KAUFFMAN BRACKET VERSUS JONES POLYNOMIAL SKEIN MODULES

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Abstract. This paper resolves the problem of comparing the skein modules defined using the skein relations discovered by P. Melvin and R. Kirby that underlie the quantum group based Reshetikhin-Turaev model for $SU(2)$ Chern-Simons theory to the Kauffman bracket skein modules. Several applications and examples are presented.

1. Motivation

In 1984 V.F.R. Jones introduced a polynomial invariant of knots and links [16]. Immediately after, L. Kauffman defined a similar polynomial knot and link invariant, the Kauffman bracket, which is in fact an invariant of framed knots and links [17]. Kauffman has shown how the Jones polynomial of a knot can be computed from the Kauffman bracket.

In 1989 E. Witten has explained in [25] the Jones polynomial by means of a quantum field theory based on the Chern-Simons functional. The Jones polynomial corresponds to the particular case of the Chern-Simons theory with gauge group $SU(2)$. By making use of physical intuition, Witten predicted the Jones polynomial to be part of a more general family of knot, link, and manifold invariants. Motivated by Witten's ideas, Reshetikhin and Turaev constructed the knot, link, and manifold invariants of the $SU(2)$ Chern-Simons theory using a quantum group associated to $sl(2,\mathbb{C})$ [23]. This theory fulfills Witten's predictions. An analogous theory was developed for the Kauffman bracket by Blanchet, Habegger, Masbaum, and Vogel in [1], and this theory parallels that of Reshetikhin and Turaev. Each of these two parallel theories have lead to significant developments and the aim of the present paper is to explain the relationship between the two models at the most general level.

Within the Reshetikhin-Turaev theory, and already present in previous works by Reshetikhin himself, lies the Jones polynomial of framed knots and links, but with a slightly different normalization. This polynomial fits exactly the quantum field theoretical model from Witten's paper, it is the polynomial that Chern-Simons theory would associate to a link whose components are colored by the 2-dimensional irreducible representation of $SU(2)$.

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We will refer to this polynomial as the Jones polynomial in the Reshetikhin-Turaev normalization (or simply the Jones polynomial, when there is no possibility of confusion). The coloring of a knot by the n-dimensional irreducible representation of $SU(2)$ yields a polynomial invariant of knots called the colored Jones polynomial, of which the Jones polynomial in the Reshetikhin Turaev normalization itself corresponds to $n = 2$. The convention is that the nth colored Jones polynomial of a knot K, denoted by $J(K, n)$, corresponds to the coloring of K by the $n + 1$st irreducible representation.

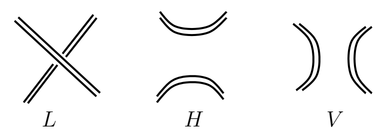

FIGURE 1

Let M be a compact, orientable, 3-dimensional manifold M, on which an orientation has been chosen. A framed link in M is an embedding of finitely many annuli.

Let us discuss first the case $M = S^3$. Both the Kauffman bracket and the Jones polynomial in the Reshetikhin-Turaev normalization of a framed knot or link in S^3 can be computed using skein relations, and these skein relations are quite similar. We denote the Kauffman bracket of a link L by $\langle L\rangle$ and this version of the Jones polynomial by J_L, both in the variable t. To write down the skein relations, let L, H, V be three framed links that coincide except in a ball where they are as shown in Figure 1. What this means is that we have an orientation preserving embedding of the ball into S^3 such that the preimage of the three links through this embedding look as depicted in the diagrams. The Kauffman bracket has the skein relations

$$\langle L\rangle = t\,\langle H\rangle + t^{-1}\,\langle V\rangle\,, \quad \langle O\rangle = -t^2 - t^{-2}.$$

Here and below O is the unknot. The first equality expresses the relation between the brackets of L, H, and V, while the second expresses the fact that every trivial link component can be replaced by multiplication by the scalar $-t^2 - t^{-2}$.

On the other hand, the skein relations of the Jones polynomial in the Reshetikhin-Turaev normalization have been computed by R. Kirby and P. Melvin in [18]; they are

$$J_L = tJ_H + t^{-1}J_V \text{ or } J_L = \epsilon(tJ_H - t^{-1}J_V), \quad J_O = t^2 + t^{-2}.$$

There are two skein relations for resolving a crossing, the one on the left is used when different link components cross, meaning that the two crossing strands in the diagram L from Figure 1 come from different link components, and the skein relation on the right is used when the diagram L corresponds to a link component crossing itself, with ϵ being the sign of the crossing.

