The action of the Kauffman bracket skein algebra of the torus on the Kauffman bracket skein module of the 3-twist knot complement

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ABSTRACT

We determine the action of the Kauffman bracket skein algebra of the torus on the Kauffman bracket skein module of the complement of the 3-twist knot. The point is to study the relationship between knot complements and their boundary tori, an idea that has proved very fruitful in knot theory. We place this idea in the context of Chern-Simons theory, where such actions arose in connection with the computation of the noncommutative version of the A-polynomial that was defined by Frohman, the first author, and Lofaro, but they can also be interpreted as quantum mechanical systems. Our goal is to exhibit a detailed example in a part of Chern-Simons theory where examples are scarce.

Keywords: Kauffman bracket, skein modules, Chern-Simons theory, 3-twist knot complement

Mathematics Subject Classification 2020: 57K31, 57K16

1. Introduction

This paper should be viewed as a piece of experimental mathematics. It describes the action of the Kauffman bracket skein algebra of the torus on the Kauffman bracket skein module of the complement of the 3-twist knot, which is listed as the 5_2 knot in the knot table. Such computations have been done before for the trefoil knot [8], the figure-eight knot [11], and $(2, 2p+1)$ torus knots [18] as the main step in the computation of the noncommutative version of the A-polynomial defined in [6]. The noncommutative version of the A-polynomial has been linked to colored Jones polynomials in [6], [9], [7], and to $SL(2,\mathbb{C})$-Chern-Simons theory [13], [4], a difficult area of mathematics that has yet to be thoroughly understood. The theory of Kauffman bracket skein modules has been linked to $SL(2,\mathbb{C})$-character varieties [2], [20], as deformations of rings of affine characters, and as such they are also supposed to be related to $SL(2,\mathbb{C})$-Chern-Simons theory, though it is not known how.

The case of the 3-twist knot is probably the most complicated example that can still be done by hand; this is why we want to present it to the public. Computational complexity grows very fast in the theory of skein modules, hence there are few examples. A striking feature exhibited in this paper is the ocurrence of Jones-Wenzl idempotents in the computations within skein modules. This feature has been observed before by the first author; it seems that (arbitrary) skein computations tend to structure themselves in terms of Jones-Wenzl idempotents.

In [21], E. Witten has related $SU(2)$-Chern-Simons theory to quantizations of moduli spaces of flat connections on surfaces, which then leads to quantum mechanical models (see [12] for a complete discussion). These quantum mechanical models have a combinatorial version that arises from quantizing Wilson lines, formulated using *reduced* skein modules (which are the building blocks of the topological quantum field theory of Blanchet, Habbegger, Masbaum and Vogel [1]). The action of the Kauffman bracket skein algebra of the torus on the skein module of the knot complement that makes the object of this paper is similar to that quantum mechanical model, but here we work with the non-reduced version of skein modules. One might ask what does the model in which we work with the actual skein modules and not their reduced versions represent? Given such questions, the lack of examples, and recent renewed interest in skein modules, we consider worth showing this particular situation, as it might help clarify the general situation.

Given that the structure of the Kauffman bracket skein modules is now known for a fairly large family of knot and link complements [16], [17], we hope that the above work will be expanded to the study of structures that arise from skein modules in other knot complements.

We start with some background material. Throughout the paper t is a variable. A *framed link* in an orientable 3-manifold M is a disjoint union of embedded annuli. If M is the cylinder over the torus, framed links are identified with curves on the torus, with the annulus being parallel to the torus. If we draw a framed link on paper, its framing is parallel to the plane of the paper, unless the link is drawn on the torus, when we use the previous convention. Let $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold M, including the empty link. Consider the free $\mathbb{C}[t, t^{-1}]$-module with basis $\mathcal{L}$, and factor it by the smallest subspace containing all expressions of the form $\asymp - t \, \smile_\frown - t^{-1} \,)($ and $\bigcirc + t^2 + t^{-2}$, where the links in each expression are identical except in a ball in which they look like depicted. This quotient is denoted by $K_t(M)$ and is called the Kauffman bracket skein module of the manifold [19]. The factorization allows us to smoothen crossings (which we can create at will using isotopy) and to replace trivial link components by a scalar. Because at each application of the first skein relation one term is replaced by two terms, the complexity of computations grows exponentially, and so the computations in this paper are quite involved.

For the cylinder over torus, $\mathbb{T}^2 \times I$ (where $I = [0, 1]$), the skein module has a multiplication induced by the operation of gluing one cylinder on top of another.

This multiplication has been explicated in [5], here it is: As a module, $K_t(\mathbb{T}^2 \times I)$ is free with basis $(p,q)_T$, $p, q \in \mathbb{Z}$, $p \geq 0$, where $(p,q)_T = T_n((p,q))$ with T_n the (normalized) Chebyshev polynomial of first kind defined by $T_0(x) = 2$, $T_1(x) = x$, and $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$, $n = \gcd(p,q)$, and (p,q) is the curve of slope p/q on the torus. We have product-to-sum formula

$$(p,q)_T * (r,s)_T = t^{|\frac{pq}{rs}|}(p+r, q+s)_T + t^{-|\frac{pq}{rs}|}(p-r, q-s)_T.$$

In this paper we focus on the 3-twist knot K drawn in bold line in Figure 1. By
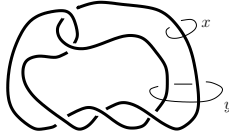
Fig. 1.

the complement of K we mean the compact orientable manifold $S^3 \backslash N(K)$ obtained by removing from the 3-sphere an open regular neighborhood $N(K)$ of K. The gluing of the cylinder over $\partial N(K) = \mathbb{T}^2$ to $S^3 \backslash N(K)$ induces a $K_t(\mathbb{T}^2 \times I)$-left module structure on $K_t(S^3 \backslash N(K))$; we want to explicate this structure.

It was shown in [3] that $K_t(S^3 \backslash N(K))$ is a free $\mathbb{C}[t, t^{-1}]$-module with basis $x^n y^k$, $n \geq 0$, $0 \leq k \leq 3$, where x, y are shown in Figure 1. It suffices to understand the action of a set of generators of $K_t(\mathbb{T}^2 \times I)$ on the basis, and as generators we have chosen $(0,1)_T$, $(1,-3)_T$ and $(1,-2)_T$. The action of $(0,1)_T$ is $(0,1)_T x^n y^k = x^{n+1}y^k$, so we focus on the other two.

Using the fact the x can be pulled back into the cylinder over the boundary as the skein $(0,1)_T$, and using the relations $(1,q)_T (0,1)_T = t(1, q+1)_T + t^{-1}(1, q-1)_T$, and $(0,1)_T (1,q)_T = t^{-1}(1, q+1)_T + t(1, q-1)_T$, we see that the action of $(1,q)_T$ on $x^n y^k$ can be found easily if we know how $(1,q)_T$ acts on the basis elements $1 = y^0, y, y^2, y^3$. It should also be noted that in computations from this paper x behaves like a *scalar*.

We change the basis of $K_t(S^3 \backslash N(K))$ to $S_n(x)S_k(y)$, where S_n is the (normalized) Chebyshev polynomial of the second kind: $S_0(x) = 1, S_1(x) = x, S_{n+1}(x) = xS_n(x) - S_{n-1}(x)$. As such, the basis elements are the curves x and y colored by *Jones-Wenzl idempotents*. There are two explanations for this, one is practical: the formulas become simpler. But there is a deeper explanation for this, namely that the polynomial S_n is the character of the $n+1$-dimensional irreducible representation of $SL(2, \mathbb{C})$, and as such the skein $S_n(x)S_k(y)$ consists of two Wilson lines (one for x and one for y) associated to irreducible representations. Note also that the colored Jones polynomials of a knot K are $(-1)^n \langle S_n(K) \rangle$, where $\langle \cdot \rangle$ denotes the Kauffman bracket of knots and links in S^3.

The goal of this paper is to find $(1,-3)_T \cdot S_k(y)$ and $(1,-2)_T \cdot S_k(y)$, $k = 0, 1, 2, 3$.

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2. Formulas in a quotient of the Kauffman bracket skein module of cylinder over the twice punctured disk

It is known that the Kauffman bracket skein module of the cylinder over the twice punctured disk, i.e. a disk with two disjoint open disks being removed, is free with basis $x^m y^n z^k$, $m, n, k \geq 0$, where x and z are curves that are parallel with the boundaries of the two open disks that have been removed, and y is a curve parallel to the boundary of the original disk. In Sections 2 and 3, we make the following convention. We schematically represent the cylinder over the twice punctured disk sideways, by drawing only the two curves that trace the punctures in the cylinder. These curves will either be represented as twisting around each other, such as in the first diagram from Figure 2, or as two parallel lines such as in the second and third diagram from the same figure. Closed curves in the diagram comprise skeins, taken with the blackboard framing. Whenever a number is written next to a curve, such as the k written next to the y-curve in the first diagram from Figure 2, that number indicates that the skein contains that many parallel copies of that curve, as such as in our example there are k parallel copies of y.

We factor the Kauffman bracket skein module of the cylinder over the twice punctured disk by the relation $x = z$ and perform all computations from this section of the paper in this quotient. All computations in this section can be used for general twist knots.

In Figure 2, we recall the skeins $X_i * y^k$ from [10] and define the skeins $Y_1 * y^k$. In the first diagram, the index i counts the crossings of the two strands that define the genus 2 handlebody. For $Y_1 * y^k$, the undercrossings can be at the bottom and the overcrossings at the top, as one diagram is mapped into the other by isotopy.
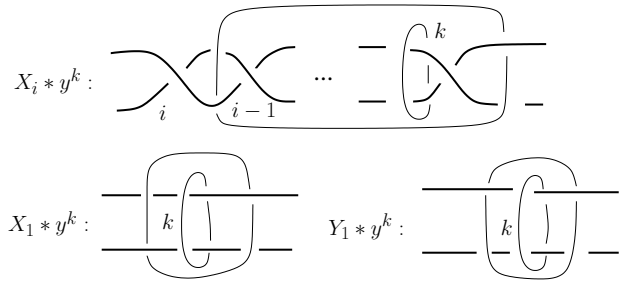

Fig. 2.

