)_T + 2t^2(2,1)_T + (0,1)_T - 2t^2(2,1)_T + t^{-4}(4,1)_T + 2t^{-2}(2,1)_T + (0,1)_T \\
&-2t^{-2}(2,1)_T - (2,0)_T + (2,1)_T\eta \\
&= t^4(4,3)_T + t^{-4}(4,1)_T + (2,1)_S\eta,
\end{aligned}
$$

and the formula is proved. □

Proposition 2.2.12. *The following formula holds*

$$(4,3)_T * (0,1)_T = t^4(4,4)_T + t^{-4}(4,2)_T + [t^2(2,2)_S + t^{-2}(0,0)_S]\eta.$$

Proof. We have

$$\begin{aligned}
&(4,3)_T * (0,1)_T = h_{L_*}(f_*[(4,1)_T * (0,1)_T]) \\
&= h_{L_*}(f_*[t^4(4,2)_T + t^{-4}(4,0)_T + t^{-2}(2,0)_S + t^2(0,0)_S]) \\
&= h_{L_*}(t^{-4}(4,-2)_T + t^4(4,0)_T + t^2(2,0)_S + t^{-2}(0,0)_S) \\
&= t^{-4}(4,2)_T + t^4(4,0)_T + t^2(2,2)_S + t^{-2}(0,0)_S.
\end{aligned}$$

So the formula is proved. □

CHAPTER 3
REPRESENTATIONS OF THE KAUFFMAN BRACKET SKEIN ALGEBRA OF THE PUNCTURED TORUS

So far, we described the basic multiplicative structure of $K_t(\Sigma_{1,1} \times I)$. Now we will focus on its representation on the vector space with a special basis called Kauffman triad defined in terms of trivalent graphs in the solid torus. As we mentioned in Chapter 2, the skein module of manifold having a punctured torus on the boundary can be endowed with a $K_t(\Sigma_{1,1} \times I)$-module structure via multiplication described , inducing a representation. We will consider the manifold as the solid torus and consider skeins defined by trivalent graphs. We will examine the action of $K_t(\Sigma_{1,1} \times I)$ on these basis elements.

3.1 Background material

3.1.1 Basic terminologies and properties

We will now explain how the basis elements are constructed. For this we need some background material. At first we will start with the definition and properties of elementary tangles, following [11].

Definition 2. *We define the elementary tangles $U_1, U_2, ..., U_{n-1}$, where each U_i is a tangle with n-input strands and n-output strands. In each U_i, the kth input is connected to the kth output for $k \neq i, i+1$, while the ith input is connected to $i+1$-st input and the ith output is connected to the $i+1$st output.*

Example 3.1.1. *The elementary tangles $1_5, U_1, U_2, U_3, U_4$ are shown in Figure 3.1.*

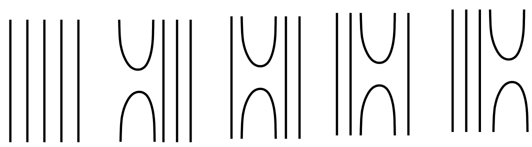

Figure 3.1.

Now we will give multiplicative structure on these n-strand elementary tangles by attaching the n output strands of the first tangles to the n input strands of second tangle. Two tangles are called equivalent if they are regularly isotopic relative to their end points. The basic properties of the tangles under this multiplication are as follows.

The U_i satisify $U_i^2 = dU_i$, where d is the value assigned to a loop. In our work $d = -t^2 - t^{-2}$. Also $U_iU_{i\pm1}U_i = U_i$, and $U_iU_j = U_jU_i$ for $|i - j| > 1$.

Definition 3. *The Temperley-Lieb algebra T_n is the free additive algebra over $\bar{\mathbb{C}}[t, t^{-1}]$ with multiplicative generator $1_n, U_1, U_2, ..., U_{n-1}$ and relations, given above and $\bar{\mathbb{C}}[t, t^{-1}] = \{ \frac{p}{q} \mid p, q \in \mathbb{C}[t, t^{-1}] \}$*

We can interpret T_n as the Kauffman bracket skein algebra $K_t(D^2, 2n)$ of $(D^2, 2n)$, namely of a disk with $2n$ boundary points, where the links in $(D^2, 2n)$ consist of arcs and closed curves within D^2 with the end points of arcs being the specified $2n$ points on the boundary. By viewing $(D^2, 2n)$ as rectangle with n points on the left edge and n points on the right edge and attaching right edge of one rectangle to left edge of another, we can define mutiplication.

Now we will define, for each n, an essential element in T_n called Jones-Wenzl idempotent. These were introduced by Jones and Wenzl (see [16]) in their studies of von Neumann algebras.

Definition 4. *Let $f_i \in T_n$ be defined inductively for $i = 0, 1, 2, ..., n - 1$ by the following*

$$\begin{aligned} f_0 &= 1_n \\ f_{k+1} &= f_k - \mu_{k+1} f_k U_{k+1} f_k. \end{aligned}$$

where $\mu_1 = d^{-1}, \mu_{k+1} = (d - \mu_k)^{-1}$. Here d is loop value defined above in T_n, and $U_i^2 = dU_i$ for each i.

If x is a n-tangle , then we let $\bar{x}$ be the standard closure of x obtained by attaching the i-th input to i-th output, and denote $tr(x) =< \bar{x} >$, where $<, >$ denotes the bracket polynomial.

Lemma 3.1.1. *The elements f_i, $(i = 0, 1, 2, ..., n - 1)$, satisify following properties.*

$$f_i^2 = f_i, \ \textit{for each } i \tag{3.1}$$

$$f_i U_j = U_j f_i = 0, \ \textit{for } j \le i \tag{3.2}$$

$$tr(f_{n-1}) = \Delta_n = \Delta_n(-t^2) \ \textit{and} \ \mu_{k+1} = \frac{\Delta_k}{\Delta_{k+1}} \ \textit{with} \ \Delta_0 = 1 \tag{3.3}$$

$$\textit{where } \Delta_n(x) = \frac{x^{n+1} - x^-(n+1)}{x - x^{-1}} \textit{ is the n-th Chebyschev polynomial .}$$

Now we can prove the existence of the Jones-Wenzl idempotent in T_n.

Proposition 3.1.2. *There exist a unique non-zero element $f \in T_n$ such that $f^2 = f$, $fU_i = U_i f = 0$.*

Proof. By Lemma 3.1.1, $f = f_{n-1}$ satisfies the desired property. □

Definition 5. *The element f from Proposition 3.1.2 in T_n is called the $n-1$st Jones-Wenzl idempotent.*

As an example, in T_2,

$$f_1 = f_0 - \mu f_0 U_1 f_0 = 1_2 - d^{-1} U_1.$$

This is equal to the tangle in Figure 3.2.

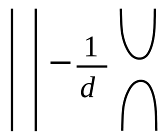

Figure 3.2.

The trace of f_1 is shown in Figure 3.3.

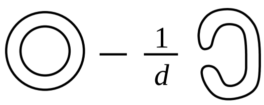

Figure 3.3.

This is futher equal to

$$d^2 - \left(\frac{1}{d}\right)^2 = d^2 - 1 = (-t^2 - t^{-2})^2 - 1$$

and this equals $t^4 + t^{-4} + 1$. Also

$$\Delta_2(x) = \frac{x^3 - x^{-3}}{x - x^{-1}} = x^2 + x^{-2} + 1$$

In this example we can check that $tr(f_1) = \Delta_2(-t^2)$.

