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THE NONCOMMUTATIVE A-IDEAL OF A $(2, 2p+1)$-TORUS KNOT DETERMINES ITS JONES POLYNOMIAL

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ABSTRACT

The noncommutative A-ideal of a knot is a generalization of the A-polynomial, defined using Kauffman bracket skein modules. In this paper we show that any knot that has the same noncommutative A-ideal as the $(2, 2p+1)$-torus knot has the same colored Jones polynomials. This is a consequence of the orthogonality relation, which yields a recursive relation for computing all colored Jones polynomials of the knot.

1. Introduction

The noncommutative A-ideal was defined in [?] using Kauffman bracket skein modules. Skein modules were introduced by Turaev (for surfaces) [?] and Przytycki (for arbitrary 3-manifolds) [?] in an attempt to generalize the polynomial invariants of knots and links in the 3-sphere to invariants of knots and links in arbitrary manifolds. The simplest skein module is the Kauffman bracket skein module, defined using the Kauffman bracket skein relation [?]. The module depends on a parameter t, the variable of the Kauffman bracket. A theorem of Bullock, Przytycki and Sikora [?], [?] shows that when $t = -1$, the Kauffman bracket skein module of a 3-manifold has a ring structure, which ring is isomorphic with the affine character ring of $SL(2, \mathbb{C})$-representations of the fundamental group of the manifold.

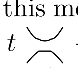
This result allows the study of $SL(2, \mathbb{C})$-invariants of knots from the point of view of quantum invariants. An example for which this is done is the A-polynomial. The A-polynomial of a knot was introduced by Cooper, Culler, Gillet, Long, and Shalen in [?] using the character variety of $SL(2, \mathbb{C})$-representations of the fundamental group of the knot complement. The construction was later generalized via Kauffman bracket skein modules in [?], giving rise to a new knot invariant, the noncommutative A-ideal. The A-ideal is a finitely generated ideal of polynomials in two noncommuting variables. It depends on a parameter t, and when $t = -1$, the radical of the one dimensional part of the ideal is generated by the A-polynomial of the knot.

In [?] was discovered a property relating the noncommutative A-ideal to the Jones polynomial. Called the orthogonality relation, this property states that every nonzero element in the A-ideal yields a matrix annihilating the vector with entries the colored Jones polynomials of the knot. As an application of the orthogonality relation, it was shown in [?] that if the A-ideal contains a polynomial of second degree in the variable corresponding to the longitude of the knot, whose coefficients satisfy a certain technical property, then the A-ideal of the knot determines all of its colored Jones polynomials. In this paper we prove that this is the case with the $(2, 2p + 1)$-torus knots, p an integer. The main result of the paper is

Theorem 1.1 *Any knot having the same noncommutative A-ideal as the* $(2, 2p+1)$*-torus knot has the same colored Jones polynomials.*

To avoid the hassle of keeping track of both the sign of p and of the types of crossings in diagrams, we do the proof for the case where p is *positive*. The case where p is negative is obtained by replacing in all formulas p by $|p|$ and t by t^{-1}.

2. Preliminary Facts

A *framed link* in an orientable manifold M is a disjoint union of annuli. If the manifold is the cylinder over the torus, framed links will be identified with curves, using the convention that the annulus is parallel to the framing. In figures annuli will be represented by curves, the framing will be considered parallel to the plane of the drawing (the so called blackboard framing). Consider the free $\mathbb{C}[t]$-module $\mathbb{C}[t]\mathcal{L}$ with basis $\mathcal{L}$ the set of all isotopy classes of links in M, including the empty link. The quotient of this module by the smallest submodule containing all expressions of the form  $-\, t$  $-\, t^{-1}$  and  $+t^2+t^{-2}$, where the links in each expression are identical except in a ball in which they look like depicted, is called the Kauffman bracket skein module of M, and is denoted by $K_t(M)$. If t is a complex number instead of the variable of a polynomial, then with the same definition the Kauffman bracket skein module becomes a complex vector space.

The Kauffman bracket skein module of a cylinder over a surface has an algebra structure. The multiplication is induced by the operation of gluing one cylinder atop another. Similarly, the operation of gluing the cylinder over ∂M to M induces a $K_t(\partial M \times I)$-left module structure on $K_t(M)$. We denote by $*$ the multiplication in $K_t(\partial M \times I)$. For a link γ in a skein module we will denote by γ^n the link consisting of n parallel copies of γ, and extend the notation to polynomials.

Two families of polynomials are of interest to us. The first family consists of the classical Chebyshev polynomials, $T_0 = 2$, $T_1 = x$, $T_{n+1} = xT_n - T_{n-1}$. Related to them are the polynomials S_n subject to the same recurrence relation but with $S_0 = 1$, $S_1 = x$. Extend both polynomials recursively to all indices $n \in \mathbb{Z}$. Note that $T_{-n} = T_n$, while $S_{-n} = -S_{n-2}$. For a knot K, $S_n(K)$ is called the coloring of K by the nth Jones-Wenzl idempotent. If K is a knot in the 3-sphere, then $S_n(K)$ as a polynomial in $K_t(S^3) = \mathbb{C}[t]$ is called the nth colored Kauffman bracket of the knot. Under the change of variable $t \to it$ this becomes the nth colored Jones polynomial. Although this is a small alteration, and so $S_n(K)$ itself could very well be called the colored Jones polynomial, we prefer the name colored Kauffman bracket, to keep track of the fact that it comes from the Kauffman bracket. As such, we denote it by $\kappa_n(K)$.