To compute ϵ, one chooses any of the two possible orientations of the link component, which then orients the two strands inside the ball, and then the sign is computed using the right hand rule. Both orientations of the link yield the same value for ϵ.

A great amount of Chern-Simons theory is dedicated to the study of the combinatorial properties of knots and links decorated by irreducible representations of quantum groups (the so called quantized Wilson lines), and the algebraic topological concept that lies at the heart of this study is that of a skein module. Following J. Przytycki [22], we construct the skein module of a compact, orientable, 3-dimensional manifold M on which an orientation has been fixed. We do this by considering first the free $\mathbb{C}[t, t^{-1}]$-module with basis the isotopy classes of framed links in M, and then factoring this module by the skein relations. In the case of the Kauffman bracket we obtain the Kauffman bracket skein module $K_t(M)$, obtained by factoring the above mentioned free module by the submodule spanned by the elements of the form $L - tH - t^{-1}V$, where L, H, V are framed links that coincide except in a ball that is embedded by an orientation preserving homeomorphism in which they look as in Figure 1, and also by the relation that states that every link that contains a trivial link component (one that bounds a disk so that the framing is inside the disk) is equivalent to the same link with that component erased, multiplied by $-t^2 - t^{-2}$.

For the Jones polynomial, the skein module of M was defined in [15]; it is denoted by $RT_t(M)$ to point out that it comes from the Reshetihin-Turaev theory. It is defined like for the Kauffman bracket, but with the Kirby-Melvin skein relations instead. We should point out that the choice of the orientation of the manifold M determines uniquely the sign of the self-crossing of a link component, exactly like in the case of S^3, and that this sign can be computed by choosing either of the two orientations of the link component and then using the right hand rule in the embedded ball. In [15] it was explained how several constructs of $SU(2)$ Chern-Simons theory can be reduced to these skein modules.

For a better understanding of the need to introduce the skein modules of the Reshetikhin-Turaev theory, let us contrast the two skein relations in the so called "classical case". When $t = -1$, the Kauffman bracket skein relation yields the trace identity for the negative of the trace of $sl(2, \mathbb{C})$ characters of the fundamental group of M:

$$(-\operatorname{tr}\rho(\alpha\beta)) + (-\operatorname{tr}\rho(\alpha))(-\operatorname{tr}\rho(\beta)) + (-\operatorname{tr}\rho(\alpha\beta^{-1})) = 0,$$

as it has been noticed in [3]. On the other hand, the skein relation of Kirby and Melvin yields, when $t = 1$, the trace identity for the trace itself

$$\operatorname{tr}\rho(\alpha\beta) - \operatorname{tr}\rho(\alpha)\operatorname{tr}\rho(\beta) + \operatorname{tr}\rho(\alpha\beta^{-1}) = 0.$$

In Chern-Simons theory $t = e^{i\pi h}$, where h is interpreted, depending on the context, as either the coupling constant or Planck's constant. Setting $t = 1$ is equivalent to setting the coupling constant or Planck's constant equal to zero, and this is predicted to correspond to the classical (nonquantized) situation, that is to the character variety. This physical interpretation as well as the fact that it is more natural to work with the trace than the negative of the trace are two of the reasons for which we have proposed in our previous work the study of the skein modules $RT_t(M)$ of the Jones polynomial. Yet another reason is that the fundamental facts of $SU(2)$ Chern-Simons theory (the Murakami Theorem [20], the Volume Conjecture [21], the AJ Conjecture [10]) are phrased in the quantum group setting. But other constructs (such as the quantum Teichmüller theory [2], [9]) are phrased in the Kauffman bracket setting, so the present paper clarifies the relationship between the two types of skein theories: $K_t(M)$ and $RT_t(M)$.

2. The main result

Let M be a compact, orientable 3-dimensional manifold on which an orientation has been chosen, and let L be a framed link in M. Consider a compact orientable 3-dimensional manifold N such that $\partial N = -\partial M$, and consider the closed manifold $M \cup N$ obtained by gluing M and N along their common boundary.

The 3-dimensional manifold $M \cup N$, being closed, can be obtained from S^3 by performing surgery along a framed link L'. Without loss of generality we may assume that the solid tori of the surgery along L' are disjoint from L. As such, $M \cup N$ is the boundary of a 4-dimensional manifold W' obtained by gluing 2-handles to the 4-dimensional ball B^4. Let us further glue 2-handles to W' along the components of the framed link L to obtain a 4-dimensional manifold W. Note that W is obtained by gluing 2-handles to B^4 as specified by the framed links L and L' (both of which can now be viewed as embedded annuli in S^3), so that 2-dimensional disks are glued along the actual link components. These 2-handles define homology classes in $H_2(W, \mathbb{Z})$, which homology classes are determined by the closed surfaces obtained by capping each disk by a Seifert surface in S^3 of the corresponding link component. Now let us focus only on the homology classes classes in $H_2(W, \mathbb{Z})$ determined by the link components of L, and let us denote by $\mathrm{tr}(L)$ the trace of the intersection matrix of these homology classes. This trace is the sum of $[L_j] \cdot [L_j]$ over the components L_j of L, where $[L_j] \cdot [L_j]$ is the algebraic intersection number of the homology class $[L_j]$ defined by L_j with itself.