Definition 6. *For a given positive integer n, define the Jones-Wenzl idempotent by the formula.* $\frac{1}{\{n\}!}\sum_{\sigma\in S_n}(t^{-3})^{t(\sigma)}\widehat{\sigma}$*, where* $\widehat{\sigma}$ *is described in Figure 3.4. Here* $\{n\}! = \sum_{\sigma\in S_n}(t^{-4})^{t(\sigma)} = \prod_{k=1}^{n}(\frac{1-t^{-4k}}{1-t^{-4}})$*. Here* S_n *denote symmetric group on n letter , so that* $\sigma \in S_n$ *may be thougth as a permutation of 1, 2,..., n and* $\widehat{\sigma}$ *denote the n-tangle obtained from any minimal representation of* σ *as a product of transposition , so that each tranaposition is replaced by a braid in the forn* σ *for* $i = 1, 2, ..., n-1$*.*

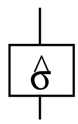

Figure 3.4.

We will denote the nth Jones-Wenzl idempotent by f_n, and in a diagram, as shown in Figure 3.5. As such, a link component decorated by the nth Jones-Wenzl idempotent consist of n parallel copies of it, with the Jones-Wenzl idempotent inserted as shown by the box. We also denote by Δ_n, the Kauffman bracket of the trivial knot decorated by the nth Jones-Wenzl idempotent. It is not hard to prove inductively that

$$\Delta_n = (-1)^n\frac{t^{2n+2} - t^{-2n-2}}{t^2 - t^{-2}}.$$

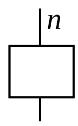

Figure 3.5.

of permutation σ . In the example above , $t(\sigma) = 2$

Now we will give some important properties related to Jones-Wenzl idempotent described in terms of diagram.

Proposition 3.1.3. *The identities described in Figure 3.6 hold.*

Now we now recall the definition of a 3-vertex from [11]

Definition 7. *A 3-vertex (also known as Kauffman triad) is defined as shown in Figure 3.7, where* $i = \frac{b+c-a}{2}$*,* $j = \frac{a+c-b}{2}$*, and* $k = \frac{a+b-c}{2}$*.*

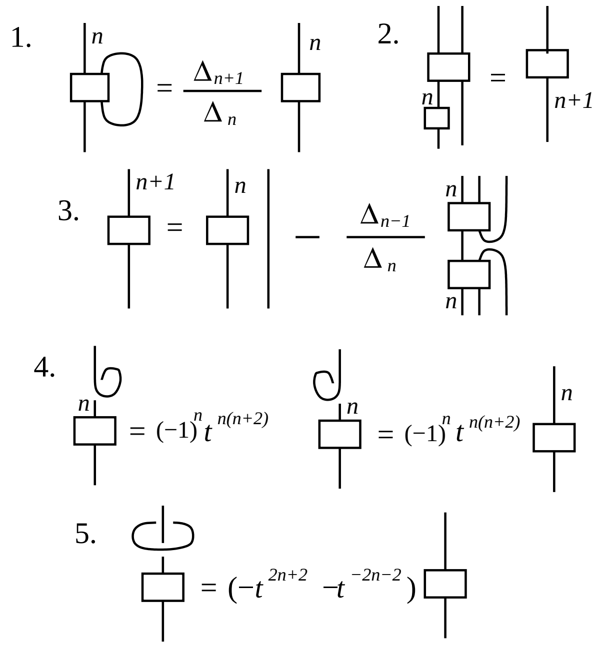

Figure 3.6.

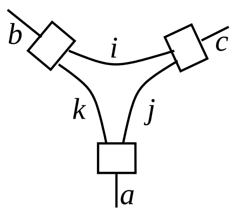

Figure 3.7.

A 3-vertex with adjacent labels a, b, c will be denoted shortly as shown in Figure 3.8. Finally, we recall the vanishing condition for the Jones-Wenzl idempotent.

Proposition 3.1.4. *If t is not a root of unity , then Jones-wenzl idempotent are defined for all n , while if $t = e^{\frac{i\pi}{2r}}$, then the Jones-Wenzl idempotents are defined only for $n = 0, 1, 2, ..., r-2$.*

3.1.2 The representations of $K_t(\Sigma_{1,1} \times I)$

From now on, throughout the paper we set $t = e^{\frac{i\pi}{2r}}$ where r is a positive integer. As such t^4 is a primitive rth root of unity. Because we work at roots of unity, both the Jones-Wenzl idempotents and the 3-vertices come with additional conditions which we recall below (a detailed discussion can be found in [11].

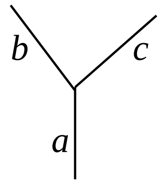

Figure 3.8.

The condition imposed on Jones-Wenzl idempotents is that the $r-1$st Jones-Wenzl idempotent is equal to zero. We require that 3-vertices are of admissible type (according to Lickorish [12]).

Definition 8. *A 3-vertex with labels a, b, c is called admissible if $a+b+c$ is even, $a+b+c \leq 2(r-2)$,$|b-c| \leq a \leq min\{b+c, 2(r-2)-b-c\}$.*

The vector spaces on which we represent the Kauffman bracket skein algebra $K_t(\Sigma_{1,1} \times I)$ of the punctured torus are parametrized by the integers n with the property that $0 \leqq 2n \leqq r-2$. These are the vector spaces $V_{r,n}$ defined as follows.

Consider a solid torus $(S^1 \times D^2, 2n)$ with a "puncture disk" on the boundary and $2n$ disjoint marked points in the punctured disk, numbered $1, 2, ...2n$. We think of these points as lying on a diameter, in this order.

Definition 9. *The Kauffman bracket skein module of the solid torus with $2n$ points on the boundary, $K_t(S^1 \times D^2, 2n)$, as the quotient of the free $C[t, t^{-1}]$-module with basis the set of isotopy classes of framed tangles with ends the $2n$ marked points by Kauffman bracket skein relations.*

The space $K_t(S^1 \times D^2, 2n)$ is spanned by elements that consist of several embedded circles together with n embedded arcs whose end-points are the $2n$ points. The skein relations allow us to remove any trivial circles and any crossings.

The topological operation of gluing the cylinder over the punctured torus $\Sigma_{1,1} \times [0,1]$ to the complement of the puncturing disk in the boundary of solid torus gives rise to an action of $K_t(\Sigma_{1,1} \times [0,1])$ on $K_t(S^1 \times D^2, 2n)$.

For $k < n$, we will define a family of inclusions of $K_t(S^1 \times D^2, 2k)$ into $K_t(S^1 \times D^2, 2n)$. Let δ be the data consisting of a function $f : \{1, 2, ..., 2k\} \to \{1, 2, ..., 2n\}$ such that $f(i) - f(i-1)$ is odd for $i = 2, 3, ..., 2k$ and a pairing of $2n - 2k$ numbers in the complement of Imf such that if (p, q) is a pair then p and q belong to same interval $(f(i-1), f(i))$ and for every two pairs (p, q) and (r, s), $(r-p)(r-q)(s-p)(s-q) > 0$.

For each such δ we define the inclusion $i_\delta : K_t(S^1 \times D^2, 2k) \to K_t(S^1 \times D^2, 2n)$ by identifying the $2k$ boundary points of $K_t(S^1 \times D^2, 2k)$ with the boundary points of $K_t(S^1 \times D^2, 2n)$ idexed by $f(i)$, $i = 1, 2, ..., 2k$, for esch pair (p, q), connecting these points by an arch isotopic to the segment $[p, q]$.