Lemma 2.1. *The skeins $X_1 * y^k$ and $Y_1 * y^k$, $k \geq 0$, satisfy the recursions:*

$$X_1 * y^{k+1} = t^4 y X_1 * y^k + (t^{-2} - t^6) Y_1 * y^k + 2(1 - t^4) x^2 y^k, \quad k \geq 0,$$
$$Y_1 * y^{k+1} = t^{-4} y Y_1 * y^k + (t^2 - t^{-6}) X_1 * y^k + 2(1 - t^{-4}) x^2 y^k, \quad k \geq 0,$$
$$X_1 * y^0 = X_1 = -t^4 y - t^2 x^2, \quad Y_1 * y^0 = Y_1 = -t^2 - t^{-2}.$$

Proof. We start computing $X_1 * y^{k+1}$ as in Figure 3. The first diagram is computed

Fig. 3.

as in Figure 4, and is equal to $t^2 y X_1 * y^k - t^4 Y_1 * y^k - 2t^2 x^2 y^k$. So the first term is $t^4 y X_1 * y^k - t^6 Y_1 * y^k - 2t^4 x^2 y^k$. The sum of the other terms is equal to $x^2 y^k + x^2 y^k + t^{-2} Y_1 * y^k$. Adding we obtain the first recursion. Similarly for the second recursion. $\qquad\square$

Applying Lemma 2.1 we obtain

$$X_1 * y = -t^8 S_2(y) - t^6 x^2 S_1(y) + 2(1 - t^4)x^2 + (t^4 - 1 - t^{-4});$$
$$Y_1 * y = -(t^6 + t^{-6})S_1(y) + (2 - t^4 - t^{-4})x^2;$$

$$X_1 * y^2 = -t^{12}S_3(y) + (-t^{10})x^2 S_2(y) + (-2t^8 + 2)x^2 S_1(y)$$
$$+ (t^8 - 2t^4 - t^{-8})S_1(y) + (-2t^6 + 2t^{-2} - t^{-6})x^2;$$
$$Y_1 * y^2 = (-t^{10} - t^{-10})S_2(y) + (2 - t^8 - t^{-8})x^2 S_1(y)$$
$$+ (-2t^6 - 2t^{-6} + 2t^2 + 2t^{-2})x^2 + (t^6 + t^{-6} - 2t^2 - 2t^{-2});$$

The following result is a straighforward generalization of Lemma 1 in [10].

Lemma 2.2. *The skeins $X_i * y^k$, $i, k \geq 0$, satisfy the recursive relation*

$$X_{i+2} * y^k = t^2 y X_{i+1} * y^k - t^4 X_i * y^k - 2t^2 x^2 y^k,$$
$$X_2 * y^k = t^2 y X_1 * y^k - t^4 Y_1 * y^k - 2t^2 x^2 y^k.$$

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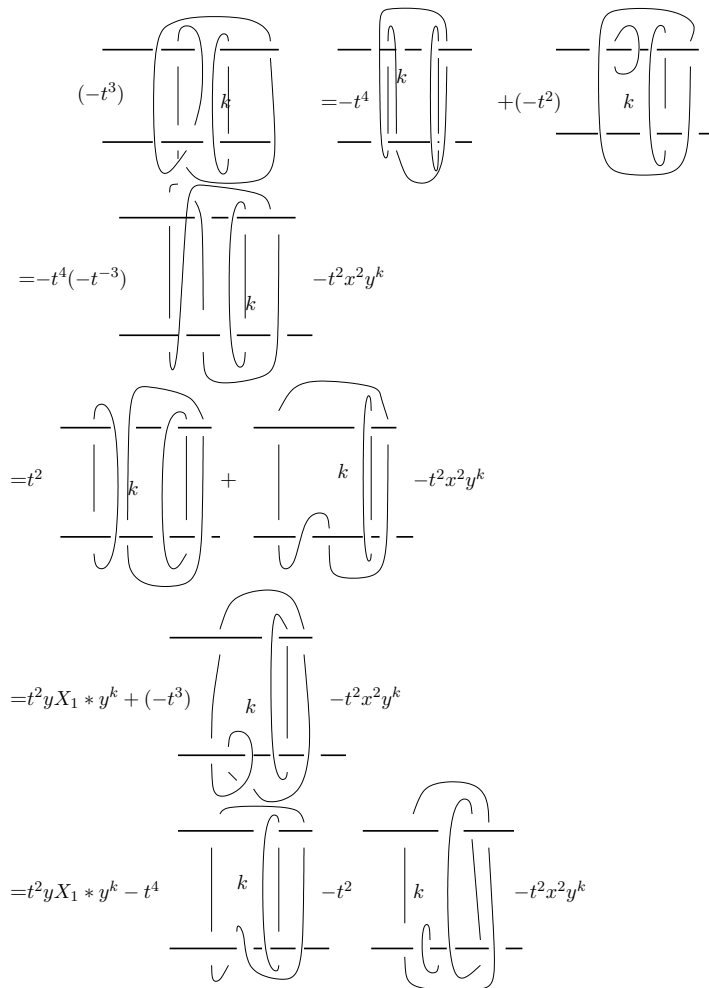
Fig. 4.

As a consequence, we obtain

$$X_2 = -t^6 S_2(y) - t^4 S_2(x)S_1(y) - t^4 S_1(y) - 2t^2 S_2(x) - t^2;$$

$$X_3 = -t^8 S_3(y) - t^6 S_2(x)S_2(y) - t^6 S_2(y) - 2t^4 S_2(x)S_1(y)$$
$$-t^4 S_1(y) - 2t^2 S_2(x) - 2t^2;$$

$$X_4 = -t^{10} S_4(y) - t^8 S_2(x)S_3(y) - t^8 S_3(y) - 2t^6 S_2(x)S_2(y)$$
$$-t^6 S_2(y) - 2t^4 S_2(x)S_1(y) - 2t^4 S_1(y) - 2t^2 S_2(x) - 2t^2;$$

$$X_2 * y = -t^{10} S_3(y) - t^8 S_2(x)S_2(y) - t^8 S_2(y) - 2t^6 S_2(x)S_1(y)$$
$$+(-t^6 - t^2)S_1(y) + (-2t^4 + 1)S_2(x) + (-2t^4 + 1);$$

$$X_3 * y = -t^{12}S_4(y) - t^{10}S_2(x)S_3(y) - t^{10}S_3(y) - 2t^8 S_2(x)S_2(y)$$
$$+(-t^8 - t^4)S_2(y) + (-2t^6 - t^2)S_2(x)S_1(y) + (-2t^6 - t^2)S_1(y)$$
$$-2t^4 S_2(x) + (-2t^4 + 1);$$
$$X_4 * y = -t^{14}S_5(y) - t^{12}S_2(x)S_4(y) - t^{12}S_4(y) - 2t^{10}S_2(x)S_3(y)$$
$$+(-t^{10} - t^6)S_3(y) + (-2t^8 - t^4)S_2(x)S_2(y) + (-2t^8 - t^4)S_2(y)$$
$$+(-2t^6 - 2t^2)S_2(x)S_1(y) + (-2t^6 - t^2)S_1(y) - 2t^4 S_2(x) - 2t^4;$$

$$X_2 * y^2 = -t^{14}S_4(y) - t^{12}S_2(x)S_3(y) - t^{-12}S_2(y) - 2t^{10}S_2(x)S_2(y)$$
$$+(-t^{10} - 2t^6)S_2(y) + (-2t^4 - 2t^8 + 2)S_2(x)S_1(y)$$
$$+(-2t^4 - 2t^8 + 2)S_1(y) + (-2t^6 - 2t^2 + 2t^{-2})S_2(x)$$
$$+(-2t^6 + t^{-2} - t^{-6});$$
$$X_3 * y^2 = -t^{16}S_5(y) - t^{14}S_2(x)S_4(y) - t^{14}S_4(y) + (-2t^{12})S_2(x)S_3(y)$$
$$+(-t^{12} - 2t^8)S_3(y) + (-2t^{10} - 2t^6)S_2(x)S_2(y) + (-2t^{10} - 2t^6)S_2(y)$$
$$+(-2t^8 - 4t^4 + 2)S_2(x)S_1(y) + (-2t^8 - 2t^4 + 1)S_1(y)$$
$$+(-2t^6 - 2t^2 + t^{-2})S_2(x) + (-2t^6 - 2t^2 + t^{-2});$$
$$X_4 * y^2 = -t^{18}S_6(y) - t^{16}S_2(x)S_5(y) - t^{16}S_5(y) - 2t^{14}S_2(x)S_4(y)$$
$$+(-t^{14} - 2t^{10})S_4(y) + (-2t^{12} - 2t^8)S_2(x)S_3(y)$$
$$+(-2t^{12} - 2t^8)S_3(y) + (-2t^{10} - 4t^6)S_2(x)S_2(y)$$
$$+(-2t^{10} - 2t^6 - t^2)S_2(y) + (-2t^8 - 4t^4 + 1)S_2(x)S_1(y)$$
$$+(-2t^8 - 4t^4 + 1)S_1(y) + (-2t^6 - 2t^2)S_2(x) + (-2t^6 - 2t^2 + t^{-2}).$$

Lemma 2.3. *The skeins $X_i * y^k$, $i, k \geq 0$, satisfy the recursive relation*

$$X_2 * y^k = t^{-2}X_1 * y^{k+1} - 2t^{-2}x^2 y^k - t^{-4}Y_1 * y^k$$

and for $i \geq 1$,

$$X_{i+2} * y^k = t^{-2}X_{i+1} * y^{k+1} - 2t^{-2}x^2 y^k - t^{-4}X_i * y^k.$$

Proof. To compute $X_i * y^{k+1}$, we separate a y from y^{k+1}, slide it so as to produce two crossings in the link diagram, then solve the crossings using the Kauffman bracket skein relation as in Figure 5.

In the first term, by sliding the strand to the right we see that this term equals $t^2 X_{i+1} * y^k$. The second and third terms are each equal to $2x^2 y^2$. The last term is equal to $X_{i-1} * y^k$ if $i \geq 2$ and to $Y_1 * y^k$ if $i = 1$. $\square$

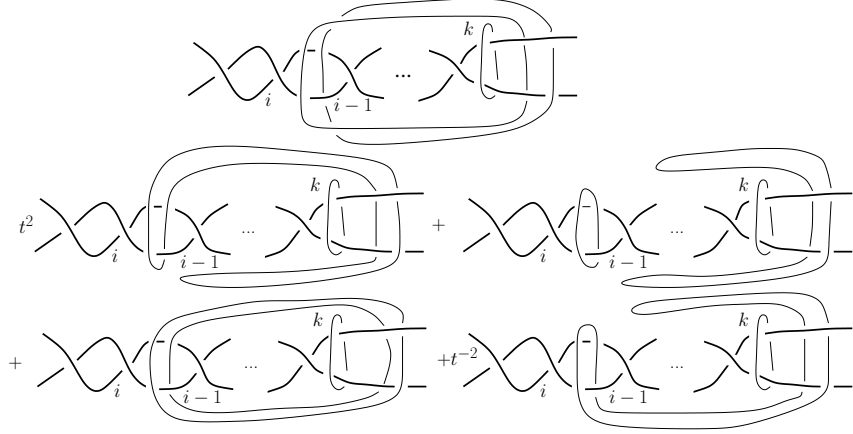Define the skeins $A * y^k$, $\overline{A * y^k}$, $B * y^k$, $\overline{B * y^k}$ as in Figure 6.
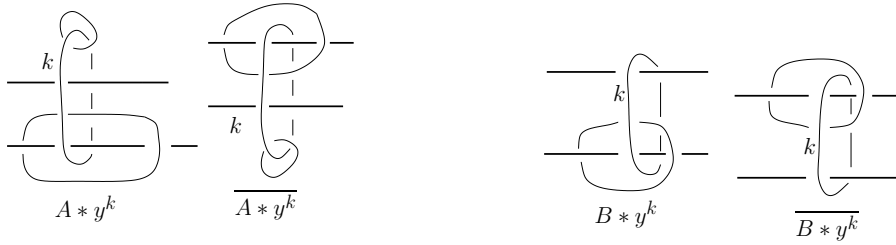
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Fig. 5.

