INTRODUCTION TO GEOMETRY

Lecture notes by Răzvan Gelca

Contents

Chapter 1

Absolute Geometry

1.1 The axioms

1.1.1 Properties of incidence

Lines and points are primary notions, they are not defined. A point can belong to a line or not.

I1. Given two points, there is one and only one line containing those points.

I2. Any line has at least two points.

I3. There exist three non-collinear points in the plane.

When a line contains a point, we also say that the line passes through that point. Points are denoted by capital letters of the Roman alphabet. Given two distinct points A and B, the unique line passing through A and B is denoted by AB. Sometimes lines will be denoted by lower case letters of the Roman alphabet. Several points belonging to the same line are called collinear. If C belongs to AB we write $C \in AB$. We identify a line with the set of all the points belonging to it.

Example 1.1.1. *The Euclidean plane.*

Example 1.1.2. *The plane consists of three noncollinear points A, B, C, and the lines are the sets $\{A, B\}$, $\{A, C\}$, $\{B, C\}$.*

Problem 1.1.1. *Show that if A, B, C are three non-collinear points, then the lines AB, BC, and AC are pairwise distinct.*

Proof. We argue by contradiction (*reductio ad absurdum*). Assume that $AB = BC$. We know that $C \in BC$, and hence $C \in AB$ as well. This implies that A, B, C are collinear, a contradiction. So our initial assumption was false, which implies that $AB \neq BC$. QED □

1.1.2 Properties of ordering

The notion of *between* is not defined, but it is given through its properties. It applies to three points A, B, C, with $A \neq C$, by saying that "B is between A and C".

O1. If B is between A and C then A, B, C are collinear and B is also between C and A.

O2. If B is between A and C, then A is not between B and C.

O3. If A, B, C are collinear and distinct such that A is not between B and C and C is not between A and B, then B is between A and C.

O4. If B is between A and C and C is between B and D, then B and C are between A and D.

O5. If A and B are distinct points, there is C such that B is between A and C.

O6. If A and B are distinct points, there is C between A and B.

Theorem 1.1.1. Every line contains infinitely many points.

Proof. Let us consider a line. By I2 it contains two points A and B. Using O5 we can find a point M_1 such that B is between A and M_1. From O1 and O2 we deduce that $M_1 \neq A$ and $M_1 \neq B$.

Using O5 we can find M_2 such that M_1 is between B and M_2. Then by O4, M_1 and B are between A and M_2. Consequently M_2 is different from A, B, M_1.

Next using O5 we can find M_3 such that M_2 is between M_1 and M_3. By O4, M_1 and M_2 are between B and M_3, and so by the same O4, B, M_1, M_2 are between A and M_3. So M_3 is different from A, B, M_1, M_2.

Inductively we construct the sequence of points $M_1, M_2, M_3, \ldots$, such that for each $k > 1$, $B, M_1, M_2, \ldots, M_{k-1}$ are between A and M_k. These points are distinct, so the line has infintely many points. $\square$

Definition. Given two points A, B, the (closed) *segment* $|AB|$ consists of A, B and all points between A and B.

Definition. Given two points A, B, the *ray* $|AB$ is the set of all points M such that A is not between M and B.

Definition. Given three noncollinear points A, B, C, the *half-plane* containing C and bounded by AB is the set of all points M such that there does not exist N on AB with N between C and M.

Definition. Given three noncollinear points A, B, C, the triangle ΔABC is the union of the segments $|AB|$, $|BC|$, and $|AC|$.

We introduce one more ordering axiom.

O7. (Pasch' axiom) If a line does not pass through any of the noncollinear points A, B, C and intersects the segment $|AB|$, then it intersects one and only one of the segments $|AC|$ and $|BC|$.

Definition. Given the points $A_1, A_2, \ldots, A_n$, the union of the segments $|A_1A_2|, |A_2A_3|, \ldots, |A_nA_1|$ is called a *polygon* ($\pi o\lambda u\gamma o\nu o\nu$). If $n = 3$, the polygon is a triangle, if $n = 4$ it is a quadrilateral. The points $A_1, A_2, \ldots, A_n$ are called *vertices*, while the segments $|A_1A_2|, |A_2A_3|, \ldots, |A_nA_1|$ are called *sides*.

If nonconsecutive sides intersect, the polygon is called *skew*. By default, we assume that polygons are nonskew. If any two consecutive vertices determine a line such that all other vertices are in the same half-plane determined by that line, then the polygon is called *convex*.

In a polygon $A_1A_2 \ldots A_n$, if A_j and A_k are *not* consecutive vertices, then the segment $|A_jA_k|$ is called a *diagonal*.

Definition. We call *angle* the union of two rays with the same origin.

If the rays are $|AB$ and $|AC$ we denote the angle they form by $\angle BAC$ or $\widehat{BAC}$.

Adjacent, supplementary, and opposite angles are defined as in Figure 1.1.

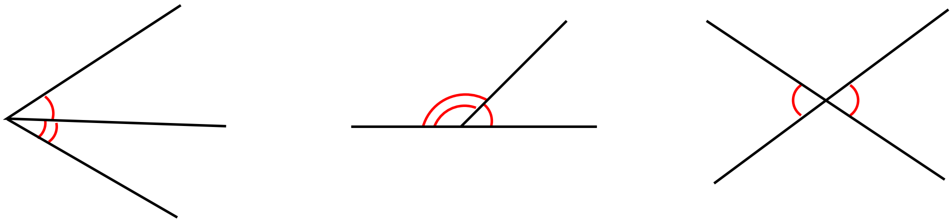

Figure 1.1: Types of angles

1.1.3 Congruence

"Congruence" is the notion of equality in Euclidean geometry, in the same way as "isomorphic" is the notion of equality in group theory. The congruence of segments and angles is again a primary notion, defined by properties, but intuitively two segments or angles are congruent if one can be overlaid on top of the other. Congruence is denoted by $\equiv$.

C1. Given a ray with origin O and a segment $|AB|$, there exists one and only one point M on the ray such that $|OM| \equiv |AB|$.

C2. $|AB| \equiv |AB|$ and $|AB| \equiv |BA|$.
If $|AB| \equiv |A'B'|$ then $|A'B'| \equiv |AB|$.
If $|AB| \equiv |A'B'|$ and $|A'B'| \equiv |A''B''|$ then $|AB| \equiv |A''B''|$.

C3. If B is between A and C and B' is between A' and C', and if $|AB| \equiv |A'B'|$ and $|BC| \equiv |B'C'|$, the $|AC| \equiv |A'C'|$.

C4. Given an angle $\angle AOB$ and a ray $|O'A'$ and given any of the half-planes bounded by $O'A'$, there is a unique ray OB' contained in this half-plane such that $\angle AOB \equiv \angle A'O'B'$.