Definition 10. *The reduced Kauffman bracket skein module $K_{t,r}(S^1 \times D^2)$ is the quotient of $K_t(S^1 \times D^2)$ by the skein relation $f^{r-1} = 0$, where $f^n, n \geq 1$ is Jones-Wenzl idempotent defined above.*

For $n \leq m \leq r - 2 - n$, define the skein $v_{2n,m}$ as shown in the picture. Now we examine the structure of $K_{t,r}(S^1 \times D^2, 2n)$

Lemma 3.1.5. *The $K_{t,r}(S^1 \times D^2, 2n)$ is a finite dimensional vector space with basis $i_\delta(v_{2k,m})$ where $0 \leq k \leq n, k \leq m \leq r - 2 - k$ and δ ranges over all possible setof data defined above.*

Proof. All elements of $K_{t,r}(S^1 \times D^2, 2n)$ consist of some circles homotopic to $(1, 0)$, some folds of arcs not homotopic to interval $[p, q]$ and/or some folds of arcs homotopic to interval $[p, q]$, without crossings in the projection onto the annulus with a puncturing disk with specific marked $2n$ points on the boundary of disk in the plane. These elements can be written as a linear combination of the skeins of the form $i_\delta(\sigma)$, where σ is a skein in some $K_{t,r}(S^1 \times D^2, 2k)$. Note that the n-th Jones-Wenzl idempotent can be expanded as the sum of $n - 1$ strands plus a sum of Temperley-Lieb elements, each containing a "turn-back". Now by the definition of the Kauffman triad, we can transform the Kauffman triad into the combination of two Jones-Wenzl idempotents. By expanding one of the Jones-Wenzl idempotents, we obtain $2k$ strands plus a sum of Temperley-Lieb elements, eaching containing a "turn-back". This "turn-back" in one Jones-Wenzl idempotent combines with the other Jones-Wenzl idempotent leaving us with circles and arcs not homotopic to interval $[p, q]$ and one $2k$-Jones-Wenzl idempotent attached to these archs for $k < n$. By Definition 6, we know that the $2k$ Jones-Wenzl idempotent can be expanded in terms of permutations of tangles, and we know that those elements are all possible "turn-backs" and arcs without crossing. This implies that $v_{2k,m}$ generate $K_{t,r}(S^1 \times D^2, 2k)$. So we know that the set $i_\delta(v_{2k,m})$, indexed by $0 \leq k \leq m, k \leq m \leq r - 2$ and δ span $K_{t,r}(S^1 \times D_2, 2n)$. Their linear independence is trivial. □

The action of $K_t(\Sigma_{1,1} \times [0, 1])$ on $K_{t,r}(S^1 \times D^2)$ induces a representation. We will consider the represetation on some subspaces. Now let ∂ be the elemet of $K_t(\Sigma_{1,1} \times [0, 1])$.

We know that this element is in the center of $K_t(\Sigma_{1,1} \times [0,1])$. This implies that the eigenspaces associated with ∂ are invariant subspaces of the representation. We also know that the eigenvalues of ∂ are $-t^{4k+2} - t^{-4k-2}, k \leq n$. The eigenspace associated with eigenvalue $-t^{4k+2} - t^{-4k-2}$ has a basis $i_\delta(v_{2k,m})$ for all δ and m, $k \leq m \leq r-2-k$. Our main concern is the case when $k = n$. Let $V_{r,n}$ be the eigenspace of ∂ associated with the eigenvalue $-t^{4n+2} - t^{-4n-2}$,with basis $v_{2n,m}$, where $n = 0, 1, 2, ..., r-2$. To describe the representation of $K_t(\Sigma_{1,1} \times [0,1])$ on $V_{r,n}$,we will start with case $r = 5, 6, 8$ to get some insight about the representation and will extend to general positive integer r.

We will start with the first non-trivial case, $r = 5$. In this case there are two possible 5-admissible 3-vertex embedded in the solid torus with a puncture, these are $v_{2,1}$ and $v_{2,2}$. Now, we will explain the representation of $K_t(\Sigma_{1,1} \times I)$ on the vector space of these basis. Since all the closed loops in the $K_t(\Sigma_{1,1} \times I)$ can be generated from the curves $(0,1)$, $(1,0)$,and $(1,1)$ curves, it suffices to explain the action of these curves on the elements described above. important information about the action of (1,0) on the $v_{2,2}$.

Proposition 3.1.6. *The action of the Kauffman bracket skein algebra of the punctured torus on $V_{5,1}$ is given by*

$$
\begin{aligned}
(1,0)v_{2,1} &= v_{2,2}, \\
(1,0)v_{2,2} &= (1 - \frac{\Delta_1}{\Delta_2} \cdot \frac{1}{d})v_{2,1} = (1 - \frac{[1][2]}{[2][3]})v_{2,1}, \\
(0,1)v_{2,1} &= -(t^4 + t^{-4})v_{2,1} \\
(0,1)v_{2,2} &= -(t^6 + t^{-6})v_{2,2} \\
(1,1)v_{2,1} &= (-t^{-5})v_{2,2} \\
(1,1)v_{2,2} &= (-t^5)(1 - \frac{[1][2]}{[2][3]})v_{2,1}.
\end{aligned}
$$

where quantized integer $[k] = \frac{t^{2k}-t^{-2k}}{t^2-t^{-2}}$ *implying* $\Delta_k = (-1)^k[k+1]$.

Proof. We proceed as in Figure 3.9, where we use the defining condition for 5-admissible 3-vertices. By the defining condition of 5-addmissible 3-vertex , we know the following.

This proves that

$$
(1,0)v_{2,2} = \left(1 - \frac{\Delta_1}{\Delta_2}\frac{1}{d}\right) v_{2,1}.
$$

To compute the action of $(1,0)$ on v_{21}, we evaluate $v_{2,2}$ as shown in Figure 3.10. This

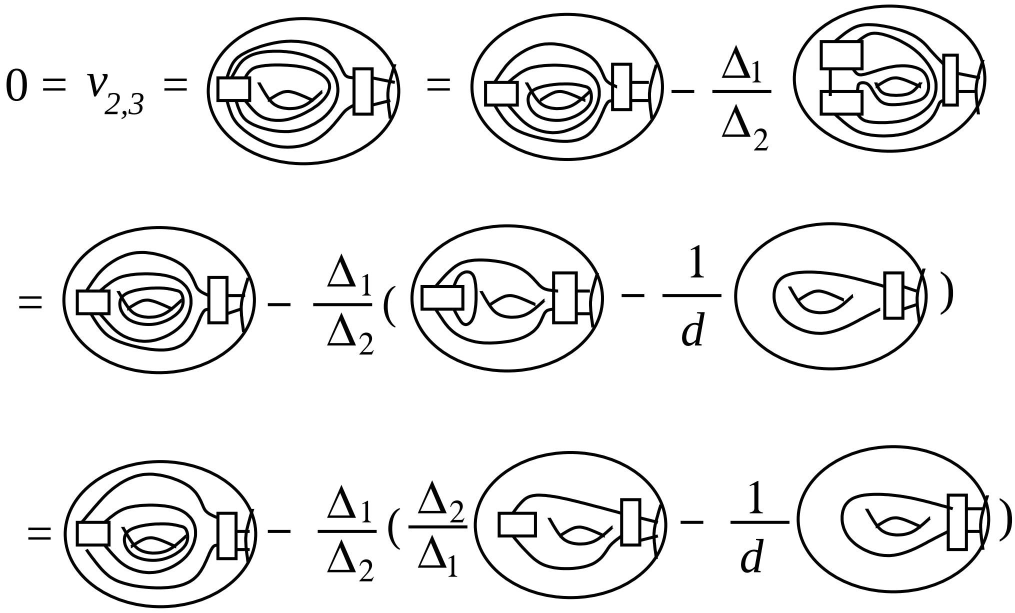

Figure 3.9.