Fig. 6.

Lemma 2.4. *The following relations hold*

$$A * S_k(y) = (-t^{2k+2} - t^{-2k-2})B * S_k(y),$$
$$\overline{A * S_k(y)} = (-t^{2k+2} - t^{-2k-2})\overline{B * S_k(y)}$$
$$B * y^k = t^2 y B * y^{k-1} + (1 - t^{-4})\overline{B * y^{k-1}},$$
$$\overline{B * y^k} = t^{-2} y \overline{B * y^{k-1}} + (1 - t^4)B * y^{k-1},$$
$$B * y^0 = \overline{B * y^0} = x.$$

Proof. The formulas for $A * S_k(y)$, $\overline{A * S_k(y)}$ follow from the standard properties of Jones-Wenzl idempotents.

For $B * y^k$ (see Figure 7) resolve the crossings specified by the arrow to obtain the first sum in this figure. Perform an isotopy of the first skein from the sum to obtain the first skein on the second row (in the process we remove and then add a positive twist), then remove a negative twist from the second term and perform an isotopy in this term. Then apply the Kauffman bracket skein relation in the place specified by the arrow to obtain the desired relation.

$\overline{B * y^k}$ is obtained by reflecting $B * y^k$ over a horizontal line, and under reflections, in the Kauffman bracket t is replaced by t^{-1}.
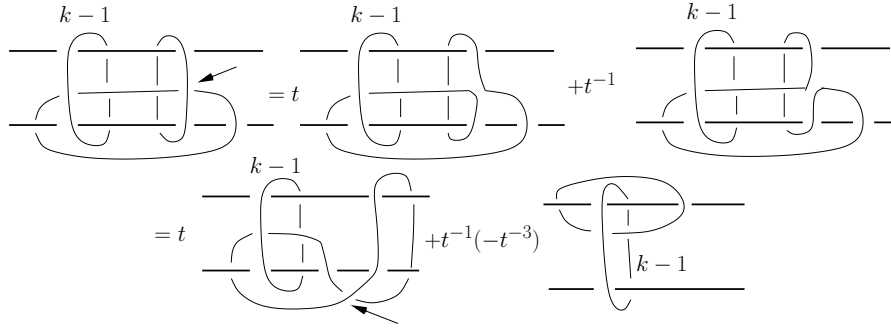

Fig. 7.

□

Corollary 2.5. *The following formulas hold*

$$A * y^0 = (-t^2 - t^{-2})x,$$

$$A * y = (-t^6 - t^{-2})xS_1(y) + (-t^4 + 1 - t^{-4} + t^{-8})x,$$

$$A * y^2 = (-t^{10} - t^{-2})xS_2(y) + (-t^8 + 1 - t^{-4} + t^{-12})xS_1(y)$$
$$+(-t^6 - t^{-6} - t^{-2} + t^{-10})x$$

$$A * y^3 = (-t^{14} - t^{-2})xS_3(y) + (-t^{12} + 1 - t^{-4} + t^{-16})xS_2(y)$$
$$+(-t^{10} - 2t^{-2} + t^2 - t^{-6} + t^{-14} - 2t^6)xS_1(y)$$
$$+(-t^8 - t^4 + 2 + t^{-12} - 2t^{-4} + t^{-8})x.$$

Corollary 2.6. *The following formulas hold*

$$\overline{A * y^0} = (-t^2 - t^{-2})x,$$

$$\overline{A * y} = (-t^2 - t^{-6})xy + (t^8 - t^4 + 1 - t^{-4})x,$$

$$\overline{A * y^2} = (-t^2 - t^{-10})xS_2(y) + (t^{12} - t^4 + 1 - t^{-8})xS_1(y)$$
$$+(-t^6 - t^{-6} - t^2 + t^{10})x$$

$$\overline{A * y^3} = (-t^2 - t^{-14})xS_3(y) + (t^{16} - t^4 + 1 - t^{-12})xS_2(y)$$
$$+(-t^{-10} + t^{-2} - 2t^2 - t^6 + t^{14} - 2t^{-6})xS_1(y)$$
$$+(-t^{-8} - t^{-4} + 2 + t^{12} - 2t^4 + t^8)x.$$

Proof. The skein $\overline{A_0 * y^k}$ is the reflection of $A_0 * y^k$ over a horizontal line. To get the formulas for $\overline{A_0 * y^k}$, swap t and t^{-1} in the formulas for $A_0 * y^k$. □

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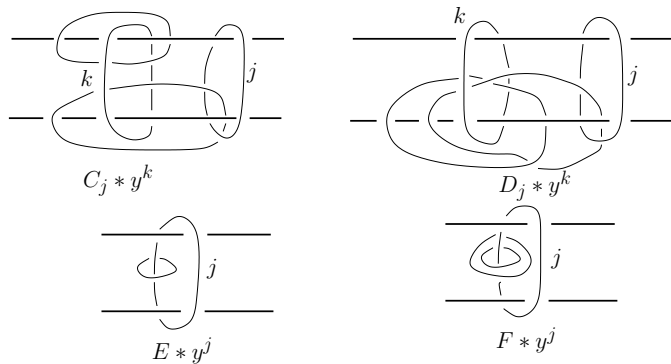We define the skeins $C_j * y^k, D_j * y^k, E_j, F_j$ as in Figure 8.

$$C_j * y^k$$

$$D_j * y^k$$

$$E * y^j$$

$$F * y^j$$

Fig. 8.

Lemma 2.7. *The skeins $C_j * y^k, D_j * y^k, E * y^j, F * y^j$ satisfy the following relations*

$$C_j * y^k = t^2 C_{j+1} * y^{k-1} + (1 - t^{-4}) D_j * y^{k-1}$$
$$D_j * y^k = t^{-2} D_{j+1} * y^{k-1} + (1 - t^4) C_j * y^{k-1}$$
$$C_j * y^0 = x\overline{B * y^j}, \quad D_j * y^0 = t^{-2} x\overline{B * y^j} + (1 - t^4) E * y^j$$
$$E * S_j(y) = (-t^{2j+2} - t^{-2j-2}) S_j(y), F * S_j(y) = (t^{2j+2} + t^{-2j-2})^2 S_j(y).$$

Proof. For $C_j * y^k$, separate a strand from y^k as in the skein on the left in Figure 9, then resolve the crossings defined by arrows to obtain

$$t^2 C_{j+1} * y^{k-1} + D_j * y^{k-1} + D_j * y^{k-1} + t^{-2}(-t^2 - t^{-2}) D_j * y^{k-1},$$

which yields the relation. For $D_j * y^k$ do the same in the skein on the right.
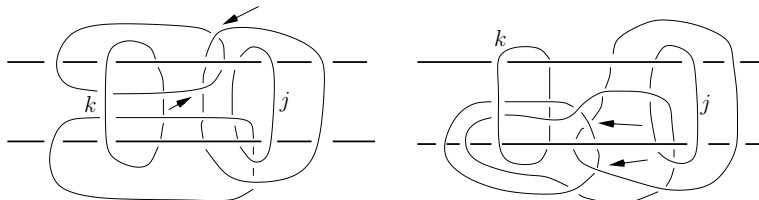
Fig. 9.

The skein $C_j * y^0$ is the mirror image of xA_j over a vertical line, so, as a skein, equals $x\overline{A_j}$. Resolving the two crossings in $D_j * y^0$ specified in Figure 10, we get
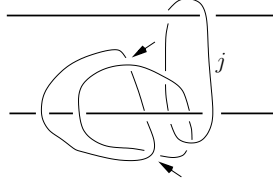
Fig. 10.

$t^2(-t^2-t^{-2})E_j+E_j+E_j+t^{-2}x\overline{A_j}$. The formulas for $E*S_j(y)$ and $F*S_j(y)$ follow from standard properties of Jones-Wenzl idempotents. $\qquad\square$

Corollary 2.8. *The following formulas hold*

$$C_0*y^0 = x^2$$
$$C_0*y = x^2S_1(y) + (t^{-2}+t^2-t^6-t^{-6})x^2 + (t^6-t^{-2}-t^2+t^{-6})$$
$$C_0*y^2 = x^2S_2(y) + (t^2+t^{-2}-t^{10}-t^{-10})x^2S_1(y) + (3-t^8-t^{-8})x^2$$
$$\quad +(-t^{-6}-t^6+t^{10}+t^{-10})S_1(y)$$
$$C_0*y^3 = x^2S_3(y) + (t^{-2}+t^2-t^{14}-t^{-14})x^2S_2(y) + (t^4+t^{-4}+4$$
$$\quad -t^{12}-t^{-12}-t^{-8}-t^8)x^2S_1(y) + (-2t^{-6}-2t^{10}+4t^{-2}+5t^2$$
$$\quad +2t^{14}-3t^6-t^{-10}+t^{-14})x^2 + (-t^{10}+t^{14}-t^{-10}+t^{-14})S_2(y)$$
$$\quad +(-3t^{-2}+3t^{-6}-3t^2+3t^6).$$

We define the skeins $G*y^k, H*y^k$, and $\overline{H*y^k}$ as in Figure 11. The next result has a proof analogous to that to Lemma 2.4.

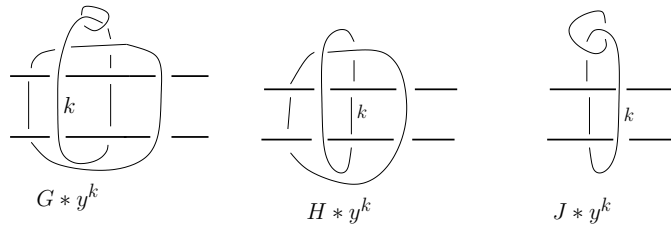$G*y^k \qquad\qquad H*y^k \qquad\qquad J*y^k$

Fig. 11.