In earnest, the intersection form on $H_2(W, \mathbb{Z})$ depends on an additional piece of information: the orientation of the surfaces that are being intersected. When restricted to the homology classes that arise from the link components of L, that additional piece of information is encoded in an orientation of the link components. But the elements on the diagonal of this matrix do not depend on the orientation, they compute self-crossings, and so $\mathrm{tr}(L)$ is well defined, and can be computed by choosing any such orientation.

Note that $\mathrm{tr}(L)$ depends on the choice of N and W, but this fact does not alter the conclusion of the following theorem, and in practical applications one should always make the simplest choice.

Additionally, for a link L, we denote by $n(L)$ the number of components of L.

Theorem 2.1. *The equality*

$$\sum_{k=1}^{n} c_k L_k = 0$$

holds in $K_t(M)$ *for some Laurent polynomials* $c_k \in \mathbb{C}[t, t^{-1}]$ *and some framed links* L_k *in* M *if and only if the equality*

$$\sum_{k=1}^{n} (-1)^{n(L_k)+\mathrm{tr}(L_k)} c_k L_k = 0$$

holds in $RT_t(M)$.

Proof. Note that to write down these formulas we have implicitly choosen orientations of the the link components, but the formulas themselves, and the proof below, do not depend on these orientations. All we have to show is that the statement of the theorem is invariant under skein relations. We have to examine three cases.

Case 1. If two components L_α and L_β of one of the links L_k cross, then after resolving the crossing the number of components dropped by 1. On the diagonal of the intersection matrix the entries $[L_\alpha] \cdot [L_\alpha]$ and $[L_\beta] \cdot [L_\beta]$ disappear, and the entry

$$[L_\alpha] \cdot [L_\alpha] + [L_\beta] \cdot [L_\beta] + 2[L_\alpha] \cdot [L_\beta] - 1$$

appears, thus the exponent $n(L_k) + \mathrm{tr}(L_k)$ changes by an even number. And indeed, the skein relation for two disjoint components that cross is the same for the Kauffman bracket and for the Jones polynomial in the Reshetikhin-Turaev normalization.

Case 2. If a component L_α of some link L_k crosses itself, the crossing can be positive or negative. Let H and V be the diagrams obtained after resolving the crossing. If the crossing is positive, then $\mathrm{tr}(V) = \mathrm{tr}(H) = \mathrm{tr}(L_k) - 1$, and V has the same number of components as L_k while the H term has one component more. Thus when passing from the Kauffman bracket to the Jones polynomial we keep the same sign in front of H, while we change the sign in front of V, exactly as in the skein relation for the Jones polynomial in the Reshetikhin-Turaev normalization. If the crossing is negative, then $\mathrm{tr}(V) = \mathrm{tr}(H) = \mathrm{tr}(L_k) + 1$, but this time the number of components stays the same in H and increases in V. And this is again consistent with the skein relation.

Case 3. If we remove a trivial component, then the Kauffman bracket is multiplied by $-t^2 - t^{-2}$, while the Jones polynomial is multiplied by $t^2 + t^{-2}$. In this case the number of link components decreases by 1, and so the exponent of -1 decreases by 1, changing the sign of the corresponding term. The theorem is proved. $\square$

If we vary N and W we just multiply by a ± 1 the entire second equation from the statement. Note also that you can swap the two relations in order to pass from $RT_t(M)$ to $K_t(M)$.

Remark 2.2. If $M \subset S^3$, we can choose $N = \overline{S^3 \backslash M}$ and let W be the 4-dimensional ball that S^3 bounds. Then the intersection matrix whose trace is $\mathrm{tr}(L)$ is just the linking matrix of L.

Remark 2.3. In the case where $M = S^3$ and L is a knot, then as we know how the Jones polynomial in the Reshetikhin-Turaev normalization relates to the original Jones polynomial [25], we obtain the particular case of Theorem 2.8 from [17].

If we work over the field of fractions $\mathbb{C}(t)$, we obtain the following immediate corollary.