C5. $\angle AOB \equiv \angle AOB$, $\angle AOB \equiv \angle BOA$.
If $\angle AOB \equiv \angle A'O'B'$ then $\angle A'O'B' \equiv \angle AOB$.
If $\angle AOB \equiv \angle A'O'B'$ and $\angle A'O'B' \equiv \angle A''O''B''$, then $\angle AOB \equiv \angle A''B''C''$.

C6. Given two triangles ΔABC and $\Delta A'B'C'$ such that $\angle BAC \equiv \angle B'A'C'$, $|AB| \equiv |A'B'|$, and $|AC| \equiv |A'C'|$ then $\angle ABC \equiv \angle A'B'C'$.

Note that there is no analogue of C3 for angles. The property is actually true, but it is a theorem provable from these axioms (Theorem1.2.6).

Definition. Let $|AB|$ and $|CD|$ be two segments and P, Q, R three points such that Q is between P and R. If $|AB| \equiv |PQ|$ and $|CD| \equiv |QR|$, we say that $|PR| \equiv |AB| + |CD|$. We say that $|AB| > |CD|$ if $|AB| \equiv |CD| + |EF|$ for some segment $|EF|$. Also $|CD| \equiv |AB| - |EF|$.

If $|AB| \equiv |A'B'| + |A'B'| + \cdots + |A'B'|$, we write $|AB| = n|A'B'|$ or $|A'B'| = \frac{1}{n}|AB|$. If $|AB| \equiv n|A'B'|$ and $|A''B''| \equiv m|A'B'|$, we write $|A''B''| \equiv \frac{m}{n}|AB|$. This can also be written as

$$\frac{|A''B''|}{|AB|} = \frac{m}{n}.$$

If M is between A and B, and $|AB| \equiv 2|AM|$, then M is called the midpoint of $|AB|$.

1.2 Congruence of triangles

1.2.1 Theorems of congruence of triangles

Definition. One says that $\Delta ABC \equiv \Delta A'B'C'$ if $\angle BAC \equiv \angle B'A'C'$, $\angle ABC \equiv \angle A'B'C'$, $\angle ACB \equiv \angle A'C'B'$ and $|AB| \equiv |A'B'|$, $|BC| \equiv |B'C'|$ and $|AC| \equiv |A'C'|$.

Theorem 1.2.1. (SAS) If in ΔABC and $\Delta A'B'C'$, $|AB| \equiv |A'B'|$, $\angle BAC \equiv \angle B'A'C'$, and $|AC| \equiv |A'C'|$, then $\Delta ABC \equiv \Delta A'B'C'$.

Proof. By C6 we have $\angle ABC \equiv \angle A'B'C'$ and $\angle ACB \equiv \angle A'C'B'$. We are left to show that $|BC| \equiv |B'C'|$.

We refer to Figure 1.2. On the ray $|BC$, choose C'' such that $|BC'' \equiv |B'C'|$ (which is possible by C1). We want to show that $C'' = C$. Applying C6 to triangles $\Delta BAC''$ and $\Delta B'A'C'$ ($\angle ABC'' \equiv \angle A'B'C'$, $|BC''| \equiv |B'C'|$, $|AB| \equiv |A'B'|$), we obtain that $\angle BAC'' \equiv \angle B'A'C'$. The later is congruent to $\angle BAC$ by hypothesis. Axiom C4 implies that $|AC'' = |AC$. But $C, C'' \in BC$ and since the line AC and BC cannot have more than one point in common, by I1, it follows that $C = C''$. Therefore $|BC| \equiv |B'C'|$, and the theorem is proved. □

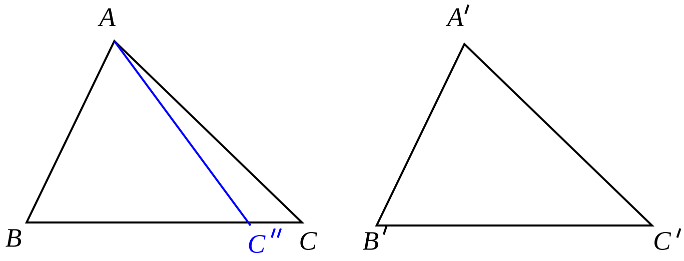

Figure 1.2: Theorem SAS

Theorem 1.2.2. (ASA) If in ΔABC and $\Delta A'B'C'$, $\angle ABC \equiv \angle A'B'C'$, $|BC| \equiv |B'C'|$, and $\angle ACB \equiv \angle A'C'B'$, then $\Delta ABC \equiv \Delta A'B'C'$.

Proof. We argue on Figure 1.3. On $|BA$ choose A'' such that $|BA''| \equiv |B'A'|$. Since $|BC| \equiv |B'C'|$, $\angle A''BC \equiv \angle A'B'C'$ and $|BA''| \equiv |B'A'|$, by applying Theorem SAS to the triangles $\Delta A''BC$ and $\Delta A'B'C'$ we deduce that $\Delta A''BC \equiv \Delta A'B'C'$. Hence $\angle BCA'' \equiv \angle B'C'A'$. But by hypothesis, $\angle BCA \equiv \angle B'C'A'$. From C4 we obtain $|CA'' = |CA$, and since CA and BA can have at most one point in common, $A = A''$. But then we have $\Delta ABC \equiv \Delta A'B'C'$, as desired. □

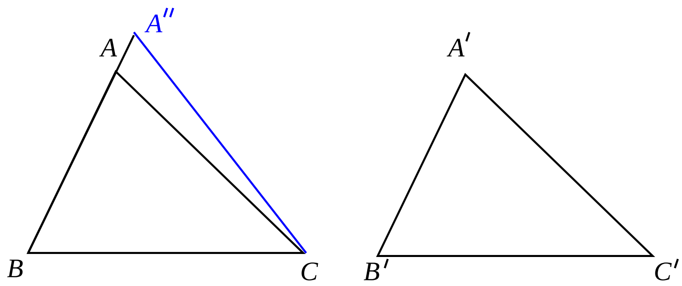

Figure 1.3: Theorem ASA

Theorem 1.2.3. In ΔABC, $|AB| \equiv |AC|$ if and only if $\angle ABC \equiv \angle ACB$.

Proof. $\Rightarrow$ Let us prove first the direct implication. If $|AB| \equiv |AC|$, by applying Theorem SAS to triangles ΔABC and ΔACB ($|AB| \equiv |AC|$, $\angle BAC \equiv \angle CAB$, $|AC| \equiv |AB|$), we obtain that these triangles are congruent. It follows that $\angle ABC \equiv \angle ACB$.