Let us describe now $K_t(\mathbb{T}^2 \times I)$, the Kauffman bracket skein algebra of the cylinder over a torus. Let p, q be two integers, with n their common divisor and $p' = p/n$, $q' = q/n$. We denote by $(p, q)_T$ be the skein $T_n((p', q'))$ in $K_t(\mathbb{T}^2 \times I)$, where (p', q') is the simple closed curve of slope p'/q' on the torus. As a module, $K_t(\mathbb{T}^2 \times I)$ is free with basis $(p, q)_T$, $p \geq 0$, $q \in \mathbb{Z}$. As shown in [**?**], the multiplication is given by the product-to-sum formula

$$(p, q)_T * (r, s)_T = t^{\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|}(p + r, q + s)_T + t^{-\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|}(p - r, q - s)_T.$$

This formula shows that there is an inclusion of $K_t(M)M$ into the ring of trigonometric functions on the noncommutative torus. Recall that this ring, denoted by $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$, consists of Laurent polynomials in l and m, where the two variables satisfy $lm = t^2 ml$. The inclusion map is defined by

$$(p, q)_T \to t^{-pq}(l^p m^q + l^{-p} m^{-q}).$$

A subring of $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$ is the quantum plane $\mathbb{C}_t[l, m]$, consisting of the polynomials in l and m, again with $lm = t^2 ml$.

If M is the complement of a regular neighborhood of a knot K, then the noncommutative A-ideal of K, $\mathcal{A}_t(K)$ is defined as follows. Denote by π the map between skein modules induced by the inclusion $\partial M \times I \subset M$ and let $I_t(K)$ be the kernel of π. $I_t(K)$ is a left ideal called the peripheral ideal of K. The noncommutative A-ideal of K is defined to be the left ideal obtained by extending $I_t(K)$ to $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$, then contracting it to $\mathbb{C}_t[l, m]$.

3. The Kauffman Bracket Skein Module of the Complement of a $(2, 2p+1)$-Torus Knot

Denote by M_p the complement of a regular neighborhood of the $(2, 2p+1)$-torus knot. Bullock proved in [**?**] that $K_t(M_p)$ is a free module with basis

$$\{x^k y^n; \quad 0 \leq k, 0 \leq n \leq p\},$$

where x and y are depicted in Fig. 3.1. Here $x^k y^n$ means k parallel copies of x together with n parallel copies of y. It is important to remark that although the skein module of the knot complement does not have a multiplicative operation, this formula makes sense.

Computations are simpler if we work instead with the basis

$$\{S_k(x) S_n(y); \quad 0 \leq k, 0 \leq n \leq p\}.$$

The following result about the Kauffman bracket skein module of the complement of the $(2, 2p+1)$-torus knot is important in itself. In addition it will be used extensively in our computations.

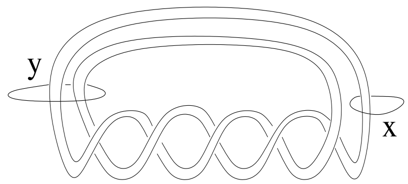


Figure 3.1.

Theorem 3.1. *For $i = 0, 1, \ldots, p+1$, one has*

$$t^{-2i-1}S_{p+i}(y) + t^{2i+1}S_{p-i-1}(y) = (-1)^i S_{2i}(x)(tS_{p-1}(y) + t^{-1}S_p(y)).$$

In order to prove the theorem we first study the skeins $A(k,n)$ and $\bar{A}(k,n)$ in the knot complement defined in Fig. 3.2. It will be seen below that $A(k,n) = \bar{A}(k,n)$ and that $A(n,k)$ is a polynomial of degree $k+n$ in y. Note that $A(i,0) = \bar{A}(2p+1-i, 0)$, which will produce the relation from the statement of the theorem.

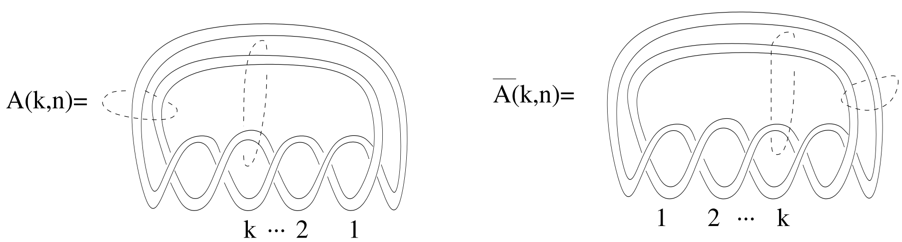


Figure 3.2.

We now prove two recursive relations.

Lemma 3.2. *For all $n \geq 0$ and all $1 \leq k \leq 2p$ one has*

$$A(k+1,n) = t^{-2}A(k,n+1) - t^{-4}A(k-1,n) + t^2x^2y^n - t^{-2}x^2y^n + x^2y^{n+1}$$
$$\bar{A}(k+1,n) = t^{-2}\bar{A}(k,n+1) - t^{-4}\bar{A}(k-1,n) + t^2x^2y^n - t^{-2}x^2y^n + x^2y^{n+1}.$$

Proof. We discuss only the first recursive relation, the second is similar and is left to the reader. The computation of $A(k+1,n)$ begins as described in Fig.3.3.

After introducing a "kink" and resolving the crossing the second skein becomes $x^2y^{n+1} + t^{-2}A(k,n+1)$. The first skein is computed as in Fig. 3.4.