Proposition 2.4. *The vector spaces $K_t(M)$ and $RT_t(M)$ are isomorphic.*

Proof. As isotopy classes of framed links span $K_t(M)$, we can find a basis consisting of framed links. But then this basis is a spanning set for $RT_t(M)$. It is either a basis, or it contains a basis. If it is not a basis, then the basis it contains is a spanning set for $K_t(M)$, a contradiction. Thus any basis of framed links of $K_t(M)$ is a basis of framed links of $RT_t(M)$. Hence the conclusion. $\square$

3. Applications and examples

Let us introduce the polynomials $T_n(\xi) = 2\cos[n \arccos(\xi/2)]$, $n \in \mathbb{Z}$, which is a normalized version of the Chebyshev polynomial polynomial of the first kind, and $S_n(\xi) = \sin[(n+1)\arccos(\xi/2)]/\sin\arccos(\xi/2)$, $n \in \mathbb{Z}$, which is a normalized version of the Chebyshev polynomial of the second kind. For a framed knot K in some compact, oriented, 3-dimensional manifold M and a positive integer m, we let K^m be the framed link consisting of m parallel copies of K, where in order to produce the parallel copies K is pushed in the direction of the framing. Given a framed link $L = L_1 \cup L_2 \cup \cdots \cup L_k$ and a k-tuple of positive integers, $(j_1, j_2, ..., j_k)$, we can construct the link $L_1^{j_1} \cup L_2^{j_2} \cup \cdots \cup L_k^{j_k}$ by taking parallel copies of each component.

In particular, for a knot K we can construct the skeins $T_n(K)$ and $S_n(K)$ in either $K_t(M)$ or $RT_t(M)$.

3.1. **The product-to-sum formula and Weyl quantization.** Here is another reason for working with the skein modules of the Jones polynomial in the Reshetikhin-Turaev normalization. If a manifold is a cylinder over a surface, then the operation of gluing one cylinder on top of the other induces an algebra structure on the skein module; this is the skein algebra of the surface. Of particular interest is the skein algebra of the the torus,

$K_t(\mathbb{T}^2)$. As a module, it is free with basis $(p,q)_T$, $p,q \in \mathbb{Z}$, $p \geq 0$, where $(p,q)_T = T_n((p/n, q/n))$, with n the greatest common divisor of p and q and $(p/n, q/n)$ the curve of slope q/p on the torus whose framing is parallel to the torus.

As shown in [7] and [15], for both the Kauffman bracket and the Jones polynomial in the Reshetikhin-Turaev normalization, the multiplication is given by the product-to-sum formula

$$(p,q)_T(r,s)_T = t^{ps-qr}(p+r, q+s)_T + t^{-ps+qr}(p-r, q-s)_T.$$

For a manifold with boundary, the operation of gluing a cylinder over the boundary to the 3-dimensional manifold induces a module structure on its skein module, over the skein algebra of the boundary. A situation that was investigated in [7] and [14] is that where the manifold is the solid torus. Let α be the curve that is the core of the solid torus (the image of $(1,0)$ under the inclusion of the boundary). The following result was proved in [14] and [15].

Proposition 3.1. *In the case of the Jones polynomial in the Reshetikhin-Turaev normalization, the action of the skein algebra of the cylinder over the torus on the skein module of the solid torus is given by*

$$(p,q)_T S_{j-1}(\alpha) = t^{-pq}[t^{2jq} S_{j-p-1}(\alpha) + t^{-2jq} S_{j+p-1}(\alpha)], \tag{3.1}$$

A consequence of Theorem 2.1 is the following.

Proposition 3.2. *For the Kauffman bracket, the action of the skein algebra of the cylinder over the torus on the skein module of the solid torus is given by*

$$(p,q)_T S_{j-1}(\alpha) = (-1)^q t^{-pq}[t^{2jq} S_{j-p-1}(\alpha) + t^{-2jq} S_{j+p-1}(\alpha)].$$

Proof. Let $p = np'$, and $q = nq'$, with p', q' coprime. Then $(p,q)_T$ is a linear combination of links, each of which having the number of components congruent to n modulo 2. Each of these components is a copy of the curve of slope q/p on the torus. Because we work with the blackboard framing of the torus, each component contributes $(p'-1)q' + q' = p'q'$ to $\mathrm{tr}(L)$, and so modulo 2, each term of $(p,q)_T$ contributes $np'q'$ to the trace. And $S_{j-1}(\alpha)$ contributes nothing to the exponent of -1 in the formula from Theorem 2.1. Also, modulo 2, the number of link components in $S_k(\alpha)$ is k. Thus when switching from the Jones polynomial picture to the Kauffman bracket picture, (3.1) becomes

$$\begin{aligned}(-1)^{np'q'+n+j-1}(p,q)_T S_{j-1}(\alpha) = t^{-pq}[&t^{2jq}(-1)^{j-np'-1} S_{j-p-1}(\alpha)\\ &+(-1)^{j+np'-1} t^{-2jq} S_{j+p-1}(\alpha)],\end{aligned}$$