$\Leftarrow$ Now let us prove the converse. If $\angle ABC \equiv \angle ACB$, then by applying Theorem ASA to triangles ΔABC and ΔACB ($\angle ABC \equiv \angle ACB$, $|BC| \equiv |CB|$, $\angle ACB \equiv \angle ABC$) we deduce that these triangles are congruent. Consequently $|AB| \equiv |AC|$. The theorem is proved. □

Definition. A triangle with two congruent sides (or equivalently two congruent angles) is called *isosceles*. A triangle with all sides congruent (or equivalently all angles congruent) is called *equilateral*.

Theorem 1.2.4. Given $\angle AOB, \angle BOC, \angle A'O'B', \angle B'O'C'$ such that $\angle AOB$ is the supplement of $\angle BOC$, $\angle A'O'B'$ is the supplement of $\angle B'O'C'$ and $\angle AOB \equiv \angle A'O'B'$, then $\angle BOC \equiv \angle B'O'C'$.

Said in plain words, "angles with congruent supplements are congruent".

Proof. Using the axiom C1, we can actually choose the points A, B, A', B' such that $|OA| \equiv |O'A'|$, $|OB| \equiv |O'B'|$, and $|OC| \equiv |O'C'|$, as in Figure 1.4.

Since $|OA| \equiv |O'A'|$, $\angle AOB \equiv \angle A'O'B'$, and $|OB| \equiv |O'B'|$, by Theorem SAS $\Delta OAB \equiv \Delta O'A'B'$. Hence $|AB| \equiv |A'B'|$ and $\angle OAB \equiv \angle O'A'B'$.

Applying Theorem SAS to triangles ΔABC and $\Delta A'B'C'$, in which $|AC| \equiv |A'C'|$, $\angle BAC \equiv \angle B'A'C'$, and $|AB| \equiv |A'B'|$, we deduce that these triangles are congruent, hence $|BC| \equiv |B'C'|$ and $\angle OBC \equiv \angle O'B'C'$.

By Theorem SAS, $\Delta O'B'C' \equiv \Delta OBC$ ($|BC| \equiv |B'C'|$, $\angle ACB \equiv \angle A'C'B'$, $|AC| \equiv |A'C'|$). It follows that $\angle BOC \equiv \angle B'O'C'$, as desired.

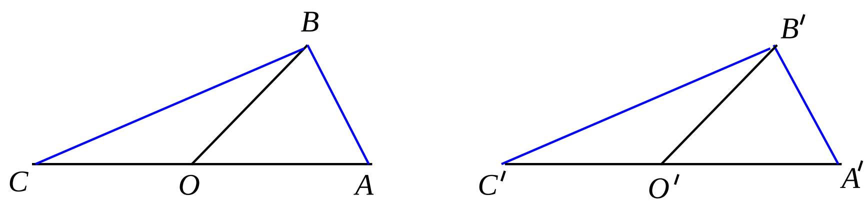

Figure 1.4: Angles with congruent supplements are congruent.

□

Theorem 1.2.5. Given the points O, A, A', B, B' such that $O \in |AA'|$ and $O \in |BB'|$, we then have $\angle AOB \equiv \angle A'O'B'$.

In other words "opposite angles are congruent".

Proof. The two angles have the common supplement $\angle AOB'$. By Theorem 1.2.4 they are congruent. □

Theorem 1.2.6. In the configuration from Figure 1.5, $\angle AOB \equiv \angle A'O'B'$ and $\angle BOC \equiv \angle B'O'C'$. Then $\angle AOC \equiv \angle A'O'C'$.

Proof. Choose A, B, C, A', B', C' such that $|OA| \equiv |O'A'|$, $|OB| \equiv |O'B'|$, A, B, C are collinear, and A', B', C' are collinear (see Figure 1.6).

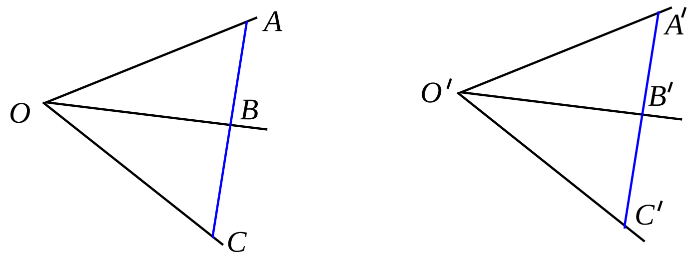

Figure 1.5: Sums of congruent angles are congruent.

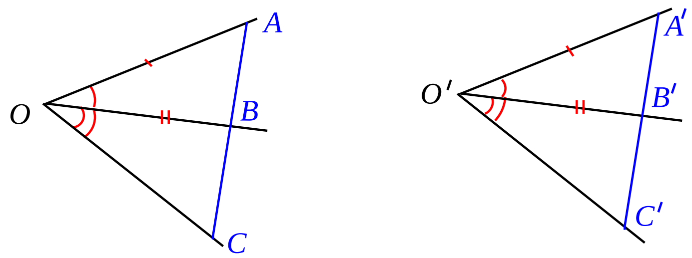

Figure 1.6: Proof that some of congruent angles are congruent.

Since $|OA| \equiv |O'A'|$, $\angle AOB \equiv \angle A'O'B'$, and $|OB| \equiv |O'B'|$, by Theorem SAS, $\Delta OAB \equiv \Delta O'A'B'$. It follows that $\angle OBA \equiv \angle O'B'A'$. Because $\angle OBC$ and $\angle O'B'C'$ have congruent supplements, by Theorem 1.2.4 they are congruent.

Hence by Theorem ASA, $\Delta OBC \equiv \Delta O'B'C'$, because $\angle BOC \equiv \angle B'O'C'$, $|OB| \equiv |O'B'|$, $\angle OBC \equiv \angle O'B'C'$. From this we obtain that $|AB| \equiv |A'B'|$ and $|BC| \equiv |B'C'|$. Adding these and using C3 we obtain $|AC| \equiv |A'C'|$.

Returning to the congruent triangles ΔOAB and $\Delta O'A'B'$, we have $\angle OAB \equiv \angle O'A'B'$. Hence in triangles ΔAOC and $\Delta A'O'C'$, $|OA| \equiv |O'A'|$, $\angle OAC \equiv \angle O'A'C'$, $|AC| \equiv |A'C'|$, and so $\Delta AOC \equiv \Delta A'O'C'$. From this congruence it follows that $\angle AOC \equiv \angle A'O'C'$, and we are done. □

Now we can introduce some notation.

If $\angle AOB = \angle A'O'B' + \angle A'O'B' + \cdots + \angle A'O'B'$, with n terms on the right, we say that $\angle AOB = n\angle A'O'B'$.

Definition. If $\angle AOB \equiv \angle A'O'B' + \angle B'O'C'$, we say that $\angle AOB > \angle A'O'B'$. Also $\angle A'O'B' \equiv \angle AOB - \angle B'O'C'$.