Resolving the crossing we obtain $t^2x^2y^n$ plus the skein from Fig. 3.5. After introducing one twist and resolving it we obtain $-t^{-2}x^2y^n - t^{-4}A(k-1,n)$ and the conclusion of the lemma follows. □

Now we determine the initial conditions of the two recurrences.

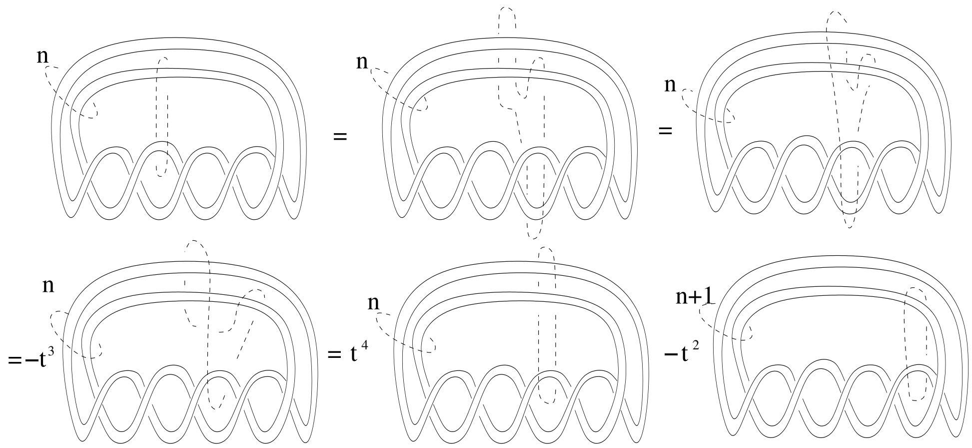


Figure 3.3.

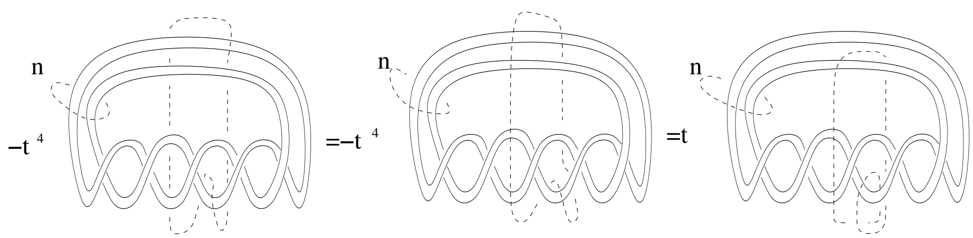


Figure 3.4.

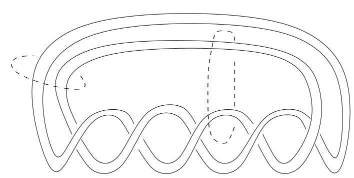


Figure 3.5.

Lemma 3.3. *For any $n \geq 0$ one has*

$$A(1,n) = \bar{A}(1,n) = y^{n+1}$$
$$A(2,n) = \bar{A}(2,n) = -(t^2+t^{-2})y^n + t^{-2}y^{n+2} + t^2x^2y^n + x^2y^{n+1}.$$

Proof. The case of $A(1,n)$ and $\bar{A}(1,n)$ is easy and is left to the reader. The computation of $A(2,n)$ begins as in Fig. 3.6. The computation of the remaining skein is described in Fig 3.7, and the desired formula is obtained by resolving the two crossings. A similar computation for $\bar{A}(2,n)$ shows that $A(2,n) = \bar{A}(2,n)$. In fact one could show directly this equality by sliding strands. □

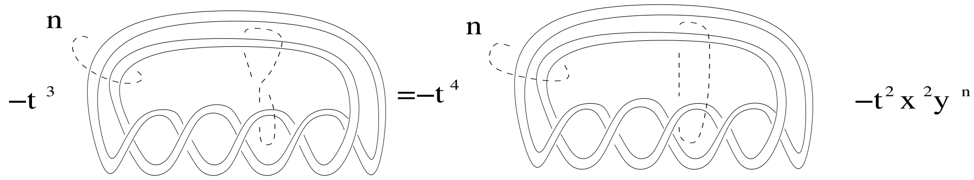


Figure 3.6.

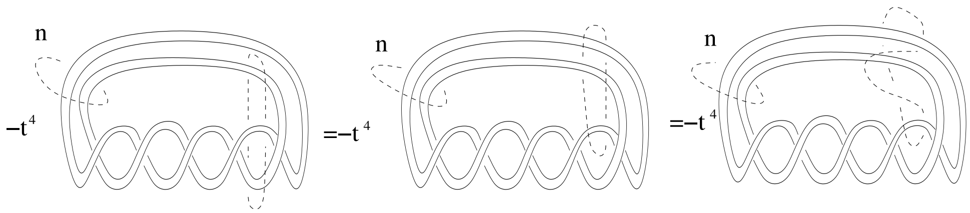


Figure 3.7.