shows that

$$(1,0)v_{2,1} = v_{2,2}$$

The action of $(0,1)$ on the basis elements is a consequence of Proposition 3.1.3 (5) Now we turn to the action of $(1,1)$ By lemma 2.2.1.

$$(1,0) * (0,1) = t(1,1) + t^{-1}(1,-1), \text{and} \tag{3.4}$$

$$(0,1) * (1,0) = t^{-1}(1,1) + t^{(}1,-1), \tag{3.5}$$

From this, we can deduce the following by multiplying (3.4) by t and (3.5) by t^{-1} and then substracting the result.

$$t[(1,0) * (0,1)] - t^{-1}[(0,1) * (1,0)] = (t^2 - t^{-2})(1,1) \text{ this gives us that}$$

$$(1,1) = \frac{t}{t^2 - t^{-2}}[(1,0) * (0,1)] - \frac{t^{-1}}{t^2 - t^{-2}}[(0,1) * (1,0)].$$

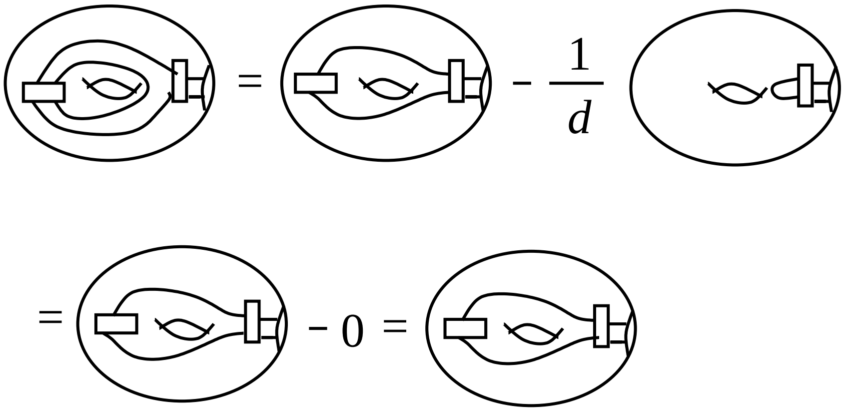

Figure 3.10.

We compute

$$
\begin{aligned}
&(1,1)v_{2,1} = \frac{t}{t^2-t^{-2}}(1,0)(0,1)v_{2,1} - \frac{t^{-1}}{t^2-t^{-2}}(0,1)(1,0)v_{2,1} \\
&= \frac{t(-t^4-t^{-4})}{t^2-t^{-2}}(1,0)v_{2,1} - \frac{t^{-1}}{t^2-t^{-2}}(0,1)v_{2,2} \\
&= \frac{-t^5-t^{-3}}{t^2-t^{-2}}v_{2,2} - \frac{(t^{-1})(-t^6-t^{-6})}{t^2-t^{-2}}v_{2,2} \\
&= \frac{-t^5-t^{-3}}{t^2-t^{-2}}v_{2,2} + \frac{t^5+t^{-7}}{t^2-t^{-2}}v_{2,2} \\
&= \frac{t^{-7}-t^{-3}}{t^2-t^{-2}}v_{2,2} \\
&= \frac{(t^{-5})(t^{-2}-t^2)}{t^2-t^{-2}}v_{2,2} = -t^{-5}v_{2,2}
\end{aligned}
$$

Similarly, we can determine the action of $(1,1)$ on $v_{2,2}$ as follows

$$
\begin{aligned}
&(1,1)v_{2,2} = \frac{t}{t^2-t^{-2}}(1,0)(0,1)v_{2,2} - \frac{t^{-1}}{t^2-t^{-2}}(0,1)(1,0)v_{2,2} \\
&= \frac{t(-t^6-t^{-6})}{t^2-t^{-2}}(1,0)v_{2,2} - (\frac{t^{-1}}{t^2-t^{-2}})(1-\frac{\Delta_1 1}{\Delta_2 d})(0,1)v_{2,1} \\
&= (\frac{-t^7-t^{-5}}{t^2-t^{-2}})(1-\frac{\Delta_1 1}{\Delta_2 d})v_{2,1} - \frac{t^{-1}(t^4-t^{-4})}{t^2-t^{-2}}(1-\frac{\Delta_1 1}{\Delta_2 d})v_{2,1} \\
&= \frac{-t^7+t^{-3}}{t^2-t^{-2}}(1-\frac{\Delta_1 1}{\Delta_2 d})v_{2,1} = \frac{-t^5(t^2-t^{-2})}{t^2-t^{-2}}(1-\frac{\Delta_1 1}{\Delta_2 d})v_{2,1} \\
&= (-t^5)(1-\frac{[1][2]}{[2][3]})v_{2,1}
\end{aligned}
$$

This completes the proof. □

As we saw from above Proposition, if $r = 5$, then the action of $(1, 0)$, $(0, 1)$, and $(1, 1)$ on the basis elements $v_{2,1}$ $v_{2,2}$ are relatively simple. We will now examine the case $r = 6$, to gain more insight about the general case.

Proposition 3.1.7. *Assume that t^4 is primitive 6th root of unity, then the action of the Kauffman bracket skein algebra of the punctured torus on $V_{6,1}$ is given by*

$$\begin{aligned}
(1,0)v_{2,1} &= v_{2,2} \\
(1,0)v_{2,2} &= v_{2,3} + (1 - \frac{\Delta_1 1}{\Delta_2 d})v_{2,1} = v_{2,3} + (1 - \frac{[1][2]}{[2][3]})v_{2,1} \\
(1,0)v_{2,3} &= (1 - \frac{\Delta_1 1}{\Delta_3\Delta_2})v_{2,2} = (1 - \frac{[1][2]}{[3][4]})v_{2,2} \\
(0,1)v_{2,1} &= -(t^4 + t^{-4})v_{2,1} \\
(0,1)v_{2,2} &= -(t^6 + t^{-6})v_{2,2} \\
(0,1)v_{2,3} &= -(t^8 + t^{-8})v_{2,3} \\
(1,1)v_{2,1} &= -t^{-5}v_{2,2} \\
(1,1)v_{2,2} &= -t^{-7}v_{2,3} + (-t^5)(1 - \frac{[1][2]}{[2][3]})v_{2,1} \\
(1,1)v_{2,3} &= -t^7(1 - \frac{[1][2]}{[3][4]})v_{2,2}.
\end{aligned}$$

Proof. The action of $(1, 0)$ on $v_{2,1}$ is done in the previous proposition 3.1.5. We will prove the remaining two formulas. First we know that

$$v_{2,3} = (1,0)v_{2,2} + \left(\frac{\Delta_1 \cdot 1}{\Delta_2 \cdot d} - 1\right) v_{2,1}$$

as seen in the proof of the Proposition 3.1.6. This implies that

$$(1,0)v_{2,2} = v_{2,3} + \left(1 - \frac{\Delta_1}{\Delta_2 d}\right) v_{2,1} = v_{2,3} + \left(1 - \frac{[1][2]}{[2][3]}\right) v_{2,1}$$

To show that

$$(1,0)v_{2,3} = \left(1 - \frac{[1][2]}{[3][4]}\right) v_{2,2}$$

we begin as shown in Figure 3.11. The second skein in the parenthesis is equal to zero.