Lemma 2.9. *The following formulas hold*

$$G*S_k(y) = (-t^{2k+2}-t^{-2k-2})H*S_k(y),$$
$$H*y^k = t^2yH*y^{k-1} + (1-t^{-4})J*y^{k-1}, \quad H*y^0 = y,$$
$$J*S_k(y) = (-t^{2k+2}-t^{-2k-2})S_k(y).$$

Corollary 2.10. *The following formulas hold*

$$G * y^0 = (-t^2 - t^{-2})S_1(y),$$
$$G * y = (-t^6 - t^{-2})S_2(y) + (-t^{-2} - t^{-10}),$$
$$G * y^2 = (-t^{10} - t^{-2})S_3(y) + (-t^2 - 2t^{-2} - t^{-14})S_1(y),$$
$$G * y^3 = (-t^{14} - t^{-2})S_4(y) + (-2t^6 - 3t^{-2} - t^{-18})S_2(y) + (-2t^{-2} - 2t^{-10}).$$

3. Formulas in the Kauffman bracket skein module of the 3-twist knot complement

For the complement of the 3-twist knot Lemma 6 in [10] gives

Lemma 3.1. *For all $k \geq 0$ we have*

$$X_4 * y^k = -t^{-4}X_3 * y^k - t^{-2}x^2y^k.$$

Lemma 3.1 yields different formulas for $X_4 * y^k$, which, combined with those in § 2, give relations that successively compute $S_4(y)$, $S_5(y)$, $S_6(y)$:

$$S_4(y) = [-t^{-2}S_2(x) - (t^{-2} + t^{-6})]S_3(y) + [-(2t^{-4} + t^{-8})S_2(x)$$
$$-(t^{-8} + t^{-4})]S_2(y) + [-2(t^{-10} + t^{-6})S_2(x) - (t^{-10} + 2t^{-6})]S_1(y)$$
$$+[-(t^{-12} + 2t^{-8})S_2(x) - (t^{-12} + 2t^{-8})];$$

$$S_5(y) = [(t^{-4}S_4(x) + (t^{-4} + t^{-8})S_2(x) + (t^{-4} + t^{-12})]S_3(y)$$
$$+[(2t^{-6} + t^{-10})S_4(x) + (3t^{-6} + 2t^{-10} + t^{-14})S_2(x) + (t^{-6} + t^{-10})]S_2(y)$$
$$+[(2t^{-12} + 2t^{-8})S_4(x) + (3t^{-12} + 4t^{-8} + 2t^{-16})S_2(x)$$
$$+(2t^{-12} + 2t^{-8} + t^{-16})]S_1(y) + (t^{-14} + 2t^{-10})S_4(x)$$
$$+(3t^{-14} + 4t^{-10} + t^{-18})S_2(x) + (2t^{-18} + 2t^{-14} + 2t^{-10});$$

$$S_6(y) = [-t^{-6}S_6(x) + (-t^{-6} - t^{-10})S_4(x) + (-t^{-6} - t^{-14})S_2(x)$$
$$+(-t^{-6} - t^{-18})]S_3(y) + [(-2t^{-8} - t^{-12})S_6(x)$$
$$+(-3t^{-8} - 2t^{-12} - t^{-16})S_4(x) + (-3t^{-8} - t^{-12} - t^{-16})S_2(x) +$$
$$+(-t^{-20} - t^{-16} - t^{-8})]S_2(y) + [(-2t^{-14} - 2t^{-10})S_6(x)$$
$$+(-4t^{-10} - 3t^{-14} - 2t^{-18})S_4(x) + (-2t^{-14} - 4t^{-10} - 2t^{-18})S_2(x)$$

$$+(-2t^{-10} - t^{-18})]S_1(y) + [(-2t^{-12} - t^{-16})S_6(x)$$
$$+(-4t^{-12} - 3t^{-16} - t^{-20})S_4(x) + (-4t^{-12} - 2t^{-16} - 3t^{-20})S_2(x)$$
$$+(-t^{-20} - 2t^{-12} - t^{-24})].$$

Also, from Lemma 3.1, we obtain

$$X_4 = t^4S_3(y) + (t^2S_2(x) + t^2)S_2(y) + (2S_2(x) + 1)S_1(y) + t^{-2}S_2(x) + t^{-2}.$$

Combining Lemma 2.3 and Lemma 3.1 we obtain the following recursive scheme that allows the writing of $X_j * S_k(y)$, $1 \leq j \leq 4, 1 \leq k \leq 6$ in terms of the basis $S_j(x)S_k(y)$, $0 \leq j, 0 \leq k \leq 3$:

$X_1 * S_{k+1}(y) = t^2 X_2 * S_k(y) + t^{-2} Y_1 * S_k(y) - X_1 * S_{k-1}(y) + (2S_2(x) + 2)S_k(y)$,

$X_2 * S_{k+1}(y) = t^2 X_3 * S_k(y) + t^{-2} X_1 * S_k(y) - X_2 * S_{k-1}(y) + (2S_2(x) + 2)S_k(y)$,

$X_3 * S_{k+1}(y) = t^2 X_4 * S_k(y) + t^{-2} X_2 * S_k(y) - X_3 * S_{k-1}(y) + (2S_2(x) + 2)S_k(y)$,

$X_4 * S_k(y) = -t^{-4} X_3 * S_k(y) - t^2 x^2 S_k(y)$.

Using also the formulas for X_1, X_2, X_3, X_4 and those that express $S_4(y)$, $S_5(y)$, and $S_6(y)$ in terms of the basis we obtain

$$X_3 * S_1(y) = t^6 S_3(y) + t^4 S_2(x)S_2(y) + t^2 S_2(x)S_1(y) + S_2(x) + 2;$$
$$X_4 * S_1(y) = -t^2 S_3(y) - S_2(x)S_2(y) + (-2t^{-2}S_2(x) - t^{-2})S_1(y)$$
$$-t^{-4}S_2(x) - 2t^{-4}.$$

$$X_2 * S_2(y) = t^8 S_3(y) + t^6 S_2(x)S_2(y) + [(2 + t^4)S_2(x) + 2]S_1(y)$$
$$+(t^2 + 2t^{-2})S_2(x) + (2t^2 + t^{-2} - t^{-6});$$
$$X_3 * S_2(y) = -t^4 S_3(y) - t^2 S_2(x)S_2(y) - t^{-2};$$
$$X_4 * S_2(y) = S_3(y) - t^{-2}S_2(y) + t^{-6}.$$

$$X_2 * S_3(y) = -t^6 S_3(y) + (-t^4 + 1)S_2(x)S_2(y) + 2S_2(y) + [2t^{-2}S_2(x)$$
$$+(-2t^{-2} - t^{-10})]S_1(y) + (2t^{-4} - t^{-8})S_2(x) + (-1 + 2t^{-4} - t^8);$$
$$X_3 * S_3(y) = t^2 S_3(y) + 2S_2(x)S_2(y) + S_2(y) + [2t^{-2}S_2(x) + 2t^{-2}]S_1(y);$$
$$X_4 * S_3(y) = [-t^{-2}S_2(x) - 2t^{-2}]S_3(y) + [-2t^{-4}S_2(x) - t^{-4}]S_2(y)$$
$$+[-2t^{-6}S_2(x) - 2t^{-6}]S_1(y);$$

$$X_2 * S_4(y) = [2S_2(x) + (t^4 + 2)]S_3(y) + [(2t^2 + 2t^{-2})S_2(x)$$
$$+(t^2 - t^{-14})]S_2(y) + [(2 + 2t^{-4} - t^{-12})S_2(x)$$
$$+(2 + 2t^{-4} - t^{-12})]S_1(y) + (2 + 2t^{-6} - 2t^{-10})S_2(x)$$
$$+(2t^{-2} + t^{-6} - t^{-10});$$
$$X_3 * S_4(y) = S_2(x)S_3(y) + t^{-2}S_2(y) - t^{-12}S_1(y) - t^{-10}S_2(x);$$
$$X_4 * S_4(y) = [t^{-4}S_4(x) + (2t^{-4} + t^{-8})S_2(x) + (2t^{-4} + t^{-8})]S_3(y)$$
$$+[(2t^{-6} + 2t^{-10})S_4(x) + (5t^{-6} + 6t^{-10})S_2(x) + (2t^{-6} + 4t^{-10})]S_2(y)$$
$$+[(2t^{-8} + 2t^{-12})S_4(x) + (10t^{-8} + 5t^{-12})S_2(x) + (3t^{-12} + t^{-16})]S_1(y)$$
$$+(2t^{-10} + t^{-14})S_4(x) + (6t^{-10} + 3t^{-14})S_2(x) + (4t^{-10} + 2t^{-14}).$$

4. The action of the skeins $(1, -3)_T$ and $(1, -2)_T$ on the skein module of the 3-twist knot complement

As said in the introduction, we compute the action of $(1, -3)_T$, $(1, -2)_T$ from $K_t(\mathbb{T}^2 \times I)$ on the basis elements $S_k(y)$, $0 \leq k \leq 3$ of $K_t(S^3 \backslash N(K))$. The skeins

$(1,-3)_T$ and $(1,-2)_T$ are depicted in Figure 12, with the cylinder $\mathbb{T}^2 \times [0,1]$ embedded as a regular neighborhood of the boundary of the knot complement. Before starting the computation we prove a lemma.
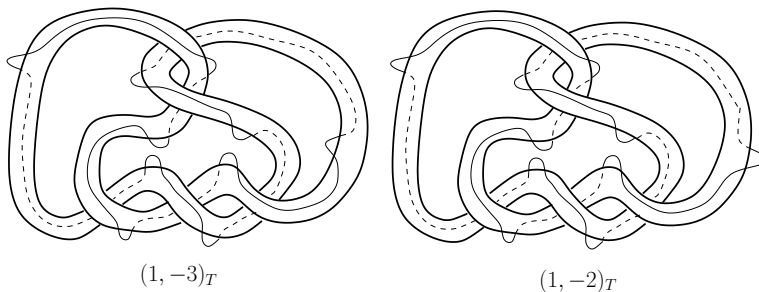

$(1,-3)_T$ $(1,-2)_T$

Fig. 12.

Lemma 4.1. *The identities from Figure 13 hold. Here the curved strand can encircle several parallel straight strands.*

Fig. 13.