An easy case check shows that if p', q' are coprime then $p'q' + 1 - p' \equiv q' \pmod 2$, so $np'q' + n - np' \equiv nq' \pmod 2$ and the formula is proved. □

We point out that for a curve γ, in the setting of the Reshetikhin-Turaev theory $S_j(\gamma)$ corresponds to γ colored by the $j+1$-dimensional irreducible representation, V^{j+1}, of the corresponding quantum group (which we denote by $V^{j+1}(\gamma)$), while in the setting of the Kauffman bracket it corresponds to the coloring of the curve by the jth Jones-Wenzl idempotent. So the relation from Proposition 3.1 has the nicer form

$$(p,q)_T V^j(\alpha) = t^{-pq}[t^{2jq}V^{j-p}(\alpha) + t^{-2jq}V^{j+p}(\alpha)].$$

This equation has been related by the second author and A. Uribe [14] to the action of the Heisenberg group on theta functions discovered by A. Weil [24], and as such to the Weyl quantization of the moduli space of $SU(2)$ connections on the tours, and this gives a second reason for our focus on the skein modules of the Jones polynomial. More explicitly, the moduli space in question is the "pillow case" obtained by factoring the complex plane $\mathbb{C}$ by the maps $z \mapsto z + m + ni$, $m, n \in \mathbb{Z}$ and $z \mapsto -z$. To perform geometric quantization we let Planck's constant be the reciprocal of an even integer $h = (2r)^{-1}$, and let ζ_j be the sections of the Chern-Simons line bundle over the moduli space that are lifted to the plane as the entire functions as

$$\zeta_j = \sqrt[4]{r}e^{-\frac{j^2\pi}{2r}}(\theta_j - \theta_j), \quad \theta_j(z) = \sum_{n=-\infty}^{\infty} e^{-\pi(2rn^2+2jn)+2\pi iz(j+2rn)}.$$

Then we let $C(p,q)$ be the operator associated by peforming equivariant Weyl quantization to the function $2\cos(2\pi(px+qy))$ on the pillow case (here $z = x + iy$). A computation with integrals yields

$$C(p,q)\zeta_j = t^{-pq}[t^{2jq}\zeta_{j-p} + t^{-2jq}\zeta_{j+p}], \text{ where } t = e^{\frac{i\pi}{2r}},$$

which has been interpreted as saying that the Weyl quantization and the quantum group quantization of the moduli space of flat $SU(2)$ connections on the torus coincide.

3.2. **The skein module of the complement of the $(2p+1,2)$ torus knot.** Let us now show an example that arises in the search for patterns in skein modules. Computations with skeins have exponential complexity, and we expected them to yield complicated results. Sometimes, for apparently no reason, the result of a lengthy computation produces a simple formula. This is the case with the following example, which we will examine, for comparison, in both situations. The Kauffman bracket skein module of the complement $S^3\backslash N(T_{2p+1,2})$ of a regular neighborhood of the $(2p+1,2)$ torus knot $T_{2p+1,2}$ is free with basis $x^n y^k$, $n \geq 0$, $0 \leq k \leq p$, as it was shown by D. Bullock in [4], where x and y are depicted in Figure 2 and are endowed with the blackboard framing. Then $RT_t(S^3\backslash T_{2p+1,2})$ is also free, with the same basis. Indeed, using Theorem 2.1 and Bullock's result we conclude that every skein in $RT_t(S^3\backslash T_{2p+1,2})$ is a linear combination of the elements $x^n y^k$, $n \geq 0, 0 \leq k \leq p$. And any nontrivial linear combination equal to 0 in $RT_t(S^3\backslash T_{2p+1,2})$ would yield a nontrivial linear combination equal to zero in $K_t(S^3\backslash T_{2p+1,2})$, which is impossible.