Definition. An angle congruent to its supplement is called a right angle.

A straight angle is twice a right angle.

Definition. Let AB and BC be two lines intersecting at B. The lines are called orthogonal (or perpendicular) if $\angle ABC$ is right.

Proposition 1.2.1. There exist right angles.

Proof. Consider an isosceles triangle ΔABC, and let M be the midpoint of $|AB|$. Then Theorem SAS implies that $\Delta ABM \equiv \Delta ACM$, so $\angle AMB \equiv \angle AMC$. This shows that $\angle AMC$ is right. □

Theorem 1.2.7. (SSS) If in ΔABC and $\Delta A'B'C'$, $|AB| \equiv |A'B'|$, $|BC| \equiv |B'C'|$, and $|AC| \equiv |A'C'|$, then $\Delta ABC \equiv \Delta A'B'C'$.

Proof. In the half-plane bounded by BC which does not contain A, choose a point A'' such that $|BA''| \equiv |B'A'|$ and $\angle CBA'' \equiv \angle C'B'A'$, as shown in Figure 1.7. We can make this choice by Axioms C1 and C4. By Theorem SAS, $\Delta BCA'' \equiv \Delta B'C'A'$, because $|BC| \equiv |B'C'|$, $\angle CBA'' \equiv \angle C'B'A'$ and $|BA''| \equiv |B'A'|$.

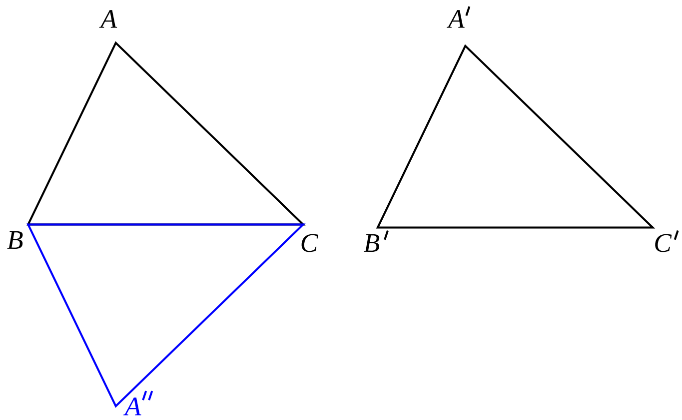

Figure 1.7: Theorem SSS

Since A, A'' do not lie in the same half-plane, the segment AA'' intersects BC. Let O be this intersection. We distinguish the following cases: $O = B$, $O = C$, $O \in |BC|$, $C \in |OB|$, $B \in |OC|$. The first two and the last two cases are similar. So we need to consider only three cases, described in Figure 1.8.

Case 1. $O = B$.

Then $\Delta CAA''$ is isosceles ($|CA''| \equiv |C'A'| \equiv |CA|$), hence $\angle BAC \equiv \angle BA''C$. Theorem SAS implies that $\Delta ABC \equiv \Delta A'B'C'$, as desired.

Case 2. $O \in |BC|$. From the isosceles triangle $\Delta BAA''$ we obtain $\angle BAO \equiv \angle BA''O$. From the isosceles triangle $\Delta CAA''$ we obtain $\angle CAO \equiv \angle CA''O$. Using Theorem 1.2.6, we obtain

$$\angle BAC \equiv \angle BAO + \angle OAC \equiv \angle BA''O + \angle OA''C \equiv \angle BA''C.$$

Now we have $|AB| \equiv |A''B|$, $\angle BAC \equiv \angle BA''C$, and $|AC| \equiv |A''C|$, so by Theorem SAS, $\Delta ABC \equiv \Delta A''BC$. Consequently $\Delta ABC \equiv \Delta A'B'C'$, so this case is solved, as well.

Case 3. $B \in |OC|$. Like before, in the isosceles triangle $\Delta BAA''$, $\angle BAO \equiv \angle BA''O$. In the isosceles triangle $\Delta CAA''$, $\angle CAO \equiv \angle CA''O$. Hence by Theorem 1.2.6,

$$\angle BAC \equiv \angle CAO - \angle BAO \equiv \angle CA''O - \angle BA''O \equiv \angle BA''C.$$

Again by Theorem SAS, $\Delta ABC \equiv \Delta A''BC \equiv \Delta A'B'C'$. The theorem is proved. □

1.2.2 Problems

1.* Prove that a line determines exactly two half planes.

2. Let $\Delta ABC \equiv \Delta A'B'C'$ and let M and M' be the midpoints of $|BC|$ respectively $|B'C'|$. Prove that $\angle BAM \equiv \angle B'A'M'$.

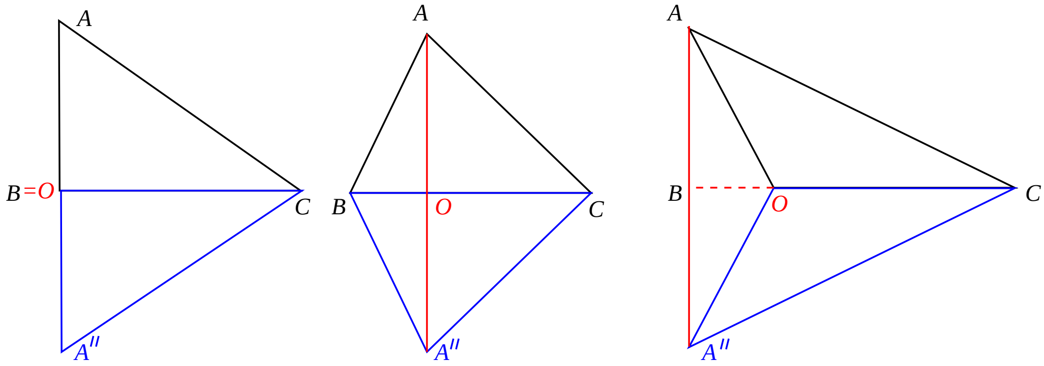

Figure 1.8: The 3 cases for the proof of Theorem SSS

3. Consider the triangle ΔABD. Suppose there is a point C such that $\angle ACB \equiv \angle ACD$ and $\angle BAC \equiv \angle DAC \equiv \frac{1}{2}\angle BAD$. Prove that the triangles ΔABD and ΔCBD are isosceles.

4. Let ΔABC be an isosceles triangle, with $|AB| \equiv |AC|$. Prove that the medians $|BN|$ and $|CP|$ are congruent.

5. In Figure 1.5 assume that $\angle AOB \equiv \angle A'O'B'$ and $\angle AOC \equiv \angle A'O'C'$. Prove that $\angle BOC \equiv \angle B'O'C'$.

6.* Let $|AB|$ be a segment. Prove that there exists $M \in |AB|$ such that $|AM| \equiv |BM|$.