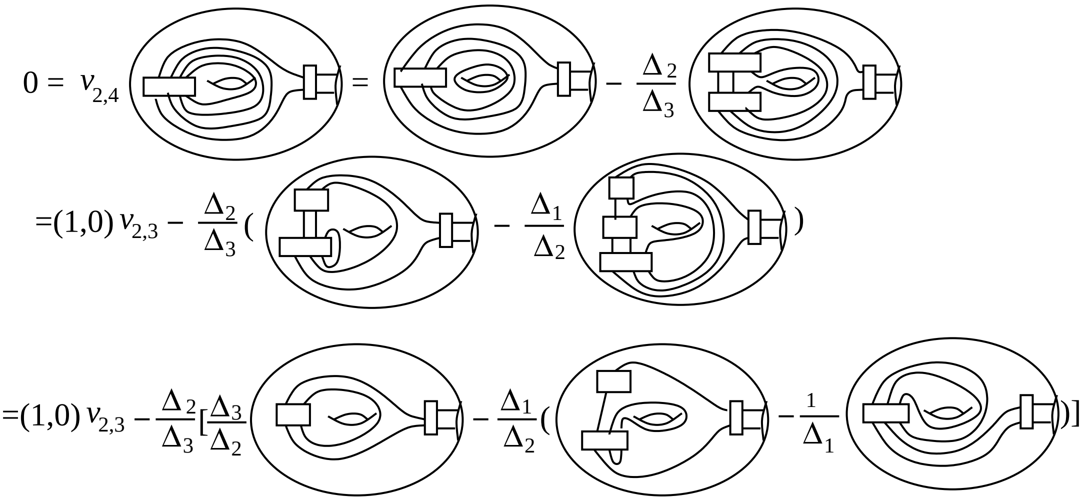

Figure 3.11.

So this is equal to the expression shown in Figure 3.12. This is further equal to

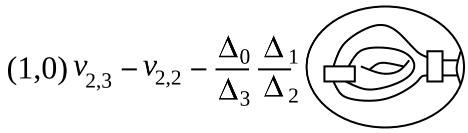

Figure 3.12.

$$
\begin{aligned}
&(1,0)v_{2,3} - v_{2,2} + \left(\frac{\Delta 1}{\Delta_3} \frac{\Delta_0}{\Delta_2} \right) v_{2,2} \\
&= (1,0)v_{2,3} - \left(1 - \frac{\Delta_1 \Delta_0}{\Delta_3 \Delta_2} \right) v_{2,2}.
\end{aligned}
$$

We obtain

$$
(1,0)v_{2,3} = v_{2,4} + \left(1 - \frac{\Delta_1 \Delta_0}{\Delta_3 \Delta_2} \right) v_{2,2} = \left(1 - \frac{[2][1]}{[4][3]} \right) v_{2,2},
$$

as desired. The action of $(0,1)$ is again a direct consequence of Proposition 3.1.3.

Now we want to describe the action of (1,1) on the basis elements $v_{2,1}, v_{2,2}, and v_{2,3}$. This is similar to the computation in Proposition 3.1.7, so we give a proof of the action of $(1,1)$ on $v_{2,2}$ only. We have

$$\begin{aligned}
(1,1)v_{2,2} &= \frac{t}{t^2-t^{-2}}(1,0)(0,1)v_{2,2} - \frac{t^{-1}}{t^2-t^{-2}}(0,1)(1,0)v_{2,2}\\
&= \frac{-t(t^6+t^{-6})}{t^2-t^{-2}}(1,0)v_{2,2} - \frac{t^{-1}}{t^2-t^{-2}}(0,1)[v_{2,3}+(1-\frac{[1][2]}{[2][3]})v_{2,1}]\\
&= \frac{-(t^7+t^{-5})}{t^2-t^{-2}}[v_{2,3}+(1-\frac{[1][2]}{[2][3]})v_{2,1}] + \frac{t^7+t^{-9}}{t^2-t^{-2}}v_{2,3}\\
&+\frac{t^3+t^{-5}}{t^2-t^{-2}}(1-\frac{[1][2]}{[2][3]})v_{2,1}\\
&= \frac{-t^{-5}+t^{-9}}{t^2-t^{-2}}v_{2,3} + \frac{-t^7+t^3}{t^2-t^{-2}}(1-\frac{[1][2]}{[2][3]})v_{2,1}\\
&= -t^{-7}v_{2,3} + (-t^5)(1-\frac{[1][2]}{[2][3]})v_{2,1}.
\end{aligned}$$

This completes the proof. □

We now turn to a situation where $n=2$. The first non-trivial case is for $r=7$, in which case $V_{7,2}$ is 2-dimensional and has the basis elements $v_{4,2}, v_{4,3}$.

Proposition 3.1.8. *The action of the Kauffman bracket skein algebra of the punctured torus on $V_{7,2}$ is given by*

$$\begin{aligned}
(1,0)v_{4,2} &= v_{4,3}\\
(1,0)v_{4,3} &= (1-\frac{[2][3]}{[3][4]})v_{4,2}\\
(0,1)v_{4,2} &= (-t^{2(2)+2}-t^{-2(2)-2})v_{4,2}\\
(0,1)v_{4,3} &= (-t^{2(3)+2}-t^{-2(3)-2})v_{4,3}\\
(1,1)v_{4,2} &= (-t^{-2(2)-3})v_{4,2}\\
(1,1)v_{4,3} &= (-t^{2(3)+1})(1-\frac{[2][3]}{[3][4]})v_{4,2}
\end{aligned}$$

Proof. The action of $(1,0)$ on $v_{4,2}$ is trivial. To determine the action of $(1,0)$ on $v_{4,3}$, we will use again the diagramatic method, as shown in Figure 3.13

$$\begin{aligned}
&(1,0)v_{4,3} - v_{4,2} + \frac{\Delta_2}{\Delta_3}\left[\frac{\Delta_1}{\Delta_2}(v_{4,2}-0)\right]\\
&= (1,0)v_{4,3} - v_{4,2} + \frac{\Delta_2\Delta_1}{\Delta_3\Delta_2}v_{4,2}
\end{aligned}$$

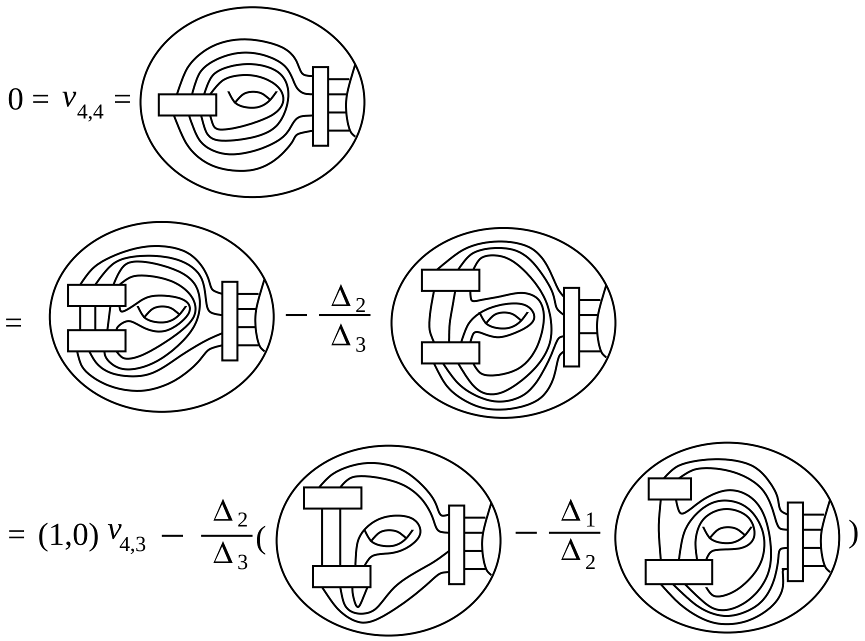

Figure 3.13.