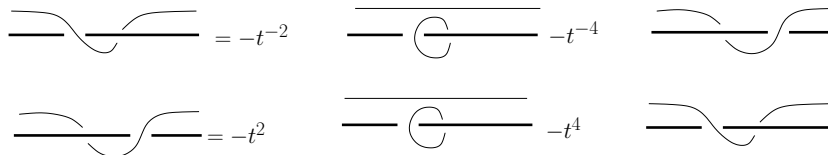**Proof.** Pull the strand in the term on the left until you create a negative twist in the first identity and a positive twist in the second identity, then resolve the crossing. $\qquad \square$

First we find $(1,-3)_T \cdot y^k$, $k = 0, 1, 2, 3$. For that we add y^k to the skein represented by the curve on the left side of Figure 12, then push this curve inside the knot complement. There is a small technical detail. The framing that the curve inherits from the torus does not coincide with the framing defined by the plane of the paper. The resulting skein (with framing defined by the plane of the paper) is the skein from Figure 14 multiplied by t^6. We compute this skein from the figure first, then multiply by the adjusting factor in the end. In Figure 14 we have labeled the 5 crossings in the order in which they are resolved. We use a boldface curve for y^k, and here and in subsequent figures no longer write the label y^k next to it.
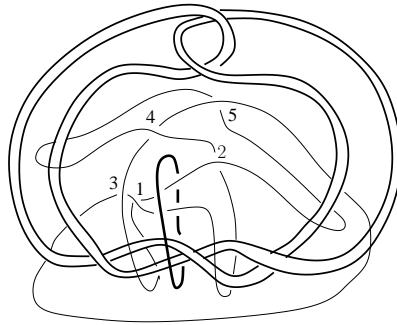
Fig. 14.

We denote by a string of length k consisting of $+$'s and $-$'s inside double brackets the skein obtained from $(1, -3)_T \cdot y^k$ by smoothening the first k crossings (in the order of labels), horizontally for a plus and vertically for a minus. For example the Kauffman bracket skein relation applied to the first crossing reads $(1, -3)_T \cdot y^k = t((+)) + t^{-1}((-))$. Applying the Kauffman bracket skein relation repeatedly we obtain

$$(1, -3)_T \cdot y^k = t^4((++++)) + t^3((+++-+)) + t((+++--)) $$
$$+ t((++-)) + ((+-)) + t^{-1}((-))$$

where the skeins from this expression are shown in Figure 15.

$((++++))$ $((+++-+))$ $((+++--))$

$((++-))$ $((+-))$ $((-))$

Fig. 15.

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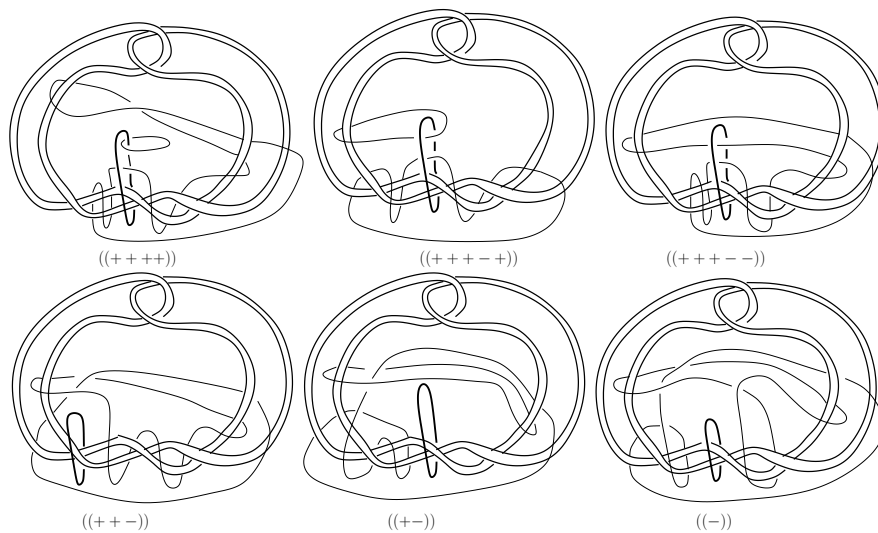
After removing two negative twists in $((-))$ and focusing on the lower part of the twist knot only, we obtain that $((-))$ is equal to the first skein in Figure 16 multiplied by t^{-6}. This new skein is computed as shown in the figure by using the skein relation and sliding the strands. In this sum the first term is $t^2 xy^{k+1}$, the third is xy^k, and the fourth is $t^{-2}xX_3 * y^k$.
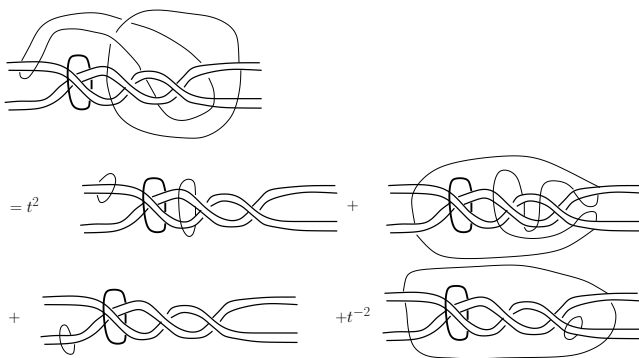

Fig. 16.

Let us focus on the second term in the sum. Applying Lemma 4.1 in the place specified by the arrow, we can transform it as in Figure 17. The first term is just $-t^2 xy^{k+1}$. By applying Lemma 4.1 at the two places specified by arrows we can transform the first term into $-t^4 x^3 y^k - t^4 xy^k - t^6 xX_4 * y^k - t^2 xX_3 * y^k$. So the term we are computing equals $-t^2 xy^{k+1} - t^4 x^3 y^k - t^4 y^k - t^6 xX_4 * y^k - t^2 xX_3 * y^k$.
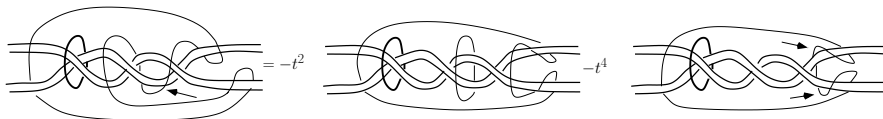

Fig. 17.

Therefore $((-)) = -xX_4 * y^k + (t^{-8} - t^{-4})xX_3 * y^k - t^{-2}x^3 y^k + (t^{-6} - t^{-2})xy^k$, which after applying Lemma 3.1 becomes

$$((-)) = t^{-8}xX_3 * y^k + (t^{-6} - t^{-2})xy^k.$$

To compute $((+ + -))$ we look again at the lower part of the twist knot and apply Lemma 4.1 in the places specified by arrows in Figure 18. In the last sum, by resolving the two crossings in each diagram by the skein relation we find that the

first term is

$$t^{-4}xX_3 * y^{k+1} + t^{-2}x^3y^{k+1} + t^{-4}xX_3 * y^{k+1} + (-t^{-4} - t^{-8})xX_3 * y^{k+1}$$
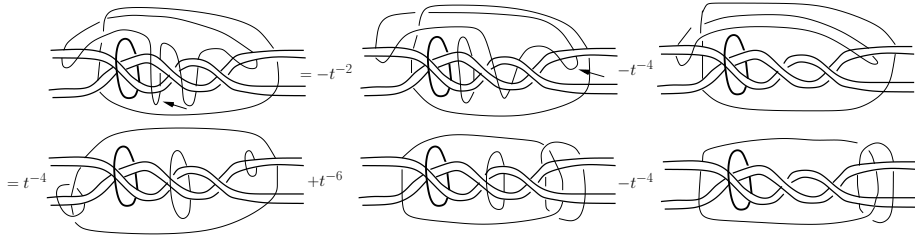$$= (t^{-4} - t^{-8})xX_3 * y^{k+1} + t^{-2}x^3y^{k+1}.$$

Fig. 18.

Resolving the crossings in the second term we obtain that it is equal to

$$t^{-4}xy^{k+1}X_1 + t^{-6}A + t^{-6}B + t^{-8}xX_3 * y^{k+1},$$

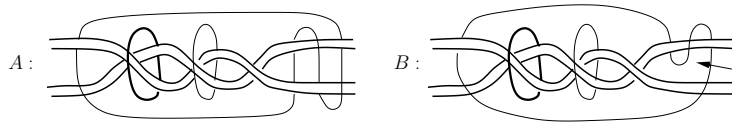where skeins A and B are as in Figure 19. Compute B by applying Lemma 4.1

Fig. 19.

at the location specified by arrow to obtain $B = -t^2xX_3 * y^{k+1} - t^4xy^{k+1}$. Then transform A by an isotopy, use Lemma 4.1 as shown in Figure 20 to obtain $A = t^{-2}xX_2 * y^{k+1} + t^2xy^{k+2} + x^3y^{k+1} + xy^{k+1}$. Substitute X_1 by $-t^4y - t^2x^2$ to conclude that the second term in the three-term sum from the second line in Figure 18 equals

$$(t^{-8} - t^{-4})xX_3 * y^{k+1} + t^{-8}xX_2 * y^{k+1} + (t^{-4} - 1)xy^{k+2}$$
$$+(t^{-6} - t^{-2})x^3y^{k+1} + (t^{-6} - t^{-2})xy^{k+1}.$$

Similarly, the third term is $(1 - t^{-4})xy^k - t^{-6}xX_3 * y^k$. Hence

$$((+ + -)) = -t^{-6}xX_3 * y^k + t^{-8}xX_2 * y^{k+1} + (t^{-4} - 1)xy^{k+2}$$
$$+t^{-6}x^3y^{k+1} + (t^{-6} - t^{-2})xy^{k+1} + (1 - t^{-4})xy^k.$$

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After applying Lemma 2.3 this becomes

$$((++-)) = t^{-10}xX_1 * y^k + (t^{-4} - 1)xy^{k+2} + t^{-6}x^3y^{k+1}$$
$$+(t^{-6} - t^{-2})xy^{k+1} + 2t^{-8}x^3y^k + (1 - t^{-4})xy^k.$$

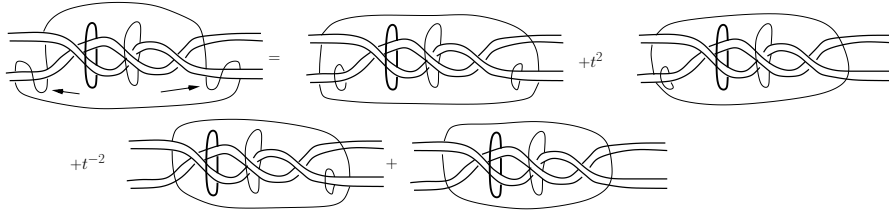We turn to $((+-))$ and by working in the lower part of the knot, we apply

Fig. 20.