7.* Prove that there is a triangle ΔABC such that if M is the midpoint of $|BC|$ then AM is not orthogonal to BC.

1.3 Inequalities in a triangle

1.3.1 The results

Theorem 1.3.1. (The exterior angle theorem) In a triangle ΔABC, the supplement of $\angle B$ is greater than $\angle A$.

Proof. We argue on Figure 1.9. Choose E such that B is between C and E, so that the supplement of $\angle B = \angle ABE$. Assume by way of contradiction that $\angle ABE \leq \angle A$, and choose the point $C' \in |BC|$ such that $\angle C''AB \equiv \angle ABE$.

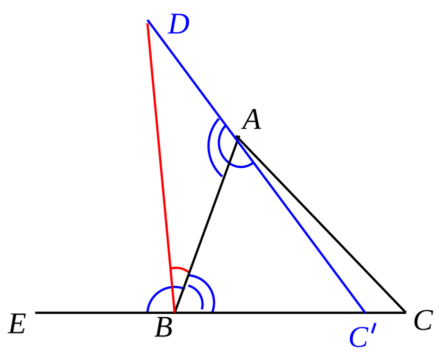

Figure 1.9: The exterior angle theorem

Let $D \in AC'$ such that A is between C' and D, and $|BC'| \equiv |AD|$. Then $\angle DAB \equiv \angle ABC'$, because they have the same supplement. This combined with $|AB| \equiv |AB|$ and $|BC'| \equiv |AD|$ implies that $\Delta ABC' \equiv \Delta BAD$ (by Theorem SAS). Hence

$$\angle ABD \equiv \angle BAC' \equiv \angle ABE.$$

By Axiom C4, $|BD$ coincides with $|BE$. So the lines AC' and BC' have two points of intersection, namely C' and D, contradicting Axiom I1. Hence our assumption was false, and the theorem is proved. □

Proposition 1.3.1. If a line forms congruent alternate interior angles with two other lines, then those lines do not intersect.

Proof. If the two lines intersected, then the two angles would be one interior and one exterior to a triangle (see Figure 1.10).

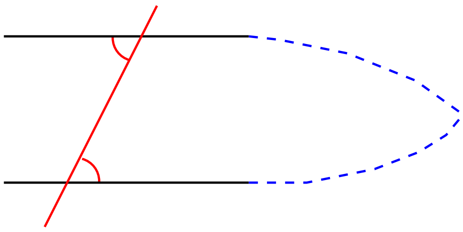

Figure 1.10: Parallel lines

But the Exterior Angle Theorem shows that this is impossible. □

Definition. Two lines that do not intersect are called parallel.

"Parallel lines exist!"

Theorem 1.3.2. In a triangle the larger side is opposite to the larger angle.

Proof. Rephrasing the statement, in ΔABC, $|AB| < |AC|$ if and only if $\angle C < \angle B$.

$\Rightarrow$ If $|AB| < |AC|$, choose $D \in |AC|$ such that $|AD| \equiv |AB|$, as shown in Figure 1.11. The triangle ΔABD is isosceles, hence $\angle ABD \equiv \angle ADB$. Using the Exterior Angle Theorem, we can write

$$\angle ABC > \angle ADB > \angle ACB.$$

Thus $\angle C < \angle B$.

$\Leftarrow$ If $\angle C < \angle B$, then $|AB| \equiv |AC|$ contradicts the isosceles triangle theorem, and $|AB| > |AC|$ contradicts what we just proved. We can only have $|AB| < |AC|$. □

Theorem 1.3.3. (The triangle inequality) In a triangle the sum of two sides is greater than the third.

Proof. Construct C' such that $B \in |CC'|$ and $|BC'| \equiv |AB|$ (see Figure 1.12). The triangle $\Delta BAC'$ is isosceles, so

$$\angle BC'A \equiv \angle BAC' < \angle CAC'.$$

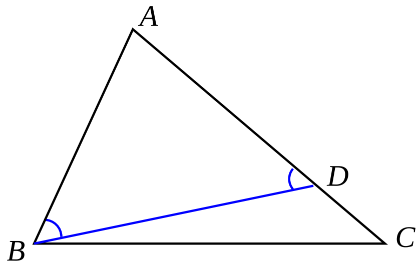

Figure 1.11: Larger side opposes larger angle.

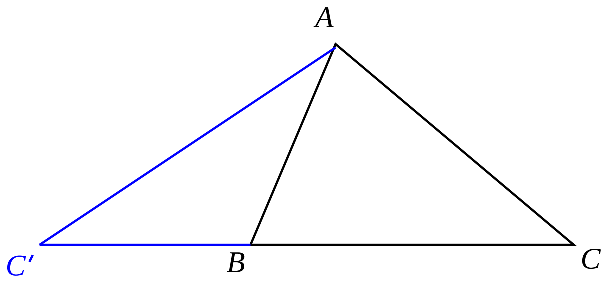

Figure 1.12: The triangle inequality

By Theorem 1.3.2,

$$|CC'| > |AC|.$$

Because $|CC'| \equiv |AB| + |BC|$, we obtain

$$|AB| + |BC| > |AC|.$$

The other two inequalities are obtained the same way. $\square$

Theorem 1.3.4. If in ΔABC, $|AB| \leq |AC|$ and $D \in |BC|$, then $|AD| < |AC|$.

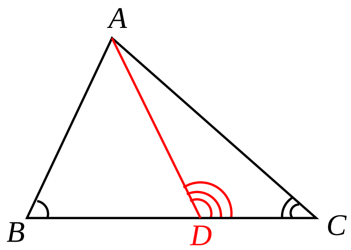

Figure 1.13: The secant is shorter than the longer side.

Proof. We argue on Figure 1.13. By Theorem 1.3.2

$$\angle B \geq \angle C.$$

By the exterior angle theorem

$$\angle ADC > \angle ABC.$$

Hence

$$\angle ADC > \angle C,$$

and so by Theorem 1.3.2 $|AD| < |AC|$, as desired. □

1.3.2 Problems

1. Let $ABCD$ be a quadrilateral. Prove that $|AB| + |BC| + |CD| \geq |AD|$.
2. Let M be a point in the interior of triangle ΔABC. Prove that $|AM|$ is shorter than the longest side of the triangle.
3. Show that a triangle can have at most one obtuse angle (you cannot use the fact that the sum of the angles of a triangle is $180°$ since this is not true in absolute geometry).
4. Let ΔABC be a right triangle, with $\angle A$ being the right angle. Prove that $|AB| + |AC| < 2|BC|$.

5.* Let ΔABC be an equilateral triangle, and $M \in |BC|$, $N \in |AC|$ and $P \in |AB|$ such that AM perpendicular to BC, BN perpendicular to AC and CP perpendicular to AB. Show that M, N and P are the midpoints of the sides, and that $|AM$, $|BN$ and $|CP$ are the angle bisectors of the triangle.