This implies that

$$(1,0)v_{4,3} = \left(1 - \frac{[2][3]}{[3][4]}\right) v_{4,2}.$$

The action of $(0,1)$ is again a consequence of Proposition 3.1.3, while for the action of $(1,1)$ we again use the multiplicative structure of the Kauffman bracket skein algebra of the punctured torus. □

We are now able to introduce the main result, which is a generalization of the previous propositions.

Theorem 3.1.9. *Let r be a positive integer greater than or equal to 5, and let n be a positive integer such that $0 \leq n \leq \frac{r-2}{2}$. The representation of $K_t(\Sigma_{1,1} \times I)$ on $V_{r,n}$ is* *given by*

$$(1,0)v_{2n,m} = v_{2n,m+1} + \left(1 - \frac{[n][n+1]}{[m][m+1]}\right) v_{2n,m-1}$$
$$= v_{2n,m+1} + \frac{[m-n][m+n+1]}{[m][m+1]} v_{2n,m-1}$$
$$(0,1)v_{2n,m} = (-t^{2m+2} - t^{-2m-2})v_{2n,m}$$
$$(1,1)v_{2n,m} = (-t^{-2m-3})v_{2n,m+1} + (-t^{2m+1})\frac{[m-n][m+n+1]}{[m][m+1]} v_{2n,m-1},$$

where $n \leq m \leq r-2-n$*, with the convention that* $v_{2n,n-1} = v_{2n,r-1-n} = 0$.

Proof. First let us recall the skein $v_{2n,m}$ shown in Figure 3.14.

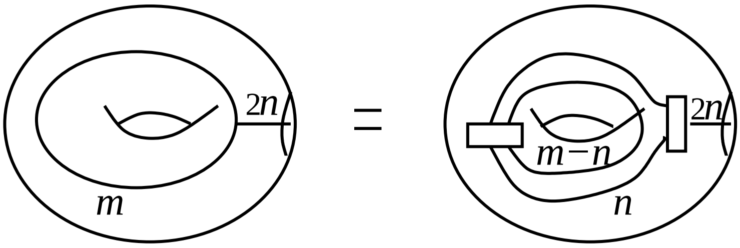

Figure 3.14.

Note that the number of strands not attached to the $2n$th Jones-Wenzl idempotent is $m-n$.

The basic strategy is to expand $v_{2n,m+1}$ to an expression in which we will recognize $(1,0)v_{2n,m}$ and $v_{2n,m-1}$. We start by expanding repeatedly the $m+1$st Jones-Wenzl idempotent (on the left in the the diagram). We begin our computation as shown in Figure 3.15.

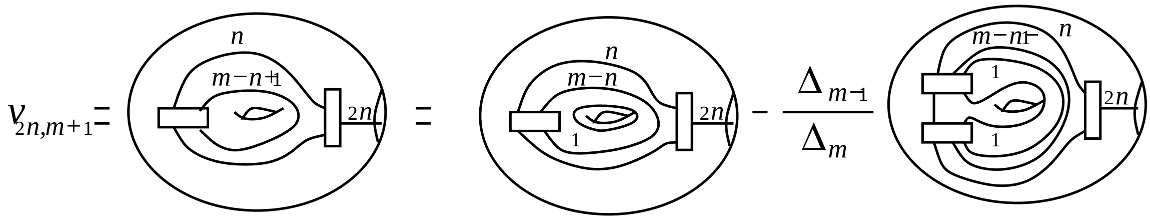

Figure 3.15.

We expand the upper Jones-Wenzl idempotent in the last diagram from Figure 3.15 as shown in Figure 3.16.

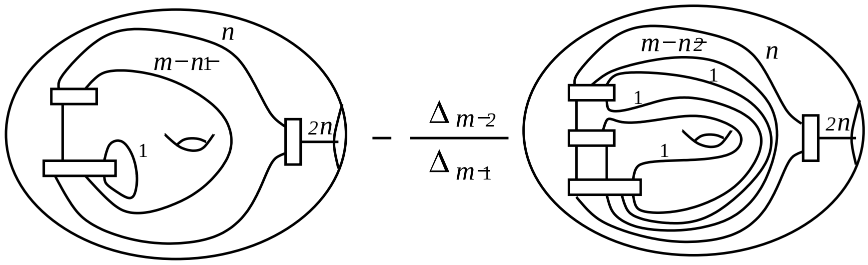

Figure 3.16.

Using Proposition 3.1.3 we deduce that this is further equal to

$$\frac{\Delta_m}{\Delta_{m-1}} v_{2n,m-1} - \frac{\Delta_{m-2}}{\Delta_{m-1}} A_0,$$

where A_k is defined in Figure 3.17. By expanding the upper Jones-Wenzl idempotent in

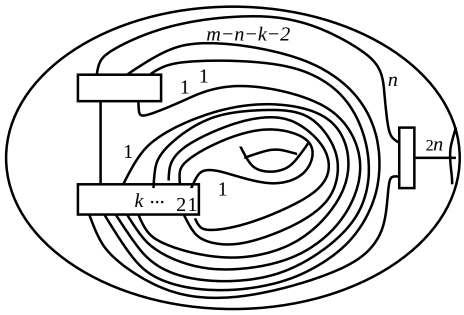

Figure 3.17.

Figure 3.17 and using the fact that skeins involving turn-backs are equal to zero we obtain the recursive relation

$$A_k = -\frac{\Delta_{m-(k+1)}}{\Delta_{m-k}} A_{k+1}$$

and consequently

$$v_{2n,m+1} = (1,0)v_{2n,m} - v_{2n,m-1} + (-1)^{m-n}\frac{\Delta_{m-1}}{\Delta_m} \cdot \frac{\Delta_{m-2}}{\Delta_{m-1}} \cdots \frac{\Delta_{n-1}}{\Delta_n} A_{m-n}. \tag{3.6}$$

It is not hard to see that A_{m-n} is the skein from Figure 3.18.

We compute the skein A_{m-n} recursively. We let B_k be the skein in Figure 3.19.

Expand the Jones-Wenzl idempotent in B_k as described in Figure 3.20. The first skein

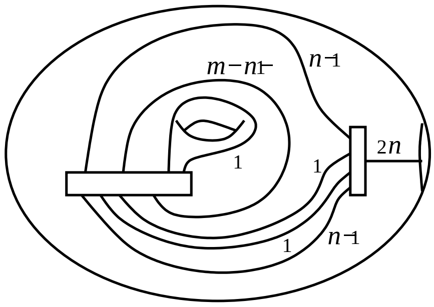

Figure 3.18.