Lemma 3.4. *For all $n \geq 0$ and $1 \leq k \leq 2p$ we have*

$$\begin{aligned} A(k,n) &= -t^{-2k+6}y^nS_{k-2}(y) - t^{-2k+4}x^2y^nS_{k-1}(d) + t^{-2k+2}y^nS_k(d) \\ &\quad -t^2x^2y^n + 2t^2x^2\sum_{r=-1}^{k-1} t^{-2r}S_r(y)y^n. \end{aligned}$$

Proof. The proof is by induction on k. The formula is valid for $k = 1$ and 2 by Lemma 3.3. Assume that it holds for $k-1$ and k and let us prove it for $k+1$. Applying Lemma 3.2 and the induction hypothesis we have

$$\begin{aligned} A(k+1,n) &= -t^{-2k+4}y^{n+1}S_{k-2}(y) - t^{-2k+2}x^2y^{n+1}S_{k-1}(y) \\ &+t^{-2k}y^{n+1}S_k(y) - x^2y^{n+1} + 2x^2\sum_{r=0}^{k-1} t^{-2r}S_r(y)y^{n+1} \\ &-(-t^{-2k+4}y^nS_{k-3}(y) - t^{-2k+2}x^2y^nS_{k-2}(y) + t^{-2k}y^nS_{k-1}(y) \\ &+2x^2\sum_{r=1}^{k-1} t^{-2r}S_{r-1}(y)y^n - t^{-2}x^2y^n) + t^2x^2y^n - t^{-2}x^2y^n + x^2y^{n+1} \\ &= -t^{-2k+4}y^nS_{k-1}(y) - t^{-2k+2}x^2y^nS_k(y) + t^{-2k}y^nS_{k+1}(y) + 2x^2y^{n+1} \\ &+t^2x^2y^n + \sum_{r=1}^{k-1} 2t^{-2r}x^2y^nS_{r+1}(y) \end{aligned}$$

$$= -t^{-2k+4}y^nS_{k-1}(y) - t^{-2k+2}x^2y^nS_k(y) + t^{-2k}y^nS_{k+1}(y) + t^2x^2y^n$$
$$+2x^2y^{n+1} + 2t^2x^2\sum_{r=2}^{k} t^{-2r}S_r(y)y^n$$
$$= -t^{-2k+4}y^nS_{k-1}(y) - t^{-2k+2}x^2y^nS_k(y) + t^{-2k}y^nS_{k+1}(y)$$
$$-t^2x^2y^n + 2t^2x^2\sum_{r=-1}^{k} t^{-2r}S_r(y)y^n,$$

and we are done. □

We now proceed with the proof of Theorem 3.1. From Lemma 3.2 and Lemma 3.3 it follows that $A(k,n) = \bar{A}(k,n)$. Also, by rotating the figure by 180° we see that $A(k,0) = \bar{A}(2p+1-k,0)$. In particular, for $1 \le i \le p$ we have

$$A(p+i,0) = \bar{A}(p-i+1,0) = A(p-i+1,0).$$

We prove the theorem by induction on i. Of course the induction can start at $i = -1$, since plugging this value into the formula produces an obvious equality. The case $i = 0$ is also easy, so let $k \ge 1$ and assume that the formula holds for $i = -1, 0, \ldots, k-1$. From $A(p+k,0) = A(p-k+1,0)$ and Lemma 3.4 it follows that

$$-t^{-2p-2k+6}S_{p+k-2}(y) - t^{-2p-2k+4}x^2S_{p+k-1}(y) + t^{-2p-2k+2}S_{p+k}(y) - t^2x^2$$
$$+2t^2x^2\sum_{r=-1}^{p+k-1} t^{-2r}S_r(y) = -t^{-2p+2k+4}S_{p-k-1}(y) + t^{-2p+2k+2}x^2S_{p-k}y$$
$$+t^{-2p+2k}S_{p-k+1}(y) - t^2x^2 + 2t^2x^2\sum_{r=-1}^{p-k-1} t^{-2r}S_r(y).$$

This yields

$$t^{-2p-2k+2}S_{p+k}(y) + t^{-2p+2k+4}S_{p-k-1}y$$
$$= (t^{-2p-2k} + 6S_{p+k-2}(y) + t^{-2p+2k}S_{p-k+1}(y)) + x^2(t^{-2p-2k+4}S_{p+k-1}(y)$$
$$+t^{-2p+2k+2}S_{p-k}(y)) - 2t^2x^2\sum_{r=p-k}^{p+k-1} t^{-2r}S_r(y)$$
$$= t^{-2p+3}[t^{-2k+3}S_{p+k-2}(y) + t^{2k-3}S_{p-k+1}(y) + x^2(t^{-2k+1}S_{p+k-1}(y)$$
$$+t^{2k-1}S_{p-k}(y))] - 2t^2x^2\sum_{j=0}^{k-1}(t^{-2p-2j}S_{p+j}(y) + t^{-2p+2j+2}S_{p-j-1}(y))$$
$$= t^{-2p+3}[(-1)^{k-2}S_{2k-4}(x)(t^{-1}S_p(y) + tS_{p-1}(y)) + (-1)^{k-1}x^2S_{2k-2}(x) \times$$
$$\times(t^{-1}S_p(y) + tS_{p-1}(y)) - 2x^2\sum_{j=0}^{k-1}(-1)^jS_{2j}(x)(t^{-1}S_p(y) + tS_{p-1}(y))].$$

Now we use the following identity, which can be proved for example by induction,

$$2x^2\sum_{r=0}^{m}(-1)^r S_{2r}(x)=(-1)^{m+1}(S_{2i-2}(x)-x^2S_{2i}(x)-S_{2i+2}(x)).$$