6. Let ΔABC be a triangle, and let M be the midpoint of $|BC|$. Prove that $|AM| \leq \frac{1}{2}(|AB| + |AC|)$. Conclude that the sum of the medians of a triangle is less than the sum of the sides.

1.4 Right angles

1.4.1 Properties of right angles

Recall that a right angle is an angle that is congruent to its supplement.

Remark 1.4.1. The supplement and the opposite of a right angle are right angles.

Definition. Two intersecting lines are called orthogonal if the four angles they determine are right. If AB and CD are the two lines, we write $AB \perp CD$.

Proposition 1.4.1. Any angle congruent to a right angle is a right angle.

Proof. Suppose that $\angle AOB \equiv \angle A'O'B'$, and $\angle A'O'B'$ is right. By Theorem 1.2.4, the supplement of $\angle AOB$ is congruent to the supplement of $\angle A'O'B'$. But the later is congruent to $\angle A'O'B'$, hence to $\angle AOB$. So $\angle AOB$ is congruent to its supplement, so it is right. □

Theorem 1.4.1. Any two right angles are congruent.

Remark 1.4.2. Euclid lists this as an axiom, but we work with a more modern system of axioms, in which this statement can be proved.

Proof. Arguing by contradiction, let us assume there exist two right angles $\angle ABC$ and $\angle A'B'C'$ such that $\angle ABC > \angle A'B'C'$. Let $\angle DBA$ be the supplement of $\angle ABC$.

Choose A'' in the half-plane bounded by BC which contains A such that $\angle A''BC \equiv \angle A'B'C'$ (see Figure 1.14). Then

$$\angle A''BC < \angle ABC,$$

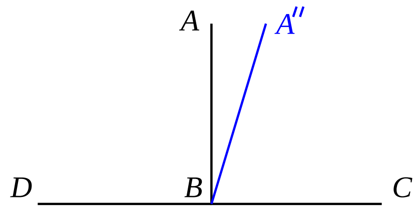

Figure 1.14: Right angles are congruent.

and so their supplements should satisfy

$$\angle DBA < \angle DBA''.$$

But since $\angle ABC$ and $\angle A''BC$ are right,

$$\angle A''BC \equiv \angle DBA'' > \angle DBA \equiv \angle ABC.$$

This contradicts $\angle A''BC < \angle ABC$. So our initial assumption was false, proving that any two right angles are congruent. □

Definition. An angle that is smaller than a right angle is called *acute*. An angle that is greater than a right angle is called *obtuse*.

Definition. If two angles add up to a right angle, they are called *complementary*. Each is the *complement* of the other.

1.4.2 Theorems of congruence for right triangles

In all five theorems below, the triangles are ΔABC and $\Delta A'B'C'$, with the right angles $\angle A$ respectively $\angle A'$.

Theorem 1.4.2. If $|AB| \equiv |A'B'|$ and $|AC| \equiv |A'C'|$, then $\Delta ABC \equiv \Delta A'B'C'$.

Proof. This is an easy application of Theorem SAS. □

Theorem 1.4.3. If $|AB| \equiv |A'B'|$ and $\angle B \equiv \angle B'$, then $\Delta ABC \equiv \Delta A'B'C'$.

Proof. This is an easy application of Theorem ASA. □

Theorem 1.4.4. If $|AB| \equiv |A'B'|$ and $|BC| \equiv |B'C'|$, then $\Delta ABC \equiv \Delta A'B'C'$.

Proof. We give three proofs to this result, the second and the third suggested by students.

I. We argue on Figure 1.15. Choose $C'' \in |AC$ such that $|AC''| \equiv |A'C'|$. Let us assume that $C \neq C''$. By Theorem SAS, $\Delta AC''B \equiv \Delta A'C'B'$. Hence $|BC''| \equiv |B'C'| \equiv |BC|$. It follows that $\Delta BC''C$ is isosceles.

But the angles of a right triangle are acute, by the Exterior Angle Theorem applied to the right angle. Thus one of the two congruent angles of $\Delta BC''C$ is acute, and the other has acute supplement, hence is obtuse. This is a contradiction, which proves that our assumption was false. It follows that $A = A''$, and hence $\Delta ABC \equiv \Delta A'B'C'$.

II. There is another way to end this proof. In the begining, we may assume $C'' \in |AC|$, or else switch the triangles. Once we have that $\Delta BCC''$ is isosceles, we notice that $|BC''| < \max(|BC|, |AB|)$ by Theorem 1.3.4. But the "larger side opposes the larger angle " shows that

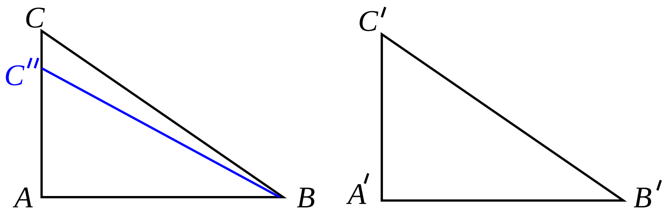

Figure 1.15: Congruence of right triangles.

in a right triangle the hypothenuse is the longest side. So $\max(|BC|, |AB|) = |BC|$. Hence $|BC''| < |BC|$, a contradiction. The conclusion follows.

III. Construct the points D and D' such that A is the midpoint of $|BD|$, and A' is the midpoint of $|B'D'|$. Then triangles ΔCAB and ΔCAD are congruent, by Theorem SAS, so $|BC| \equiv |DC|$. Similarly $\Delta C'A'B' \equiv \Delta C'A'D'$, so $|B'C'| \equiv |D'C'|$. Then triangles ΔABD and $\Delta A'B'D'$ are congruent by Theorem SSS, because $|DC| \equiv |BC| \equiv |B'C'| \equiv |D'C'|$ and $|BD| \equiv 2|AB| \equiv 2|A'B'| \equiv |B'D'|$. We obtain that $\angle B \equiv \angle B'$, and then Theorem SAS implies $\Delta ABC \equiv \Delta A'B'C'$, as desired. □

Theorem 1.4.5. If $|AB| \equiv |A'B'|$ and $\angle C \equiv \angle C'$, then $\Delta ABC \equiv \Delta A'B'C'$.

Proof. We argue again on Figure 1.15. Choose $C'' \in |AC$ such that $|AC''| \equiv |A'C'|$. Then $\Delta ABC'' \equiv \Delta A'B'C'$, by Theorem SAS. Hence $\angle BC''A \equiv \angle B'C'A' \equiv \angle BCA$. This would contradict the Exterior Angle Theorem, unless $C'' \equiv C$. We conclude that $\Delta ABC \equiv \Delta A'B'C'$. □

Theorem 1.4.6. If $|BC| \equiv |B'C'|$ and $\angle B \equiv \angle B'$, then $\Delta ABC \equiv \Delta A'B'C'$.