Lemma 4.1 as specified by an arrow in Figure 21, then remove a negative twist in each resulting diagram, to obtain the two-term sum in Figure 21. A close examination shows that the last diagram in the figure is the same as the last diagram in Figure 18. Adjusting for the different coefficient, we deduce that the second term from the sum in Figure 21 is $(-t^{-3} + t^{-7})xy^k + t^{-9}xX_3 * y^k$. Computing similarly by resolving both crossings with the Kauffman bracket skein relation, we find that the first term is $t^{-7}xX_3 * y^{k+1} + (t^{-5} - t^{-1})xy^{k+1}$. Combining, we get

$$((+-)) = t^{-7}xX_3 * y^{k+1} + t^{-9}xX_3 * y^k + (t^{-5} - t^{-1})xy^{k+1} + (t^{-7} - t^{-3})xy^k.$$
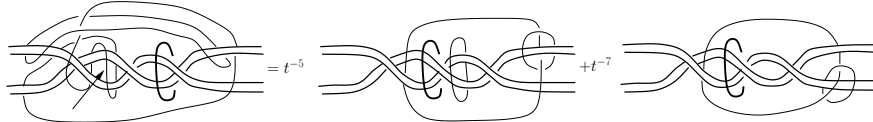

Fig. 21.

Now turn to the computation of $((+++--))$. After an isotopy at the top part of the twist knot we obtain the first diagram in Figure 22, where in this diagram we ignore the top part of the twist knot, as we will not use it again. We apply Lemma 4.1 as specified by the arrow to obtain the sum on the right. The first term is computed by applying Lemma 4.1 at the point specified by the arrow, as in Figure 23. In the sum from Figure 23, the second term is $t^{-6}xY_1 * y^k$. For the first term, we perform an isotopy to make it look like in Figure 24, then apply
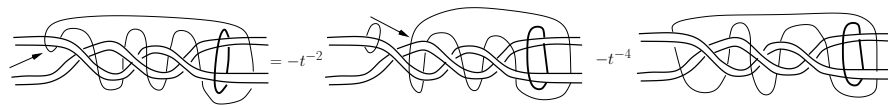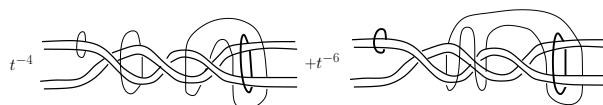
Fig. 22.

Fig. 23.

Lemma 4.1 to obtain that it is equal to $-t^{-6}x^3y^{k+1} - t^{-8}xyX_1 * y^k$. Thus the first term of the sum in Figure 22 equals $-t^{-6}x^3y^{k+1} - t^{-8}xyX_1 * y^k + t^{-6}xY_1 * y^k$.

Fig. 24.

The second term in Figure 22 can be transformed by an isotopy into the first skein in Figure 25. Then apply Lemma 4.1 as specified by the arrow to obtain the sum on the right. The second term is $t^{-8}xy^k$. The first term can be computed by applying the Lemma 4.1 as specified, and is $-t^{-8}x^3y^k - t^{-10}X_1 * y^k$.
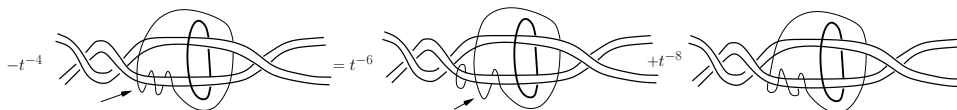

Fig. 25.

Combining the results we obtain

$$((+++--)) = -t^{-6}x^3y^{k+1} - t^{-8}xyX_1 * y^k + t^{-6}xY_1 * y^k - t^{-8}x^3y^k$$
$$-t^{-10}xX_1 * y^k + t^{-8}xy^k.$$

Using Lemma 2.2 we write this as

$$((+++--)) = -t^{-10}xX_2 * y^k - t^{-10}xX_1 * y^k - t^{-6}x^3y^{k+1} - 3t^{-8}x^3y^k$$
$$+t^{-8}xy^k.$$

To compute the term $((+ + + - +))$ we slide the skein to an area where the two strands of the twist knot are parallel, as in the first diagram from Figure 26, then apply Lemma 4.1 at the point specified by the arrow, to obtain the first sum in this figure.
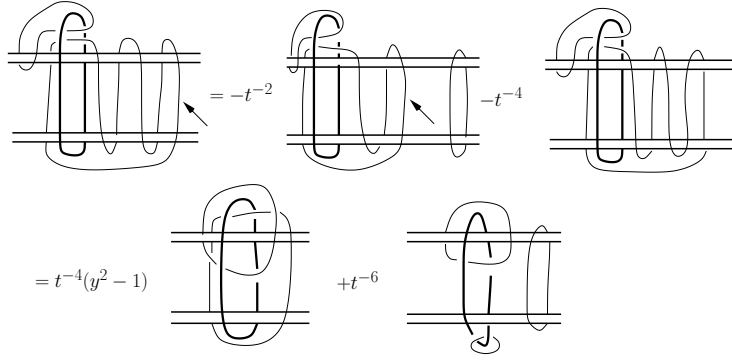

Fig. 26.

Next, apply Lemma 4.1 at the point specified by the second arrow in Figure 26 to obtain the sum on the second row. Resolve the crossings in the first diagram to obtain that the first term is $t^{-4}(y^2 - 1)(t^2 xy^k + A * y^k + xy^{k+1} + t^{-2} xy^k)$. The second term is $y\overline{A * y^k}$. Hence

$$((+ + + - +)) = (t^{-4}y^2 - t^{-4})A * y^k + t^{-6}y\overline{A * y^k} + t^{-4}xy^{k+3}$$
$$+ (t^{-2} + t^{-6})xy^{k+2} - t^{-4}xy^{k+1} + (-t^{-2} - t^{-6})xy^k.$$

Finally, for $((+ + ++))$, remove the twist and multiply the skein by $-t^3$, slide the skein over the top of the diagram to get the first skein from Figure 27 (again only the bottom of the diagram of the twist knot is shown, and the skein has been moved to the left off the area where the crossings occur).

Apply Lemma 4.1 as specified by first arrow to obtain the first equality, then apply the lemma again as specified by second arrow to obtain (after arranging the terms) the last sum in Figure 27. The second term is $-t^{-3}yA * y^k$. After applying Lemma 4.1 as specified by the arrow, the first term is equal to $t^{-1}(-y^2+1)(-t^{-2}yA* y^k - t^{-4}\overline{A * y^k})$. So

$$((+ + ++)) = t^{-3}(y^3 - 2y)A * y^k - t^{-5}(-y^2 + 1)\overline{A * y^k}.$$

To simplify the formulas we set $u_i = S_{2i+1}(x)$ and $q = t^4$.

Theorem 4.2. *The action of* $(1, -3)_T$ *on* $K_t(\mathbb{T}^2 \times I)$ *is given by*

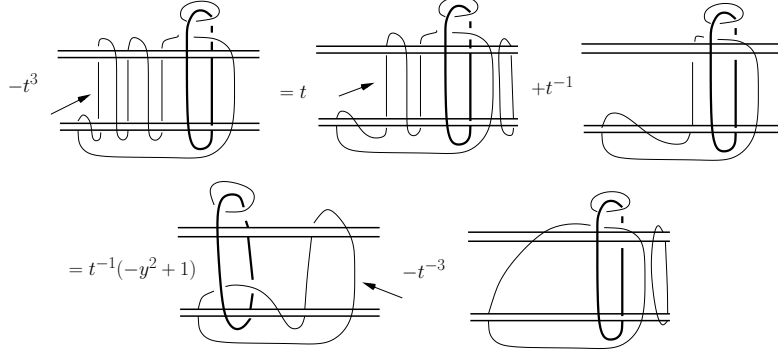$$(1, -3)_T \cdot S_k(y) = \sum_{0 \le j \le 3} t^{2k+2j-1}\alpha_{kj}S_j(y)$$

Fig. 27.

where

$$\alpha_{0,3} = (-q-1)u_0,\ \alpha_{0,2} = -u_1 - 2(q+1)u_0,\ \alpha_{0,1} = -2u_1 + (q-1)u_0$$

$$\alpha_{0,0} = -2u_1 + (-6q-1)u_0,\ \alpha_{1,3} = qu_1 + (1+q^{-1})u_0,$$

$$\alpha_{1,2} = (2q+1+q^{-1})u_1 + (q+q^{-1}-4q^{-2})u_0,\ \alpha_{1,1} = (q+2)u_1 + (q+3)u_0,$$

$$\alpha_{1,0} = (2q+1+q^{-1})u_1 + (2q+4+2q^{-1})u_0,\ \alpha_{2,3} = -qu_2 + (q-1)u_1$$

$$+(-q-2q^{-2})u_0,\ \alpha_{2,2} = (-2q-1)u_2 + (-3q-1-q^{-1}-2q^{-2})u_1$$

$$+(-3q-2-q^{-2}-q^{-3}+q^{-4})u_0,\ \alpha_{2,1} = (-2q-2)u_2 + (2q-3-2q^{-1})u_1$$

$$+(q^3-q^2-6q-1-3q^{-1}-3q^{-2}-q^{-3})u_0,\quad \alpha_{2,0} = (-2q-1)u_2$$

$$+(-4q-5+q^{-1})u_1 + (q^3-2q^2-3q+2-5q^{-1}-2q^{-2})u_0,$$

$$\alpha_{3,3} = qu_3 + u_2 + (-1+q^{-1})u_2 + (2q^{-3}-3q^{-5})u_0,\quad \alpha_{3,2} = (2q+1)u_3$$

$$+(q+1+q^{-1})u_2 - u_1 + (q^{-1}+q^{-2}+q^{-4}-2q^{-5})u_0,\quad \alpha_{3,1} = (2q+2)u_3$$

$$+(2q+1+2q^{-1})u_2 + (-1+2q^{-1}+2q^{-3})u_1 + (-q^{-2}+2q^{-3}-q^{-4})u_0,$$

$$\alpha_{3,0} = (2q+1)u_3 + (4q+2+q^{-1})u_2 + (6q-1+2q^{-1})u_1$$

$$+(4q-1-3q^{-1}+q^{-2}-2q^{-3}+q^{-4})u_0.$$

Proof. Combining the terms computed above, applying Lemmas 2.3 and 3.1, and multiplying with the frame adjusting factor t^6, we obtain:

$$(1,-3)_T \cdot y^k = t^{-3}xX_3 * y^k + (t^7S_3(y) + t^5S_2(y))A * y^k$$
$$+(t^3y + t^5S_2(y))\overline{A * y^k} + [t^5S_3(y) + 2t^3S_2(y) + (-t^5+2t)S_1(y)$$
$$+(-2t^3 + 2t^{-1})]xy^k.$$

Substituting the formulas from § 2 and § 3, switching to the basis $S_j(x)S_k(y)$, we obtain the formulas from the statement. □

Let us compute $(1,-2)_T \cdot y^k$, $k = 0,1,2,3$. Again we push the $(1,-2)_T$ skein inside the knot complement and adjust the framing from the plane of the torus

to the plane of the paper, to get the skein from Figure 28 multiplied by $-t^9$. We compute first the skein from the figure, then adjust framing. In the figure we label the 5 crossings in the order they are resolved, and use a boldface curve for y^k, as before.
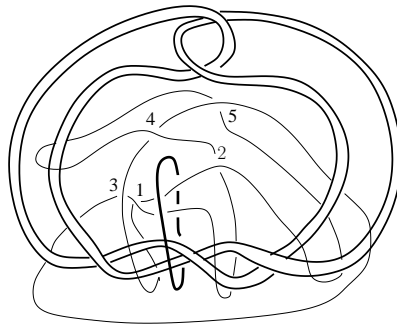

Fig. 28.