Applying it we further transform the above expression into

$$\begin{aligned}&t^{-2p+3}(t^{-1}S_p(y)+tS_{p-1}(y))[(-1)^{k-2}S_{2k-4}(x)+(-1)^{k-1}x^2S_{2k-2}(x)\\&-(-1)^k(S_{2k-4}(x)-x^2S_{2k-2}(x)-S_{2k}(x))]\\&=(-1)^k t^{-2p+3}S_{2k}(x)(t^{-1}S_p(y)+tS_{p-1}(y)),\end{aligned}$$

and the induction is complete. All what remains to prove is the case $i=p+1$. To this end we extend formally the recursive relation and define $A(0,0)$ and $A(2p+1,0)$. From

$$\begin{aligned}A(2,0)&=t^{-2}A(1,1)-t^{-4}A(0,0)+(t^2-t^{-2}+y)x^2\\A(2p+1,0)&=t^{-2}A(2p,1)-t^{-4}A(2p-1,0)+(t^2-t^{-2}+y)x^2\end{aligned}$$

and the formulas for $A(2,0), A(1,1), A(2p,1)$ and $A(2p-1,0)$ we obtain

$$A(2p+1,0)=A(0,0)=t^6+t^2-t^2x^2.$$

But this is all that we need for the above induction argument to work up to the case $i=p+1$, and the theorem is proved.

4. Two Curves on the Boundary

For proving Theorem 1.1 we need to determine the images in $K_t(M_p)$ through the inclusion map of the skeins $(1,k)_T$ from the skein module of the boundary torus. The particular case of the product-to-sum formula

$$(0,1)_T*(1,k)_T=t^{-1}(1,k+1)_T+t(1,k-1)_T$$

becomes after projecting to $K_t(M_p)$

$$x\pi((1,k)_T)=t^{-1}\pi((1,k+1)_T)+t\pi((1,k-1)_T),$$

where again, the multiplication on the left-hand side means that after we expand $\pi((1,k)_T)$ in terms of basis elements, the powers of x are raised by 1. Hence we have a recursive formula for computing $(1,k)_T$.

This shows that it suffices to compute $(1,k)_T$ and $(1,k-1)_T$ for some value of k. In the following two propositions the reasoning will be done on figures depicting the $(2,5)$-torus knot.

Proposition 4.1. *For all $p\in\mathbb{Z}$,*

$$\pi((1,-4p-2)_T)=2+(-1)^{p+1}\left(t^{-2p-3}S_{2p+2}(x)-t^{-2p+1}S_{2p-2}(x)\right)\times(tS_{p-1}(y)+t^{-1}S_p(y)).$$

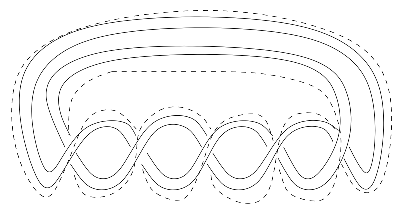


Figure 4.1.

Proof. The curve $(1,-4p-2)_T$ is depicted with dotted line in Fig. 4.1.

The computation of $\pi((1,-4p-2)_T)$ is simplified by the following observation. If we remove from M_p a regular neighborhood of the Möbius band that the knot bounds, then the resulting 3-manifold N is a solid torus that contains $\pi((1,-4p-2)_T)$ in its interior. So if we express the skein $\pi((1,-4p-2)_T)$ in terms of the basis of $K_t(M_p)$ only powers of y should appear. The coefficients of these powers are computed as for the solid torus N and the image of the curve $(2p+1,2)_T$ in its boundary. Note that when we turn the torus inside out the meridian and the longitude are exchanged. Using the formula deduced in Section 5 of [**?**], we find that this image, whether considered in $K_t(M_p)$ or $K_t(N)$, is

$$(-1)^2 t^{-2(2p+1)}\left(t^{-4}S_{2p+1}(y)-t^4S_{2p-1}(y)\right).$$

Using Theorem 3.1 for $i=p+1$ and $i=p-1$ we deduce that this is further equal to

$$\begin{aligned}&-S_{-2}(y)+(-1)^{p+1}S_{2p+2}(x)(tS_{p-1}(y)+t^{-1}S_p(y))\\ &\quad+S_0(y)-(-1)^{p-1}t^{-2p+1}S_{2p-2}(x)(tS_{p-1}(y)+t^{-1}S_p(y))\\ &=2+(-1)^{p+1}(t^{-2p-3}S_{2p+2}(x)-t^{-2p+1}S_{2p-2}(x))(tS_{p-1}(y)+t^{-1}S_p(y)).\end{aligned}$$

and we are done. □

For expanding the second boundary curve in terms of the basis we will need the following formula.

Lemma 4.2. *For any integer p the following holds*

$$\sum_{k=1}^{p-1} S_{2p-4i-2} = -\frac{S_{2p-3}}{S_1}.$$

Proof. First, since $S_n=-S_{-n-2}$,

$$\sum_{k=1}^{p-1} S_{2p-4i-2} = \sum_{k=1}^{p-1}(-1)^k S_{2p-2k-2}.$$

Recall that $S_n(2\cos x)=\sin(n+1)x/\sin x$ and that

$$\sum_{k=1}^{n}\sin(2k+1)x=\sin(2n+2)x/2\cos x.$$

We have

$$\sum_{k=1}^{p-1}(-1)^k S_{2p-2k-2}(2\cos x) = \sum_{k=1}^{p-1}(-1)^k \frac{\sin(2p-2k-1)x}{\sin x}$$
$$= -\frac{\sin(2p-2)x}{2\sin x\cos x} = -\frac{\sin(2p-2)x}{\sin 2x}$$
$$= -\frac{\sin(2p-2)x}{\sin x}\cdot\frac{\sin x}{\sin 2x} = -\frac{S_{2p-3}(2\cos x)}{S_1(2\cos x)}$$

and the identity follows. □

Proposition 4.3. *For all $p \in \mathbb{Z}$,*

$$\pi((1,-4p-1)_T) = tm + (-1)^{p+1}\left(t^{-2p-2}S_{2p+1}(x) - t^{-2p+2}S_{2p-3}(x)\right) \times (tS_{p-1}(y) + t^{-1}S_p(y)).$$

Proof. To make our figures easier to read we replace the drawing of the $(2,2p+1)$-torus knot as described in Fig. 4.2, and ask the reader to use the imagination and think of the 3-valent graph as being the torus knot.