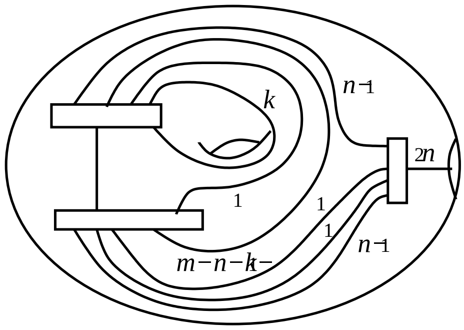

Figure 3.19.

is zero because it contains a “turn back”. This shows that B_k satisfies the recursive relation

$$B_k = -\frac{\Delta_{m-k-2}}{\Delta_{m-k-1}} B_{k+1}. \tag{3.7}$$

It is not hard to see that $B_{m-n} = v_{2n,m-1}$.

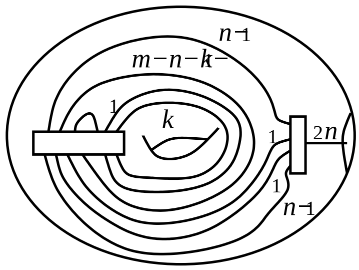

Figure 3.20.

Combining (3.6) and (3.7), we obtain

$$(1,0)v_{2n,m} = (1 - \frac{\Delta_n}{\Delta_m}\frac{\Delta_{n-1}}{\Delta_{m-1}})v_{2n,m-1} = (1 - \frac{[n][n+1]}{[m][m+1]})v_{2n,m-1}.$$

Now we will simplify this by the following computation:

$$\begin{aligned}
&1 - \frac{[n][n+1]}{[m][m+1]} = 1 - \frac{(t^{2n} - t^{-2n})}{(t^{2m} - t^{-2m})}\frac{(t^{2n+2} - t^{-2n-2})}{(t^{2m+2} - t^{-2m-2})} \\
&= 1 - \frac{t^{4n+2} - t^{2} - t^{-2} + t^{-4n-2}}{t^{4m+2} - t^{2} - t^{-2} + t^{-4m-2}} \\
&= \frac{t^{4m+2} + t^{-4m-2} - t^{4n-2} - t^{-4n-2}}{t^{4m+2} - t^{2} - t^{-2} + t^{-4n-2}} \\
&= \frac{t^{2m+2n+2}(t^{2m-2n} - t^{-2m+2n}) + t^{-2m-2n-2}(t^{-2m+2n} - t^{2m-2n})}{(t^{2m} - t^{-2m})(t^{2m+2} - t^{-2m-2})} \\
&= \frac{(t^{2m+2n+2} - t^{-2m-2n-2})(t^{2m-2n} - t^{-2m-2n})}{(t^{2m} - t^{-2m})(t^{2m+2} - t^{-2m-2})} \\
&= \frac{[m-n][m+n+1]}{[m][m+1]}
\end{aligned}$$

This prove the first equation from the statement. The second is a direct consequence of Proposition 3.1.3.

Using these two relations and the expression of $(1,1)$ in terms of $(1,0)$ and $(0,1)$ we have

$$(1,1)v_{2n,m} = \frac{t}{t^2 - t^{-2}}(1,0)(0,1)v_{2n,m} - \frac{t^{-1}}{t^2 - t^{-2}}(0,1)(1,0)v_{2n,m},$$

and by setting $k = \frac{[m-n][m+n+1]}{[m][m+1]}$ we can write

$$\begin{aligned}
&(1,1)v_{2n,m} = \frac{t(-t^{2m+2} - t^{-2m-2})}{t^2 - t^{-2}}(1,0)v_{2n,m} - \frac{t^{-1}}{t^2 - t^{-2}}(0,1)(v_{2n,m} + kv_{2n,m-1}) \\
&= \frac{-t^{2m+3} - t^{-2m-1}}{t^2 - t^{-2}}(v_{2n,m+1} + kv_{2n,m-1}) + \frac{t^{-2m-5} - t^{-2m-1}}{t^2 - t^{-2}}v_{2n,m+1} \\
&+ \frac{t^{2m-1} + t^{-2m-1}}{t^2 - t^{-2}}kv_{2n,m-1} \\
&= \frac{t^{-2m-5} - t^{-2m-1}}{t^2 - t^{-2}}v_{2n,m+1} + \frac{t^{2m-1} - t^{2m+3}}{t^2 - t^{-2}}kv_{2n,m-1} \\
&= (-t^{2m-3})v_{2n,m+1} + (-t^{2m+1})(\frac{[m-n][m+n+1]}{[m][m+1]})v_{2n,m-1}
\end{aligned}$$

This proves the third equation, and we are done. □

Remark. After this result has been announced at the Knots in Washington XXXII Conference [3], it has also been announced in [13].

CHAPTER 4
THE RESHETIKHIN-TURAEV REPRESENTATION OF THE MAPPING CLASS GROUP OF THE PUNCTURED TORUS

In Chapter 3 we described the representation of Kauffman bracket skein algebra of a punctured torus $K_t(\Sigma_{1,1} \times [0,1])$ on $V_{r,n}$. In this section, we will show a method of calculating the matrices of the Reshetikhin-Turaev representation of mapping class group of the punctured torus from the representation of Kauffman bracket skein algebra of a punctured torus $K_t((\Sigma_{1,1} \times [0,1])$ described in Theorem 3.18. A different approach for deriving these formulas was taken in [6], and in particular a different formula for the S-map was found there.

4.1 Representation of mapping class group of $\Sigma_{1,1}$ on $V_{r,n}$

Because a linear trasformation is best understood by its action on the basis elements, an element of mapping class group can be best understood by its action on the simple closed curves. It is known that the mapping class group of a punctured torus $\Sigma_{1,1}$ is isomorphic to the special linear group $SL_2(Z)$. Also it is known that the mapping class group of a punctured torus $\Sigma_{1,1}$ is generated by the maps S, T, and T_1 described in Figure 4.1.

The Reshetikhin-Turaev representation ρ of the mapping class group of the punctured torus $\Sigma_{1,1}$ on $V_{r,n}$ is determined by the representation of the Kauffman bracket skein algebra $K_t(\Sigma_{1,1} \times [0,1])$ by the relation

$$h(\gamma) = \rho(h)\gamma\rho(h)^{-1},$$

where γ is a skein in the cylinder over the punctured torus, and $h(\gamma)$ is the image of that skein under the homeomorphism $h \times id$ of the cylinder over the torus. This relation has been identified to be an exact Egorov identity in [8]. The main goal of this chapter is to determine the matrices of the S-map and T-map. We apply these equations for the particular cases where $\gamma = (1,0)$ and $\gamma = (0,1)$.