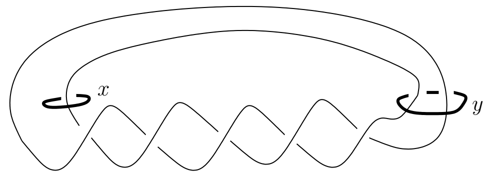

FIGURE 2

For the Kauffman bracket skein module, the following surprising formula was discovered by J. Sain in [12]

$$t^{-2i-1}S_{p+i}(y)+t^{2i+1}S_{p-i-1}(y)=(-1)^iS_{2i}(x)(t^{-1}S_p(y)+tS_{p-1}(y)),$$

for $i=1,2,...,p+1$, which allows the reduction of higher "powers" of y to lower powers. By Theorem 2.1, in the quantum group setting of the Reshetikhin-Turaev theory we have the slightly simpler identity

$$t^{-2i-1}V^{p+i+1}(y)-t^{2i+1}V^{p-i}(y)=V^{2i+1}(x)(t^{-1}V^{p+1}(y)-tV^p(y)).$$

3.3. **The colored Jones polynomials and the noncommutative A-polynomial of a knot.** If $K\subset S^3$ is a framed knot with framing zero, then, in $RT_t(S^3)$, the skein $S_n(K)$ is equal to the colored Jones polynomial of K corresponding to the coloring of K by the $n+1$st irreducible representation of the the quantum group of $SU(2)$ multiplied by the empty link:

$$S_n(K)=J(K,n)\emptyset.$$

Theorem 2.1 shows that if we evaluate $S_n(K)$ in the Kauffman bracket skein module $K_t(S^3)$ instead, we obtain $(-1)^nJ(K,n)\emptyset$, because the trace of each term of $S_n(K)$ is zero and the number of componets is congruent to n modulo 2. In other words, the nth colored Jones polynomial is equal to $(-1)^n$ times the nth colored Kauffman bracket:

$$J(K,n)=(-1)^n\langle S_n(K)\rangle$$

a fact that is being used widely (see for example [19]).

There are two versions of the definition of the noncommutative generalization of the A-polynomial of a knot and the aim of this paragraph is to give a better understanding of the relationship between the two. The first was defined by the second author in joint work with Ch. Frohman and W. Lofaro in [8] and is based on the Kauffman bracket. The construction uses the action of the Kauffman bracket skein algebra of the cylinder over the torus, $K_t(\mathbb{T}^2)$, on the Kauffman bracket skein module $K_t(S^3\backslash(N(K))$ of the complement of the regular neighborhood of a knot K, which arises from gluing the cylinder to the knot complement. The annihilator of the empty link, which is a left ideal in $K_t(\mathbb{T}^2)$, is called the peripheral ideal of the knot and is denoted by $\mathcal{I}_t(K)$. It consists of the linear combinations of framed curves on the boundary torus that become equal to zero when "pushed" inside the skein module of the knot complement. If we extend this ideal to a left ideal in the ring

$$\mathbb{C}_t[l,l^{-1},m,m^{-1}] = \mathbb{C}\left\langle l,l^{-1},m,m^{-1}\right\rangle/(lm = t^2ml)$$

using the inclusion of $K_t(\mathbb{T}^2)$ into this latter ring defined by $(1,0)\mapsto l+l^{-1}$, $(0,1)\mapsto(0,1)$ (see [7]), then restrict it to $\mathbb{C}_t[l,m]$, we obtain what is called the non-commutative A-ideal of K [8]. The reason for the definition is that for $t=-1$ this ideal is principal, and modulo a normalization, it is generated by the A-polynomial defined in [6]. Moreover, it has been observed in [8] and [11] that every element in the non-commutative A-ideal yields a recursive relation for the colored Kauffman brackets $< S_n(K) >= (-1)^n J(K,n)$.

The second construction of the noncommutative generalization of the A-polynomial has its origin in [10] and is based on quantum groups, being therefore related to the Jones polynomial in the Reshetikhin-Turaev normalization. The idea is to view the family of colored Jones polynomials as a function $f:\mathbb{Z}\to\mathbb{C}[t,t^{-1}]$, $f(n)=J(K,n)$ and consider the operators L and M on such functions $Lf(n)=f(n+1)$ and $Mf(n)=t^{2n}f(n)$. These operators satisfy $LM=t^2ML$, and so they generate the ring

$$\mathbb{C}_t[L,L^{-1},M,M^{-1}] = \mathbb{C}\left\langle L,L^{-1},M,M^{-1}\right\rangle/(LM = t^2ML).$$

The recurrence ideal of the knot K is the left ideal consisting of the polynomials $P(L,M)$ satisfying $P(L,M)f=0$, where f is the function defined above. It has been shown in [10] that this ideal is always nonzero. The two constructions are related because any recursive relation for $< S_n(K) >= (-1)^n J(K,n)$ can be transformed into a recursive relation for $J(K,n)$, so every element in the peripheral ideal defined in [8] can be transformed into an element in the recurrence ideal, but this transformation is somewhat ad hoc because it requires several sign adjustments.