As this skein looks similar to $(1, -3)_T \cdot y^k$, we expand it in the same way:

$$(1, -2)_T \cdot y^k = t^4((+++)) + t^3((+++-+)) + t((+++--))$$
$$+ t((++-)) + ((+-)) + t^{-1}((-)),$$

but now the diagrams of the 6 skeins are different (see Figure 29).

$((++++))$ $((+++-+))$ $((+++--))$

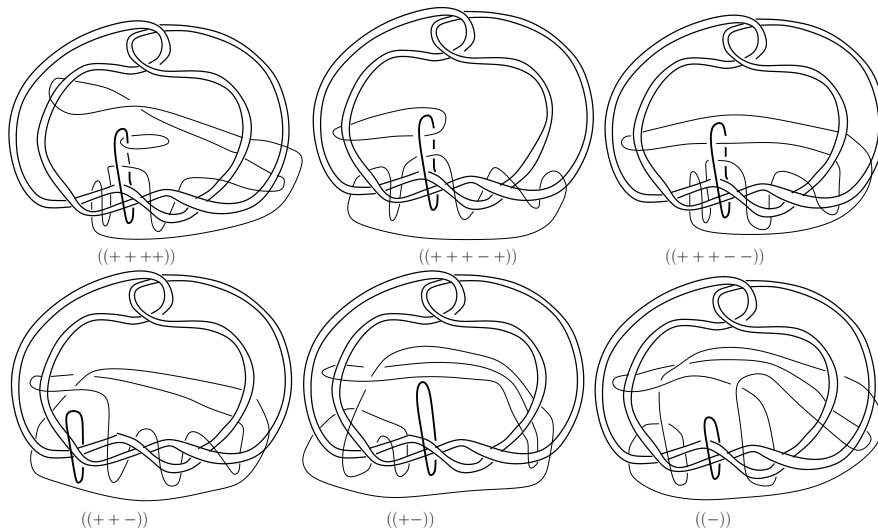$((++-))$ $((+-))$ $((-))$

Fig. 29.

To compute $((-))$ we remove the two negative twists (and multiply the skein by t^{-6}), then resolve the two crossings using the Kauffman bracket skein relation. We obtain

$$((-)) = t^{-6}[t^2 x^2 y^k + X_3 * y^k + X_3 * y^k + t^{-2}(-t^2 - t^{-2})X_3 * y^k]$$
$$= t^{-4}x^2 y^k + (t^{-6} - t^{-10})X_3 * y^k.$$

Next, we focus on $((+-))$. After removing a twist and performing an isotopy, we obtain the first skein from Figure 30. Now use Lemma 4.1 to obtain the sum in the figure.
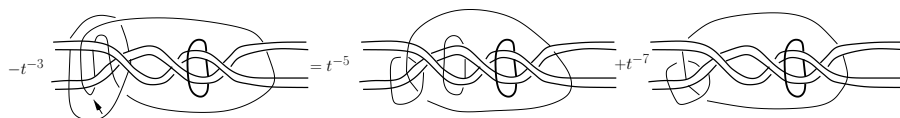

Fig. 30.

Resolving the crossings with the skein relation, we find the first term:

$$t^{-5}[t^2(-t^2 - t^{-2})X_4 * y^{k+1} + X_4 * y^{k+1} + X_4 * y^{k+1} + t^{-2}x^2 y^{k+1}]$$
$$= (t^{-5} - t^{-1})X_4 * y^{k+1} + t^{-7}xy^{k+1},$$

and the second term:

$$t^{-7}(t^2 x^2 y^k + X_3 * y^k + X_3 * y^k + t^{-2}(-t^2 - t^{-2})X_3 * y^k)$$
$$= (t^{-7} - t^{-11})X_3 * y^k + t^{-5}x^2 y^k.$$

Combining we obtain

$$((+-)) = (t^{-5} - t^{-1})X_4 * y^{k+1} + (t^{-7} - t^{-11})X_3 * y^k + t^{-7}xy^{k+1} + t^{-5}x^2 y^k.$$

Next we compute $((+ + -))$, which after an isotopy becomes the first skein in Figure 31.
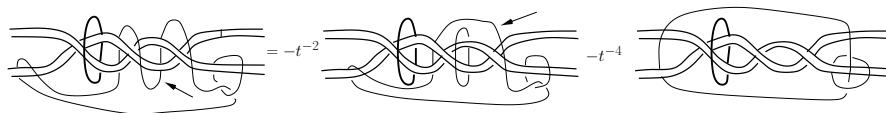

Fig. 31.

Apply Lemma 4.1 to transform this into the sum on the right side of Figure 31. Apply Lemma 4.1 in the first term on the right, then resolve the crossings to obtain

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that this term equals

$$-t^{-2}[(-t^{-2})(t^2x^2y^{k+2} + t^{-2}(-t^2-t^{-2})X_3*y^{k+2} + X_3*y^{k+2} + X_3*y^{k+2}))$$
$$-t^{-4}(t^2(-t^2-t^{-2})X_4*y^{k+1} + t^{-2}x^2y^{k+1} + X_4*y^{k+1} + X_4*y^{k+1})]$$
$$= (t^{-4}-t^{-8})X_3*y^{k+2} + (-t^{-2}+t^{-6})X_4*y^{k+1} + t^{-2}x^2y^{k+2} + t^{-8}x^2y^{k+1}.$$

Then resolve the crossings in the second term to obtain that it is equal to

$$-t^{-4}[t^2x^2y^k + t^{-2}(-t^2-t^{-2})X_3*y^k + 2X_3*y^k]$$
$$= (-t^{-4}+t^{-8})X_3*y^k - t^{-2}x^2y^k.$$

Combining, we obtain that

$$((++-)) = (t^{-4}-t^{-8})X_3*y^{k+2} + (-t^{-4}+t^{-8})X_3*y^k$$
$$+(t^{-6}-t^{-2})X_4*y^{k+1} + t^{-2}x^2y^{k+2} + t^{-8}x^2y^{k+1} - t^{-2}x^2y^k.$$

Which after applying Lemma 2.3 for the first and third terms becomes

$$((++-)) = (-t^{-4}+t^{-8})X_3*y^k + (t^{-6}-t^{-10})X_2*y^{k+1}$$
$$+t^{-2}x^2y^{k+2} + (2t^{-4}-t^{-8})x^2y^{k+1} - t^{-2}x^2y^k.$$

Compute $((+++--))$ by performing an isotopy over the top of the knot to obtain the first skein in Figure 32 (again, ignore the top of the knot), then apply Lemma 4.1 as specified by arrow to obtain the next sum. Continue applying Lemma 4.1 to each term as specified, to obtain the sum on the second row in Figure 32. Applying Lemma 4.1 three more times yields

$$((+++--)) = -t^{-6}x^2y^{k+2} - t^{-8}x^2y^{k+1} + t^{-6}x^2y^k - t^{-8}x^2y^{k+1}$$
$$-t^{-10}x^2y^k - t^{-10}x^2y^k - t^{-12}Y_1*y^k$$
$$= -t^{-12}Y_1*y^k - t^{-6}x^2y^{k+2} - 2t^{-8}x^2y^{k+1} + (t^{-6}-2t^{-10})x^2y^k.$$
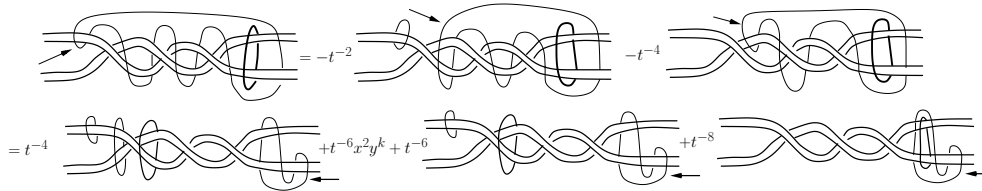

Fig. 32.

To compute $((+++-+))$, apply Lemma 4.1 as in Figure 33. The first diagram on the right is x times the second diagram in Figure 15 (that denoted by $((+++-+))$

in that figure). So the first term on the right is

$$(-t^{-6}y^2 + t^{-6})xA * y^k - t^{-8}xy\overline{A * y^k} - t^{-4}x^2y^{k+2} - t^{-8}x^2y^{k+2}$$
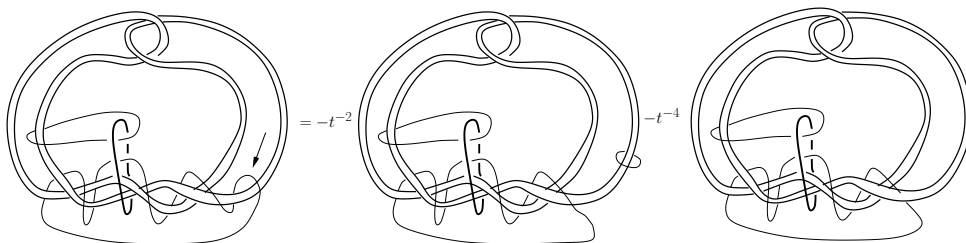$$-t^{-6}x^2y^{k+3} + t^{-4}x^2y^k + t^{-8}x^2y^k + t^{-6}x^2y^{k+1}.$$

Fig. 33.

The second term on the right can be transformed by an isotopy into the first skein in Figure 34. Apply Lemma 4.1 (see arrow) to transform it into the sum on the right. Now apply Lemma 4.1 in each term as specified by the arrows to obtain the sum on the second row.
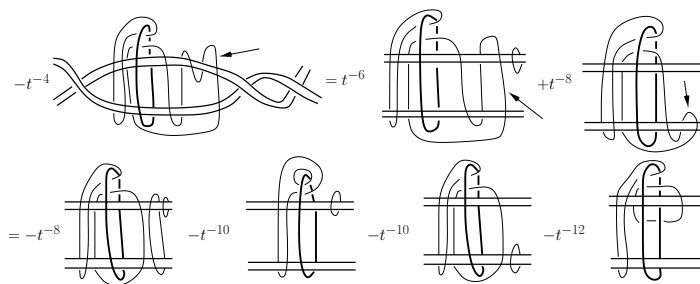

Fig. 34.