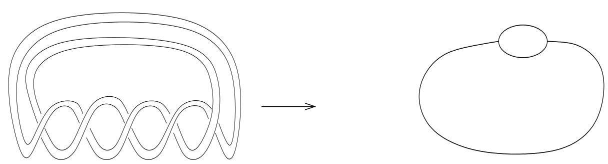


Figure 4.2.

As such, the dotted curve from Fig. 4.3 multiplied by t^{-6p} represents the skein $\pi((1,-4p-1)_T)$. The reason for the multiplcation by a power of t is that, as drawn, the curve has the blackboard framing, while the framing of $(1,-4p-1)_T$ isparallel to the boundary torus. Let us write the skein from Fig. 4.3 in terms of the basis of the module $K_t(M_p)$.

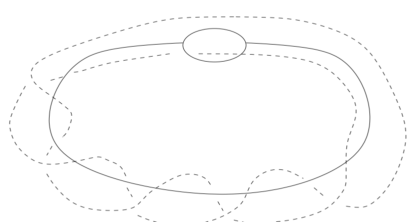


Figure 4.3.

We consider the skeins a_k and b_k described in Fig. 4.4 where a_k has k self crossings, $k \leq 2p+1$. Resolving the rightmost crossing of a_k and performing the twistings we obtain the recursive relations

$$\begin{aligned} a_{2k} &= t(-t^3)^{2k-1} b_{2p-2k+1} + t^{-1} a_{2k-1} \\ a_{2k+1} &= t(-t^3)^{2k} b_{2p-2k+1} + t^{-1} a_{2k}. \end{aligned}$$

Also, $a_0 = (-t^2 - t^{-2}) b_{2p+1}$. Let us determine b_k in terms of x and y.

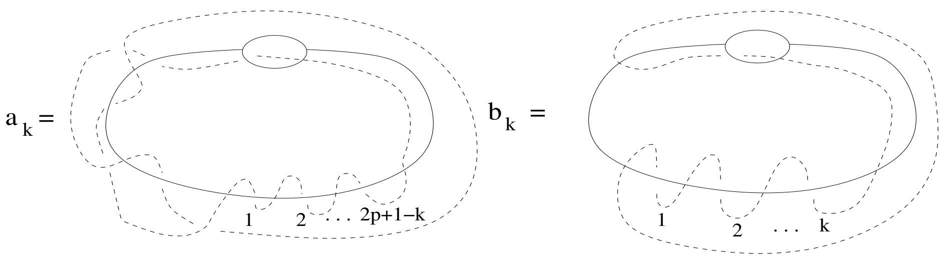


Figure 4.4.

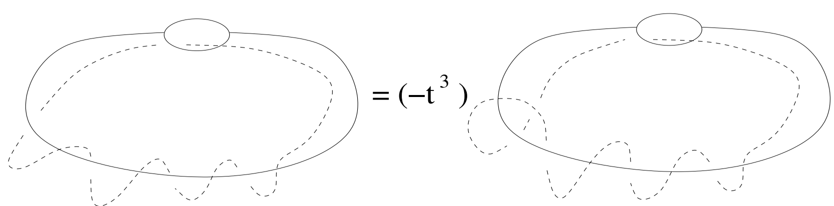


Figure 4.5.

Transform b_k as in Fig. 4.5, then resolve the crossing to obtain the recursive relation

$$b_{k+1} = (-t)^3 (t b_{k-1} + t^{-1} b_{k-2} y) = -t^2 y b_k - t^4 b_{k-1}.$$

If we let $x_k = (-t^{-2})^k b_k$, then $x_{k+1} = y x_k - x_{k-1}$, and $x_0 = m$, $x_1 = t^{-2} m$. Solving the second order recurrence we obtain $x_k = -t^{-2} x S_{k-1}(y) - x S_{j-2}(y)$, hence

$$b_k = (-1)^{k+1} t^{2k-2} x S_{k-1}(y) + (-1)^{k+1} t^{2k} x S_{k-2}(y), \quad \text{for all } k \geq 0.$$

Let us compute a_{2p+1} by writing the solution for the nonhomogeneous first order recursive relation that we deduced for a_k. We have

$$\begin{aligned} a_{2p+1} &= t^{-2p-1}a_0 + t\sum_0^p t^{-2i}(-t^3)^{2p-2i}b_{2i+1} + t\sum_1^p t^{-2i+1}(-t^3)^{2p-2i+1}b_{2i-1} \\ &= t^{-2p-1}a_0 + t\sum_0^p t^{-2i}(-t^3)^{2p-2i}b_{2i+1} + t\sum_0^{p-1} t^{-2i-1}(-t^3)^{2p-2i-1}b_{2i+1} \\ &= t^{-2p-1}(-t^2-t^{-2})b_{2p+1} + t^{-2p+1}b_{2p+1} + t^{6p+1}\sum_0^{p-1} t^{-8i}(1-t^{-4})b_{2i+1} \\ &= -t^{2p-3}b_{2p+1} + t^{6p+1}\sum_0^{p-1} t^{-8i}(1-t^4)b_{2i+1}. \end{aligned}$$