However, if we use for the definition of the noncommutative A-ideal the skein modules of the Jones polynomial, thus working instead with the action of $RT_t(\mathbb{T}^2)$ on $RT_t(S^3\backslash N(K))$, then the ideal resulting from extending the peripheral ideal to $\mathbb{C}_t[l,l^{-1},m,m^{-1}]$ and then restricting to $\mathbb{C}_t[l,m]$ is automatically included in the recurrence ideal under the identification $l=L, m=M$; no more change of signs. Moreover, Theorem 2.1 implies that to pass from the peripheral ideal for the case of the Kauffman bracket to that for the case of the Jones polynomial in the Reshetikhin-Turaev normalization, one has to substitute each $(p,q)_T$ by $(-1)^p(p,q)_T$.

3.4. **The skein module of the complement of the figure-eight knot.** We illustrate these facts with the example of the figure-eight knot K_8. Let $N(K_8)$ be an open regular neighborhood of this knot. Consider the left action of the skein algebra of the torus, $RT_t(\mathbb{T}^2)$, on $RT_t(S^3\backslash N(K_8))$ defined by gluing the cylinder over the torus to the boundary of the knot complement such that the curve $(1,0)$ is identified with the longitude and the curve $(0,1)$

is identified with the meridian of the knot. To understand the $RT_t(\mathbb{T}^2)$-module structure of $RT_t(S^3\backslash N(K_8))$, we need to explicate the action of the elements $(p,q)_T$ from the boundary.

The Kauffman bracket skein module of the figure-eight knot complement was found by D. Bullock and W. Lofaro in [5] to be free with basis $x^n, x^n y, x^n y^2$ where $n \geq 0$, or equivalently $x^n, x^n y, x^n z$, where $n \geq 0$, the framed curves x, y and z being shown in Figure 3 and being endowed with the blackboard framing. For the same reason as in the case of the torus knots discussed above, $RT_t(S^3\backslash N(K_8))$ is also free, with the same basis.

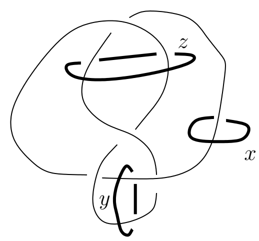

Figure 3

From the work in [13] one can infer that the action of the algebra $K_t(\mathbb{T}^2)$ on $K_t(S^2\backslash N(K_8))$ is determined by the following

$$
\begin{aligned}
&(1,q)_T\emptyset = t^q[(t^2S_{2+q}(x)+t^{-2}S_{-q}(x))Y + (t^2S_q(x)+t^{-2}S_{2-q}(x))Z \\
&\quad -(t^2S_{-4-q}(x)+t^{-2}S_{-4+q}(x))], \\
&(1,q)_TY = t^{q+4}[-(t^2S_{4+q}(x)+t^{-2}S_{-4-q}(x))Y - (t^2S_{q+2}(x) \\
&\quad +t^{-2}S_{-q}(x))Z + (t^2S_{-6-q}(x)+t^{-2}S_q(x))], \\
&(1,q)_TZ = t^{q-4}[-(t^2S_q(x)+t^{-2}S_{2-q}(x))Y - (t^2S_{-4+q}(x) \\
&\quad +t^{-2}S_{4-q}(x))Z + t^{q-4}(t^{-2}S_{-6+q}(x)+t^2S_{-q}(x))].
\end{aligned}
$$

where $Y = t^2y+1$, $Z = t^{-2}z+1$ and $q \in \mathbb{Z}$. We point out that the action of $K_t(\mathbb{T}^2)$ on elements of the form x^n, as well as on $x^n y$ and $x^n z$ for $n > 0$ can be derived from these using the product-to-sum formula (since x is the image of the $(0,1)$ curve on the boundary). This also only explicates the action of elements of the form $(1,q)_T$, but the product-to-sum formula allows the computation of the action of a general $(p,q)_T$, albeit without a nice closed form formula like the ones above.

Applying Theorem 2.1, and noticing that the computation of the signs requires just the counting link components modulo 2, we obtain the following result.