For the S-matrix, the exact Egorov identity becomes

$$(1,0)Sv_{2n,n+j} = S(0,1)v_{2n,n+j},$$
$$(0,1)Sv_{2n,n+j} = S(1,0)v_{2n,n+j}.$$

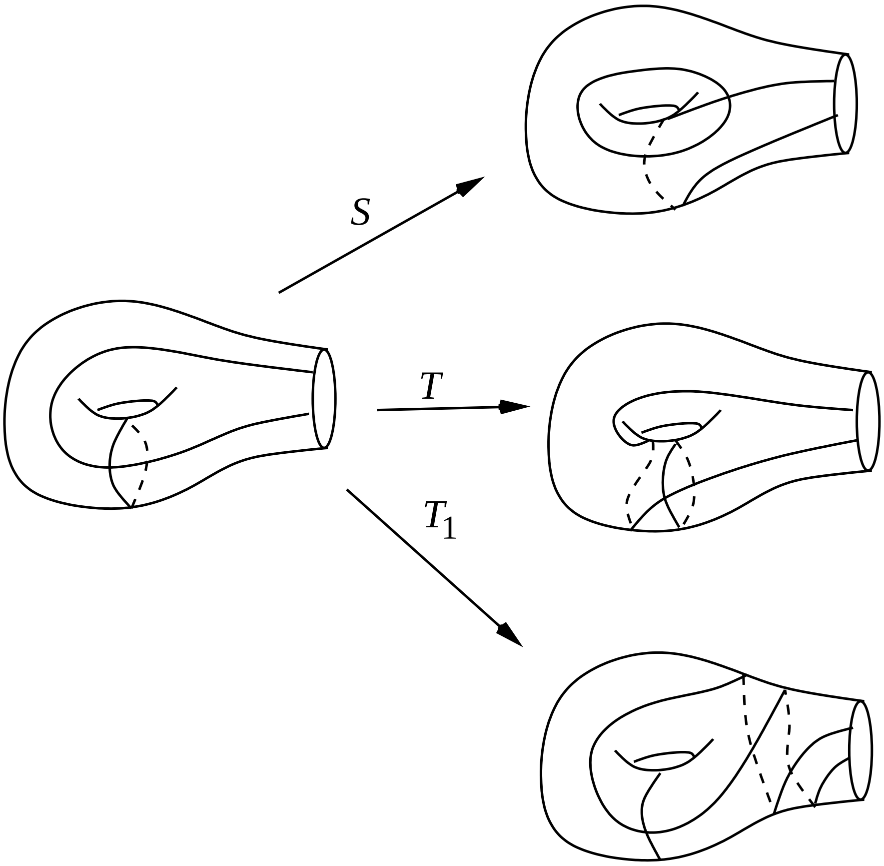

Figure 4.1.

Proposition 4.1.1. *Let* $S = (a_{j,k})$ *be the* S*-matrix, where* $0 \leq j, k \leq r - 2n - 2$*. Then* S *is given by following recursive relation.*
$a_{j-1,k} = (-t^{2n+2k+2} - t^{-2n-2k-2})a_{j,k} - \frac{[j+1][2n+j+2]}{[n+j+1][n+j+2]}a_{j+1,k}$
$a_{j,k-1} = (-t^{2n+2j+2} - t^{-2n-2j-2})a_{j,k} - \frac{[k+1][2n+k+2]}{[n+k+1][n+k+2]}a_{j+1,k}$

Proof. Let $Sv_{2n,k+n} = \sum_{j=1}^{n-2r-2} a_{j,k} v_{2n,n+j}$. Then
$(1,0)Sv_{2n,n+k} = \sum_{j=1}^{r-2n-2} a_{j,k}[(1,0)v_{2n,n+j}]$, which is further computed as follows

$$\sum_{j=1}^{r-2n-2} a_{j,k}[v_{2n,n+j+1} + \frac{[j][2n+j+1]}{[j+n][j+n+1]} v_{2n,j-1+n}]$$
$$= \sum_{j=1}^{r-2n-2} [a_{j-1,k} + a_{j+1,k}(\frac{[j+1][2n+j+2]}{[n+j+1][n+j+2]})] v_{2n,n+j}$$

We also know that

$$
\begin{aligned}
S(0,1)v_{2n,n+k} &= S(-t^{2(n+k)+2} - t^{-2(n+k)-2})v_{2n,n+k} \\
&= \sum_{j=1}^{r-2n-2} (-t^{2n+2k+2} - t^{-2n-2k-2})a_{j,k}v_{2n,n+j}
\end{aligned}
$$

By comparing the coefficients, we notice that
$a_{j-1,k} = (-t^{2n+2k+2} - t^{-2n-2k-2})a_{j+1,k} - \frac{[j+1][2n+j+2]}{[n+j+1][n+j+2]}a_{j+1,k}$.

Now we can prove second fact. Since $(0,1)Sv_{2n,n+j} = S(1,0)v_{2n,n+j}$, we have

$$
\begin{aligned}
S(1,0)v_{2n,k+n} &= S[v_{2n,k+1+n} + \frac{[k][2n+k+1]}{[k+n][k+n+1]}v_{2n,k-1+n}] \\
&= \sum_{j=1}^{r-2n-2} a_{j,k+1}v_{2n,j+n} + \frac{[k][2n+k+2]}{[k+n][k+n+1]} \sum_{j=1}^{r-2n-2} a_{j,k-1}v_{2n,j+n} \\
&= \sum_{j=1}^{r-2n-2} (a_{j,k+1}v_{2n,j+n} + \frac{[k][2n+k+1]}{[k+n][k+1+n]}a_{j,k-1})v_{2n,j+n}
\end{aligned}
$$

On the other hand, we know that

$$
\begin{aligned}
(0,1)Sv_{2n,n+k} &= (0,1) \sum_{j=1}^{r-2n-2} a_{j,k}v_{2n,j+n} \\
&= \sum_{j=1}^{r-2n-2} a_{j,k}(-t^{2j+2n+2} - t^{-2j-2n-2})v_{2n,j+n}
\end{aligned}
$$

Comparing the coefficient we obtain

$$
a_{j,k+1} = (-t^{2j+2n+2} - t^{-2j-2n-2})a_{j,k} - \frac{[k][2n+k+1]}{[k+n][k+1+n]}a_{j,k+1}.
$$

The conclusion follows. □

To compute T matrix, we will use again the two instances of the exact Egorov identity, which now read

$$
\begin{aligned}
(1,0)Tv_{2n,n+j} &= T(1,1)v_{2n,n+j}, \\
(0,1)Tv_{2n,n+j} &= T(0,1)v_{2n,n+j}.
\end{aligned}
$$

Proposition 4.1.2. *Let $T = (b_{j,k})$ be the T-matrix, where $0 \leq j, k \leq r - 2n - 2$. Then T is diagonal matrix $(b_{j,j})$, where*

$$b_{j,j} = (-1)^{n+j} t^{(n+j)^2-1}.$$

Proof. First, we know that $T(0,1) = (0,1)T$ and $(0,1)$ has 1-dimensional eigenspaces. From the commutativity, we deduce that T is diagonal. Since

$$(1,0)Tv_{2n,n+j} = T(1,1)v_{2n,n+j},$$

by comparing the respective coefficients, we obtain the recursive relation

$$b_{j,j} = -t^{2n+2j+1} b_{j-1,j-1}.$$

Setting $b_{1,1} = (-1)^n t^{n^2-1}$, we obtain $b_{j,j} = (-1)^{n+j} t^{(n+j)^2-1}$, as desired. □

The twist T_1 on the boundary commutes with all operators in $K_t(\Sigma_{1,1} \times [0,1])$. Because the representation of $K_t(\Sigma_{1,1} \times [0,1])$ on $V_{r,n}$ is irreducible, it follows that T_1 acts as multiplication by a scalar. We may choose this scalar to be $t^{(2n)^2-1}$.

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