If we replace in this the value we computed for b_k we obtain

$$\begin{aligned} a_{2p+1} &= -t^{-2p-3}\left[t^{4p}xS_{2p}(y) + t^{4p+2}xS_{2p-1}(y)\right] \\ &+t^{6p+1}(1-t^{-4})x\sum_0^{p-1}\left[t^{-4i}S_{2i}(y) + t^{-4i+2}S_{2i-1}(y)\right]. \end{aligned}$$

Factoring out an t^{1-2p} and changing the order of summation for the second of the two sums we obtain that this is further equal to

$$\begin{aligned} &-t^{-2p-3}\left(t^{4p}xS_{2p}(y) + t^{4p+2}xS_{2p-1}(y)\right) \\ &+t^{4p+2}(1-t^{-4})x\sum_0^{p-1}\left[t^{2p-4i+1}S_{2i}(y) + t^{-2p+4i+1}S_{p+(p-1-2i)}(y)\right]. \end{aligned}$$

Note that $p-(p-1-2i)-1=2i$ hence we can replace each square bracket using Theorem 3.1. We also replace $S_{2p}(y)$ and $S_{2p-1}(y)$ to deduce that

$$\begin{aligned} a_{2p+1} &= -t^{2p-3}x\left[(-1)^p t^{2p+1}S_{2p}(x)(tS_{p-1}(y)+t^{-1}S_p(y))\right] \\ &-t^{2p-1}x\left[-t^{4p-2} + (-1)^{p-1}t^{2p-1}S_{2p-2}(x)(tS_{p-1}(y)+t^{-1}S_p(y))\right] \\ &+t^{4p+2}(1-t^{-4})x\sum_{i=0}^{p-1}(-1)^{p-1-2i}S_{2(p-1-2i)}(x)(tS_{p-1}(y)+t^{-1}S_p(y)). \end{aligned}$$

Since $S_{2p}-S_{2p-2}=T_{2p}$ we conclude that

$$\begin{aligned} a_{2p+1} = t^{6p+1}x + (-1)^{p+1}\left[t^{4p-2}xT_{2p}(x) + (t^{4p+2}-t^{4p-2})x\sum_0^{p-1}S_{2p-4i-2}(x)\right] \\ \times(tS_{p-1}(y)+t^{-1}S_p(y)). \end{aligned}$$

But $\pi((1,-4p-1)_T) = t^{-6p}a_{2p+1}$, so

$$\begin{aligned} \pi((1,-4p-1)_T) &= tx + (-1)^{p+1}[t^{-2p-2}xT_{2p}(x) \\ &+(t^{-2p+2}-t^{-2p-2})x\sum_1^{p-1}S_{2p-4i-2}(x)](tS_{p-1}(y)+t^{-1}S_p(y)). \end{aligned}$$

Applying Lemma 4.2 we get

$$\pi((1,-4p-1)_T) = tx + (-1)^{p+1}[t^{-2p-2}xT_{2p}(x)$$
$$-(t^{-2p+2}-t^{-2p-2})S_{2p-3}(x)](tS_{p-1}(y)+t^{-1}S_p(y)),$$

and the desired formula follows from $xT_{2p}(x) = T_{2p+1}(x) + T_{2p-1}(x) = S_{2p+1}(x) - S_{2p-3}(x)$. □

Combining the two propositions we obtain

Proposition 4.4. *For all $k \in \mathbb{Z}$ one has*

$$\pi((1,k)_T) = t^{4p+k+2}T_{4p+k+2}(x) + (-1)^{p+1}(t^{2p+k-1}S_{-2p-k}(x)$$
$$-t^{2p+k+3}S_{-2p-k-4}(x)) \times (tS_{p-1}(y)+t^{-1}S_p(y)).$$

Proof. The particular case of the product-to-sum formula

$$(1,k+1)_T = t(0,1)_T * (1,k)_T - t^2(1,k-1)_T$$

descends in the knot complement to the recursive relation

$$\pi((1,k+1)_T) = tx\pi((1,k)_T) - t^2\pi((1,k-1)_T).$$

Hence the values of all $\pi((1,k)_T$ can be computed from $\pi((1,-4p-2)_T)$ and $\pi((1,-4p-1)_T)$. An easy induction proves the proposition. □

5. Proof of the Main Result

We start by constructing an element in the peripheral ideal of the $(2,2p+1)$-torus knot using the formulas from Section 4.

Lemma 5.1. *The element*

$$(1,-2p-3)_T - t^{-8}(1,-2p+1)_T + t^{2p-5}(0,2p+3)_T - t^{2p-1}(0,2p-1)_T$$

is in the peripheral ideal of the $(2,2p+1)$-torus knot.