Theorem 3.3. *The action of $RT_t(\mathbb{T}^2)$ on $RT_t(S^2\backslash N(K_8))$ is determined by*

$$\begin{aligned}
&(1,q)_T\emptyset = t^q[(t^2S_{2+q}(x)+t^{-2}S_{-q}(x))Y+(t^2S_q(x)+t^{-2}S_{2-q}(x))Z\\
&\quad +(t^2S_{-4-q}(x)+t^{-2}S_{-4+q}(x))],\\
&(1,q)_TY = t^{q+4}[(t^2S_{4+q}(x)+t^{-2}S_{-4-q}(x))Y+(t^2S_{q+2}(x)\\
&\quad +t^{-2}S_{-q}(x))Z+(t^2S_{-6-q}(x)+t^{-2}S_q(x))],\\
&(1,q)_TZ = t^{q-4}[(t^2S_q(x)+t^{-2}S_{2-q}(x))Y+(t^2S_{-4+q}(x)\\
&\quad +t^{-2}S_{4-q}(x))Z+t^{q-4}(t^{-2}S_{-6+q}(x)+t^2S_{-q}(x))].
\end{aligned}$$

where $Y=t^2y-1$, $Z=t^{-2}z-1$ and $q\in\mathbb{Z}$.

Using this module structure, after a tedious computation, one obtains the following example of an element in the peripheral ideal of K_8 in the version that uses the Jones polynomial in the Reshetikhin-Turaev normalization:

$$\begin{aligned}
&t^{-6}(2,3)_T-t^6(2,-1)_T-t^3(1,7)_T+t(1,5)_T+(t^{11}-t^3+t^{-1}+t^{-5})(1,3)_T\\
&+(-t^9+t^5+t^{-7})(1,1)_T+(t^{11}-2t^7-t^3+t^{-1}-t^{-9})(1,-1)_T\\
&+(-t^{13}-t)(1,-3)_T+t^{-1}(1,-5)_T+t^8(0,7)_T+(-2t^8+t^4-t^{-4})(0,5)_T\\
&+(-t^{12}+t^8-t^4-1+t^{-4})(0,3)_T+(t^{12}-t^8+1+t^{-4})(0,1)_T.
\end{aligned}$$

This should be contrasted with

$$\begin{aligned}
&t^{-6}(2,3)_T-t^6(2,-1)_T+t^3(1,7)_T-t(1,5)_T+(-t^{11}+t^3-t^{-1}-t^{-5})(1,3)_T\\
&+(t^9-t^5-t^{-7})(1,1)_T+(-t^{11}+2t^7+t^3-t^{-1}+t^{-9})(1,-1)_T\\
&+(t^{13}+t)(1,-3)_T-t^{-1}(1,-5)_T+t^8(0,7)_T+(-2t^8+t^4-t^{-4})(0,5)_T\\
&+(-t^{12}+t^8-t^4-1+t^{-4})(0,3)_T+(t^{12}-t^8+1+t^{-4})(0,1)_T
\end{aligned}$$

which was obtained in [13] as an element of the peripheral ideal defined using the Kauffman bracket. The former gives rise to the following recursive relation for colored Jones polynomials $y_n=J(K_8,n)$ of the figure-eight knot

$$\begin{aligned}
&(t^{6n+6}-t^{-2n+2})y_{n+2}+(-t^{14n+24}+t^{10n+16}+t^{6n+20}-t^{6n+12}+t^{6n+8}\\
&+t^{6n+4}-t^{2n+12}+t^{2n+8}+t^{2n-4}+t^{-2n+8}-2t^{-2n+4}-t^{-2n}+t^{-2n-4}\\
&-t^{-2n-12}-t^{-6n+4}-t^{-6n-8}+t^{-10n-16})y_{n+1}+(t^{14n+22}-2t^{10n+18}\\
&+t^{10n+14}-t^{10n+6}-t^{6n+18}+t^{6n+14}-t^{6n+10}-t^{6n+6}+t^{6n+2}+t^{2n+14}\\
&-t^{2n+10}+t^{2n+2}+t^{2n-2}+t^{-2n+10}-t^{-2n+6}+t^{-2n-2}+t^{-2n-6}-t^{-6n+6}\\
&+t^{-6n+2}-t^{-6n-2}-t^{-6n-6}+t^{-6n-10}-2t^{-10n-2}+t^{-10n-6}-t^{-10n-14}\\
&+t^{-14n-6})y_n+(t^{10n+4}-t^{6n+16}-t^{6n+4}+t^{2n+12}-2t^{2n+8}-t^{2n+4}+t^{2n}\\
&-t^{2n-8}-t^{-2n+8}+t^{-2n+4}+t^{-2n-8}+t^{-6n+8}-t^{-6n}+t^{-6n-4}+t^{-6n-8}\\
&+t^{-10-4}-t^{-14n-4})y_{n-1}+(t^{2n+6}+t^{-6n-6})y_{n-2}=0.
\end{aligned}$$

If we use the construction based on the Kauffman bracket, we obtain a recursive relation for $(-1)^nJ(K,n)$ instead.

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