Proof. Choose $A'' \in |BA$ such that $|BA''| \equiv |B'A'|$, as shown in Figure 1.16. Suppose that $A'' \neq A$. Then Theorem SAS implies that $\Delta BCA'' \equiv \Delta B'C'A'$, so $\angle CA''A$ is right. It follows that in $\Delta CAA''$, there is an exterior angle and an interior angle not adjacent to it, both of which are right. This contradicts the Exterior Angle Theorem. Hence $A'' = A$ and $\Delta ABC \equiv \Delta A'B'C'$. □

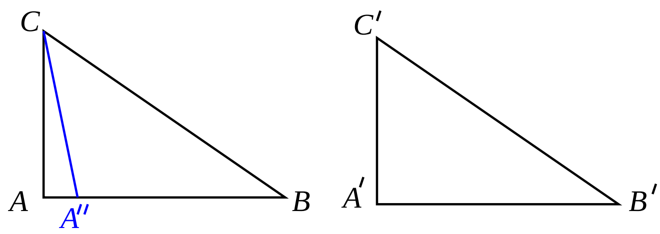

Figure 1.16: Congruence of right triangles.

We can summarize these results as follows.

Theorem 1.4.7. (The Theorem of Congruence of Right Triangles) If in two right triangles, two pairs of corresponding elements are congruent, one of which is a pair of sides, then the triangles are congruent.

Definition. The *bisector* of an angle $\angle AOB$ consists of those points M with the property that $\angle MOA \equiv \angle MOB \equiv \frac{1}{2}\angle AOB$.

Theorem 1.4.8. In a triangle the three angle bisectors intersect at one point.

Proof. We argue on Figure 1.17. Let I be the point of intersection of the bisectors from $\angle A$ and $\angle B$. By Exercise 2, there are $D \in BC$, $E \in AC$, and $F \in AB$ such that $ID \perp BC$, $IE \perp AC$, and $IF \perp AB$.

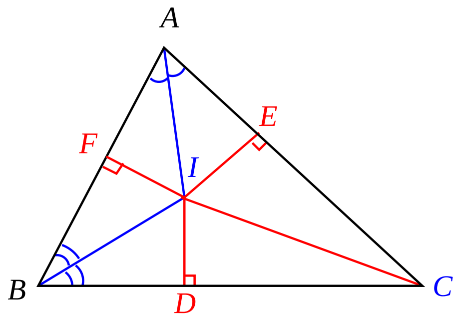

Figure 1.17: The angle bisectors intersect.

By Theorem 1.4.6, $\Delta BID \equiv \Delta BIF$, so $|ID| \equiv |IF|$. Also by Theorem 1.4.6, $\Delta AIF \equiv \Delta AIE$, and so $|IF| \equiv |IE|$. Hence $|ID| \equiv |IE|$.

In triangles ΔCID and ΔCIE, $|ID| \equiv |IE|$, $|IC| \equiv |IC|$, so by Theorem 1.4.4 they are congruent. It follows that $\angle ICE \equiv \angle ICD$. Hence $|CI$ is the angle bisector of $\angle C$. We conclude that the angle bisectors intersect at one point. □

1.4.3 Problems

1. Let AB be a line and C a point that does not belong to it. Show that there is a point $D \in AB$ such that $CD \perp AB$.

2.* Let l be a line and let C be a point that does not belong to l. Prove that on l there exists two distinct points A and B such that $|AC| \equiv |BC|$.

3. Let ABC be an isosceles triangle with $|AB| \equiv |AC|$. Let $E \in AC$ and $F \in AB$ be such that $BE \perp AC$ and $CF \perp AB$. Prove that $|BE| \equiv |CF|$. (Hint: The angle $\angle BAC$ might be acute or obtuse).

1.5 The axioms of continuity

We conclude the discussion of absolute geometry by adding two axioms that allow us to establish a one-to-one correspondence between the points of a line and the real numbers that preserves the ordering.

R1. (Archimedes) If A and B are two points of a ray $|OX$, then there is a finite set of points $\{A_1, A_2, \ldots, A_k\}$ on $|OX$ such that

$$A \in |OA_1|, A_1 \in |OA_2|, \ldots, A_{k-1} \in |OA_k|,$$
$$|OA| \equiv |AA_1| \equiv |A_1A_2| \equiv \cdots \equiv |A_{k-1}A_k|,$$

and $B \in |OA_k|$.

R2. (Cantor-Dedekind) Given a line and two sequences of points $A_1, A_2, A_3, \ldots$, $B_1, B_2, B_3, \ldots$ on this line such that for every j the segment $|A_{j+1}B_{j+1}|$ is contained in the segment $|A_jB_j|$, then there exists a point P contained in all of these segments.

Chapter 2

Euclidean Geometry

2.1 Euclid's fifth postulate

2.1.1 Parallel lines

E1. (Euclid's fifth postulate) Given a line l and a point A that does not belong to l, there is a unique line l' passing through A such that l and l' are parallel.

Notation: $l||l'$.

Theorem 2.1.1. Given two lines l_1 and l_2 that are intersected by a third line l as shown in Figure 2.1, then $l||l'$ if and only if $\angle\alpha \equiv \angle\beta$, where $\angle\alpha$ and $\angle\beta$ are a pair of alternate interior angles.

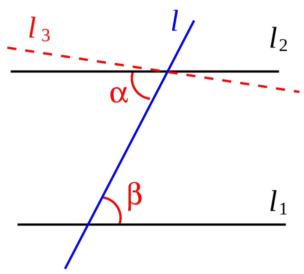

Figure 2.1: Characterization of parallel lines

Proof. $\Rightarrow$ We start with $l_1||l_2$. Construct a line l_3 through the intersection of l_2 and l such that l_3 and l_1 form congruent alternate interior angles with l. Then l_3 and l_1 are parallel, by Proposition 1.3.1, so by Euclid's fifth postulate, $l_3 = l_2$. Hence $\angle\alpha \equiv \angle\beta$.

$\Leftarrow$ The converse statement follows from Proposition 1.3.1. $\square$

Corollary 2.1.1. Two pairs of parallel lines form congruent angles.

Theorem 2.1.2. The sum of the angles of a triangles is congruent to a straight angle.