Proof. Applying Proposition 4.4 and using the fact that $S_{-1}=0$ we obtain

$$\pi((1,-2p-3)_T) = t^{2p-1}T_{2p-1}(x) + (-1)^{p+1}(t^{-4}S_3(x)(tS_{p-1}(y)+t^{-1}S_p(y))$$

and

$$\pi((1,-2p+1)_T) = t^{2p+3}T_{2p+3}(x) + (-1)^{p+1}(-t^4S_{-5}(x))(tS_{p-1}(y)+t^{-1}S_p(y))$$
$$= t^{2p+3}T_{2p+3}(x) + (-1)^{p+1}t^4S_3(x)(tS_{p-1}(y)+t^{-1}S_p(y)).$$

Multiplying the second equality by t^{-8} and subtracting it from the first we obtain

$$(1,-2p-3)_T - t^{-8}(1,-2p+1)_T = t^{2p-1}T_{2p-1}(x) - t^{2p-5}T_{2p+3}(x).$$

But $T_n(x) = \pi((0,n)_T)$ for all n. Hence the image through π of

$$(1,-2p-3)_T - t^{-8}(1,-2p+1)_T + t^{2p-5}(0,2p+3)_T - t^{2p-1}(0,2p-1)_T$$

is zero. This shows that this element is in the kernel of π, that is in the peripheral ideal. □

Proposition 5.2. *The polynomial*

$$(l - t^{-4}lm^4 + t^{4p-2} - t^{4p+10}m^4)(l - t^{4p+2}m^{4p+2})$$

is in the noncommutative A-ideal of the $(2, 2p+1)$*-torus knot.*

Proof. Through the inclusion

$$K_t(\mathbb{T}^2 \times I) \hookrightarrow \mathbb{C}_t[l, l^{-1}, m, m^{-1}]$$

an element $(a, b)_T$ transforms into $t^{-ab}(l^a m^b + l^{-a}m^{-b})$. Applying this to the element provided by Lemma 5.1 we deduce that the extension of the peripheral ideal $I_t(K)$ to $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$ contains the element

$$t^{2p+3}lm^{-2p-3} + t^{2p+3}l^{-1}m^{2p+3} - t^{2p-9}lm^{-2p+1} - t^{2p-9}l^{-1}m^{2p-1}$$
$$+t^{2p-5}m^{2p+3} + t^{2p-5}m^{-2p-3} - t^{2p-1}m^{2p-1} - t^{2p-1}m^{-2p+1}.$$

Now contract the extension $I_t(K)$ to $\mathbb{C}_t[l, m]$. The above element gives rise through multiplication to the left by lm^{2p+3} to a polynomial in the noncommutative A-ideal. This polynomial is

$$t^{-2p-3}l^2 + t^{6p+9}m^{4p+6} - t^{-2p-14}l^2m^4 - t^{6p-3}m^{4p+2} + t^{2p-5}lm^{4p+6}$$
$$+t^{2p-5}l - t^{2p-1}lm^{4p+2} - t^{2p-1}lm^4.$$

After multiplying by t^{2p+3} and factoring this becomes the polynomial from the statement. □

Let us proceed with the proof of Theorem 1.1. To this end we rephrase slightly Theorem 2 in [**?**]:

Let K be a knot whose A-ideal $\mathcal{A}_t(K)$ contains a polynomial $\sum_{p,q} \gamma_{p,q} l^p m^q$ of degree 2 in l such that there exists no $n \geq 0$ for which the expression $\sum_q \gamma_{2,q}(-1)^q t^{(2n+2)q}$ is identically equal to zero. Assume in addition that this polynomial arises from a skein in the peripheral ideal of the knot. Then for any knot K' with the property that $\mathcal{A}_t(K) = \mathcal{A}_t(K')$, one has $\kappa_n(K) = \kappa_n(K')$ for all $n \geq 1$.

Here the assumption that the polynomial comes from a skein in the peripheral ideal yields to a simpler version of the relation among the coefficients of the polynomial to be checked than the one given in [**?**], and the same proof works. Recall also that we denoted by $\kappa_n(K)$ the nth colored Kauffman bracket of K, which is (up to a change of variable) the same as the nth colored Jones polynomial as defined in [**?**].

Expand the polynomial from Proposition 5.2 to get

$$l^2 + t^{8p+12}m^{4p+6} - t^{-12}l^2m^4 - t^{8p}m^{4p+2} + t^{4p-2}lm^{4p+6}$$
$$+t^{4p-2}l - t^{4p+2}lm^{4p+2} - t^{4p+2}lm^4.$$

The only coefficients $\gamma_{2,q}$ that are not zero are $\gamma_{2,0} = 1$ and $\gamma_{2,4} = -t^{-12}$. The expression

$$1 - t^{-12}t^{4(2n+2)}$$

is identically zero if and only if $8n-4=0$ for some n. This has to be so since t is the variable of a polynomial. The equality cannot hold, which proves Theorem 1.1.

Through the procedure described in [**?**] we can find the following recursive relation for the colored Kauffman brackets of the $(2,2p+1)$-torus knot

$$\begin{aligned}
&(-t^{-4np-6n-6p-9}+t^{-4np+2n-6p-5})\kappa_{n+1}(K)\\
&+(-t^{4np+6n+6p+1}-t^{-4np-6n-2p-11}+t^{4np-2n+6p-3}+t^{-4np+2n-2p+1})\kappa_n(K)\\
&+(t^{-4np-6n+2p+3}-t^{4np+6n+2p+3}-t^{-4np+2p+2n-9}+t^{4np-2n+2p-9})\kappa_{n-1}(K)\\
&+(t^{4np+6n-2p-11}+t^{-4np-6n+6p+1}-t^{4np-2n-2p+1}-t^{-4np+2n+6p-3})\kappa_{n-2}(K)\\
&+(t^{4np+6n-6p-9}-t^{4np-2n-6p-5})\kappa_{n-3}(K)=0.
\end{aligned}$$

Recall that $\kappa_{-3}(K)=-\kappa_1(K)$, $\kappa_{-2}(K)=-1$, $\kappa_{-1}(K)=0$, $\kappa_0(K)=1$, so this recurrence allows us indeed to compute all colored Kauffman brackets.

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