Introduction to Topology – Homework 2

1. Show that the boundary of the square and the circle are homeomorphic.

2. Define $D : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by
$$D(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|.$$
Prove that D is a metric and that it induces the standard topology on $\mathbb{R}^n$.

3. Show that $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by
$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$
is indeed a metric.

4. Consider the 2-dimensional torus $S^1 \times S^1$ as a subspace of $\mathbb{C}^2 = \mathbb{R}^4$. Show that the subspace topology is the same as the quotient topology defined in Example 1 from §1.3.7.

5. Let X be a topological space. The suspension ΣX is defined as the quotient $X \times [-1, 1]/\sim$, where the equivalence relation is the following
   - for $\lambda \neq -1, 1$, $(x, \lambda) \sim (y, \mu)$ if and only if $x = y$, $\lambda = \mu$;
   - $(x, 1) \sim (y, 1)$ for all x, y;
   - $(x, -1) \sim (y, -1)$ for all x, y.

   Show that S^{n+1} is homeomorphic to ΣS^n, where S^n is the n-dimensional sphere.

6. Show that the sphere
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$
is a 2-dimensional manifold. (Hint. Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ be the North and the South poles. Take the stereographic projections onto the equatorial plane $\pi_N : S^2 \backslash \{N\} \to \mathbb{R}^2$ and $\pi_S : S^2 \backslash \{S\} \to \mathbb{R}^2$, then use the maps $f_N = \pi_N^{-1} : \mathbb{R}^2 \to S^2$ and $f_S = \pi_S^{-1} : \mathbb{R}^2 \to S^2$.)

7. Show that, as a real 2-dimensional manifold, $\mathbb{C}P^1$ is homeomorphic to the sphere S^2.