The first and third terms can be combined into $(-t^{-8}xy - t^{-10}x)Z * y^k$, where $Z * y^k$ is the skein in Figure 35. We resolve the crossings marked by arrows using the skein relation to obtain that $Z * y^k = t^2xy^k + A * y^k + xy^{k+1} + t^{-2}xy^k$. The second term from the last sum in Figure 34 can be slid back to the left over the crossing of the twist knot. Then the second term is just $-t^{-10}x\overline{A * y^k}$, while the last is $-t^{-12}$ times the 180° rotation of $C_0 * y^k$ in the plane of the paper, so it is in

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fact equal to $-t^{-12}C_0 * y^k$. Thus

$$((+++-+)) = (-t^{-6}y^2 - t^{-8}y - t^{-10} + t^{-6})xA * y^k$$
$$+(-t^{-8}y - t^{-10})x\overline{A * y^k} - t^{-12}C_0 * y^k - t^{-6}x^2y^{k+3}$$
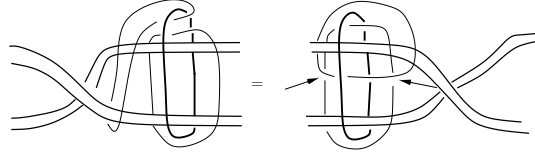$$+(-t^{-4} - 2t^{-8})x^2y^{k+2} - 2t^{-10}x^2y^{k+1} + (t^{-4} - t^{-12})x^2y^k.$$

Fig. 35.

For $((+++))$, remove the positive twist, multiply by $-t^3$, then slide through the top of the knot diagram to obtain the skein in Figure 36. Apply Lemma 4.1 as shown by arrow to obtain the first sum, then apply the lemma again in each term, as shown by arrows, to obtain the second sum. Apply again Lemma 4.1 in each
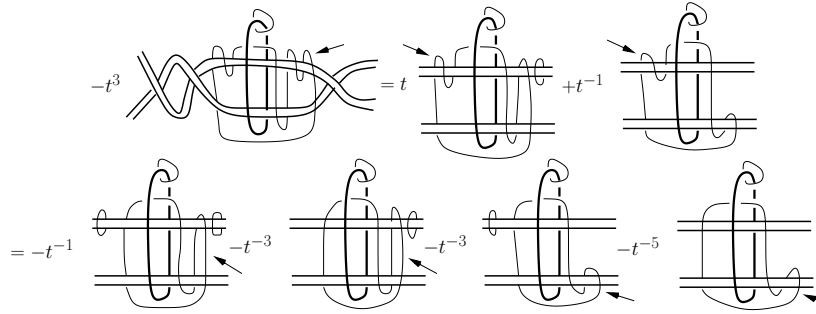

Fig. 36.

term to obtain

$$((+++)) = t^{-3}x^2yG * y^k + t^{-5}x^2F * y^k + t^{-5}xyA * y^k$$
$$+t^{-7}x\overline{A * y^k} + t^{-5}x^2G * y^k + t^{-7}xA * y^k + t^{-7}xA * y^k + t^{-9}F * y^k$$
$$= (t^{-3}y + t^{-5})x^2G * y^k + (t^{-5}x^2 + t^{-9})F * y^k + (t^{-5}y + 2t^{-7})xA * y^k$$
$$+t^{-7}x\overline{A * y^k}.$$

We set $v_i = S_{2i}(x)$ and $q = t^4$.

Theorem 4.3. *The action of* $(1, -2)_T$ *on* $K_t(\mathbb{T}^2 \times I)$ *is given by*

$$(1, -2) \cdot S_k(y) = \sum_{j=0}^{3} t^{-2k-2j} \beta_{kj} S_j(y),$$

where

$\beta_{0,3} = v_1 + qv_0$, $\beta_{0,2} = (q^2 + q + 1)v_1 + (q^2 + 2q)v_0$, $\beta_{0,1} = (2q^2 + 2q)v_1$
$+ (2q^2 + 2q)v_0$, $\beta_{0,0} = (2q^2 + 2q)v_1 + (q^2 - 1 - q^{-1})v_0$,

$\beta_{1,3} = -q^{-1}v_2 + (q^2 - q + 1 - q^{-1} - q^{-2})v_1 + (q^2 - 1 + q^{-1} - q^{-2})v_0$,

$\beta_{1,2} = (-2q^{-1} - q^{-2})v_2 + (2q^2 - q - q^{-1} - 3q^{-2})v_1 + (2q^2 + 2q^{-1})v_0$,

$\beta_{1,1} = (-2q^{-1} - 2q^{-2})v_2 + (3q^2 - 3q - 5 - q^{-1} - q^{-2})v_1$
$+ (q^2 - 3q - 7 + 3q^{-1} - 6q^{-2})v_0$, $\beta_{1,0} = (-2q^{-1} - q^{-2})v_2$
$+ (-q^2 - 4q + 1 - 3q^{-1} - 4q^{-2})v_1 + (-q^2 - 5q + 2 - 4q^{-1} - 4q^{-2})v_0$,

$\beta_{2,3} = q^{-2}v_3 + (-q^2 + q - q^{-1} + q^{-2} + q^{-3})v_2 + (-q^2 + 1 - q^{-1} + q^{-4})v_1$
$+ (-1 + 3q^{-1} - q^{-2} - q^{-3})v_0$, $\beta_{2,2} = (2q^{-1} + q^{-2})v_3 + (-2q^3 + q^2 + q$
$- 2 - 4q^{-1} + 2q^{-2} + q^{-3})v_2 + (-2q^3 + q - 4q^{-1} + 2q^{-3})v_1 + (-q^3 - q^2 - 2$
$+ 3q^{-1} - q^{-2} - q^{-4})v_0$, $\beta_{2,1} = (2q^{-2} + 2q^{-3})v_3 + (-2q^2 + 2 - 2q^{-1} + 2q^{-2}$
$+ 3q^{-3} + 2q^{-4})v_2 + (-q^2 + 1 + q^{-1} - q^{-2} - q^{-3} + 4q^{-4})v_1 + (-q^2 - q - 1$
$+ q^{-1} - q^{-2} - q^{-3} + 2q^{-4})v_0$, $\beta_{2,0} = (2q^{-2} + q^{-3})v_3 + (-2q^2 + q + 1 - 2q^{-1}$
$+ 3q^{-2} + 3q^{-3} + q^{-4})v_2 + (-3q^2 - 3q + 6 - q^{-1} - 5q^{-2} + 3q^{-3} + 4q^{-4})v_1$
$+ (-4q^2 - 3q + 1 + 3q^{-1} - 9q^{-2} - 3q^{-3})v_0$, $\beta_{3,3} = -q^{-3}v_4 + (q^2 - q$
$+ q^{-2} - q^{-3} - q^{-4})v_3 + (q^2 - q + 2q^{-2} - 4q^{-3} + 2q^{-5} + q^{-6} - q^{-7})v_2 + (q^3 + q^2$
$- 3q + 3 - q^{-1} + 4q^{-2} - 11q^{-3} + 5q^{-4} + 7q^{-5} - q^{-6} - 2q^{-7} - q^{-8})v_1$
$(-q^3 + 3q^2 + 1 + 5q^{-1} - q^{-2} - 6q^{-3} + 2q^{-4} + 6q^{-5} - 2q^{-6} - 2q^{-7} - q^{-8})v_0$,

$\beta_{3,2} = (-2q^{-3} - q^{-4})v_4 + (2q^2 - q - 1 + 2q^{-2} - 2q^{-3} - 2q^{-4} - q^{-5})v_3$
$+ (-q^3 + 10q^2 - 2q - q^{-1} - 5q^{-2} - 4q^{-3} + 2q^{-6} - 7q^{-7} - q^{-8})v_2 + (-3q^3$
$+ 24q^2 - 8q + 4 - 3q^{-1} + 9q^{-2} - 11q^{-3} + q^{-4} - q^{-5} + 8q^6 - 20q^{-7} - q^{-8})v_1$
$+ (-2q^3 + 16q^2 - 6q + 3 - 4q^{-1} + 6q^{-2} - 12q^{-3} - 2q^{-5} + 6q^{-6} - 13q^{-7})v_0$,

$\beta_{3,1} = (-2q^{-3} - 2q^{-4})v_4 + (2q^2 - 2 + 2q^{-2} - 2q^{-3} - 3q^{-4} - 2q^{-5})v_3$
$+ (-2q^3 + 3q^2 - 1 - 2q^{-1} + 5q^{-2} - 2q^{-4} + 2q^{-6} - 2q^{-8})v_2 + (-5q^3 - 2q$
$+ 6 + 8q^{-1} + 6q^{-2} - 9q^{-3} - 3q^{-4} + 6q^{-5} - q^{-7} - 5q^{-8})v_1 + (-q^2 + 5$
$+ 12q^{-1} - 5q^{-2} + 3q^{-3} - 2q^{-4} + 4q^{-5})v_0$, $\beta_{3,0} = (-2q^{-3} - q^{-4})v_4 + (2q^2 - q$
$- 1 - 2q^{-1} + 8q^{-2} + 10q^{-3} - q^{-4} - 2q^{-5} + 2q^{-6} - 3q^{-7} - q^{-8})v_2 + (-3q^3$
$+ 5q^2 + 3q + 6 + 9q^{-2} + 24q^{-3} + t^{-4} - 5t^{-5} + 5t^{-6})v_1 + (-3q^3 + 3q^2 + 3q + 6$
$+ 9q^{-2} + 19q^3 - 4q^{-5} + 3q^{-6} + 3q^{-7})v_0$.

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Proof. Adding the terms and adjusting framing by $-t^9$ we obtain

$$(1, -2)_T \cdot y^k = (t^8 - t^4)X_4 * y^{k+1} + (t^6 + 2t^{-2} - 3t^2)X_3 * y^k$$
$$+(-t^4 + 1)X_2 * y^{k+1} + t^{-2}Y_1 * y^k + [t^6 y^2 + (-t^8 + t^4)y$$
$$+(-3t^6 + t^2)]xA * y^k + (t^4 y + t^2 - t^6)x\overline{A * y^k} + C_0 * y^k$$
$$+(-t^8 x^2 - t^4)F * y^k + (-t^{10}y - t^8)x^2 G * y^k + t^6 x^2 y^{k+3}$$
$$+3t^4 x^2 y^{k+2} + (4t^2 - 2t^6)x^2 y^{k+1} + (-3t^4 + 3)x^2 y^k.$$

Then use the formulas in § 2, § 3 and switch to the basis $S_j(x)S_k(y)$. $\square$

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