Proof. Let ΔABC be a triangle. On AB choose a point D such that A is between B and D. Take the only line through A that is parallel to BC, as shown in Figure 2.2. This line divides $\angle CAD$ into two angles. Let these angles be $\angle CAE$ and $\angle EAD$. Then by Theorem 2.1.1,

$$\angle CAE \equiv \angle ACB \text{ and } \angle EAD \equiv \angle ABC.$$

The conclusion follows, since the angles $\angle BAC$, $\angle CAE$ and $\angle EAD$ add up to a straight angle. $\square$

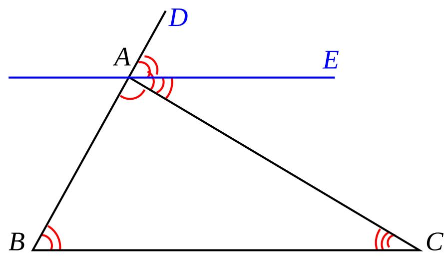

Figure 2.2: The sum of the angles of a triangle is a staight angle

Definition. A *trapezoid* is a quadrilateral that has a pair of opposite sides that are parallel.

Definition. A *parallelogram* is a quadrilateral whose opposite sides are parallel.

Proposition 2.1.1. A quadrilateral is a parallelogram if and only if one of the following properties holds:

(i) Two opposite sides are parallel and congruent.

(ii) The two pairs of opposite sides are congruent.

(iii) The diagonals intersect at their midpoint.

Definition. A parallelogram whose angles are right is called a *rectangle*. A parallelogram whose sides are congruent is called *rhombus*. A quadrilateral that is both a rectangle and and a rhombus is called a *square*.

2.1.2 Problems

1. Show that if $l||l'$ and $l'||l''$, then $l||l''$.

2. Prove Proposition 2.1.1.

3. Show that if a parallelogram has one right angle, then all of its angles are right.

4. What is the sum of the angles of a polygon with n sides?

2.2 Similarity

2.2.1 The Theorems of Thales

Before we state Thales' Theorems, we introduce the following notation. We say that $|AB|/|CD| = x$, where x is a positive real number, if $|AB| \equiv x|CD|$. It is easy to see how this works if x is rational; the axioms of continuity imply that for any choice of a segment $|CD|$ and any positive real number x, there is a segment $|AB|$ such that $|AB| \equiv x|CD|$.

Theorem 2.2.1. (Thales) Let l and l' be two distinct lines in the plane, and $A, A', A'' \in l$, $B, B', B'' \in l'$ such that $AB||A'B'$, $B' \in |BB''|$, and $|AA'| \equiv |A'A''|$. Then $A'B'||A''B''$ if and only if B' is the midpoint of the segment $|BB''|$.

Proof. $\Rightarrow$ Construct $M \in |A'B'|$ and $M' \in |A''B''|$ such that $BM||AA'$ and $B'M'||A'A''$ (see Figure 2.3). Applying repeatedly Theorem 2.1.1 we deduce that

$$\angle BMB' \equiv \angle AA'B' \equiv \angle AA''B'' \equiv \angle B'M'B'',$$

and

$$\angle MBB' \equiv \angle M'B'B''.$$

Also, by Proposition 2.1.1,

$$|BM| \equiv |AA'| \equiv |A'A''| \equiv |B'M'|.$$

Hence by Theorem ASA, $\Delta B'BM \equiv \Delta B''B'M'$. It follows that $|BB'| \equiv |B'B''|$, so B' is the midpoint of $|BB''|$.

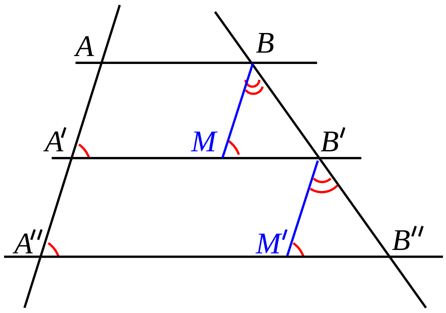

Figure 2.3: Proof of Thales' Theorem, the particular case

$\Leftarrow$ Choose $B_1 \in l'$ such that $BB_1||A'B'$. Then by what we just proved, $|B'B_1| \equiv |BB'|$. It follows that B_1 is on the ray $|B'B''$, and $|B'B_1| \equiv |B'B''|$. This can only happen if $B_1 = B''$, and we are done. □

Theorem 2.2.2. (Thales) Let l, l' be two distinct lines in the plane, $A, A', A'' \in l$ and $B, B', B'' \in l'$ such that $A' \in |AA''|$, $B' \in |BB''|$ and $A'B'||AB$. Then $A''B''||A'B'$ if and only if

$$\frac{|AA'|}{|A'A''|} = \frac{|BB'|}{|B'B''|}.$$

Proof. Case 1. $|AA'|/|A'A''| = 1/n$, n a positive integer. We argue on Figure 2.4.

$\Rightarrow$ Choose $A_1, A_2, \ldots, A_{n-1} \in |A'A''|$ such that

$$|A'A_1| \equiv |A_1A_2| \equiv \cdots \equiv |A_{n-1}A''| \equiv |AA'|.$$

Choose also $B_1, B_2, \ldots, B_{n-1} \in |B'B''|$ such that

$$A_1B_1||A_2B_2|| \cdots ||A_{n-1}B_{n-1}||A'B'$$

Applying the previous theorem successively, we obtain

$$|B'B_1| \equiv |BB'|, |B_1B_2| \equiv |B'B_1|, \ldots |B_{n-1}B''| \equiv |B_{n-2}B_{n-1}|.$$

It follows that $|B'B''| \equiv n|BB'|$, as desired.

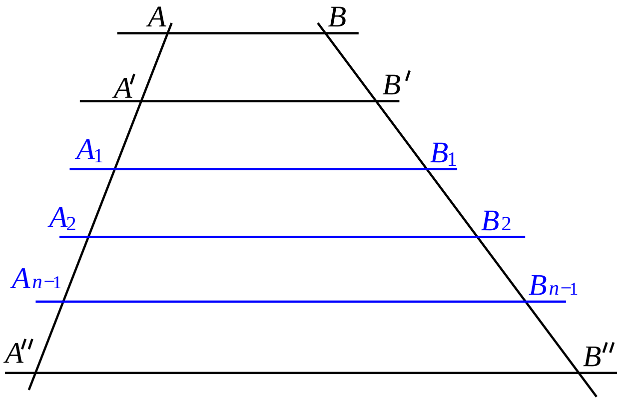

Figure 2.4: Proof of Thales' Theorem

$\Leftarrow$ We use the same figure but this time we choose the B_i's such that

$$|B'B_1| \equiv |B_1B_2| \equiv \cdots \equiv |B_{n-1}B''| \equiv |BB'|.$$

Applying again successively the previous theorem we deduce that

$$A_1B_1||A'B', A_2B_2||A_1B_1, \ldots, A''B''||A_{n-1}B_{n-1}.$$

We obtain that $A''B''||A'B'$, which proves this case.
Case 2. $|AA'|/|A'A''| = m/n$, m, n positive integers, $m > 1$. We argue on Figure 2.5.

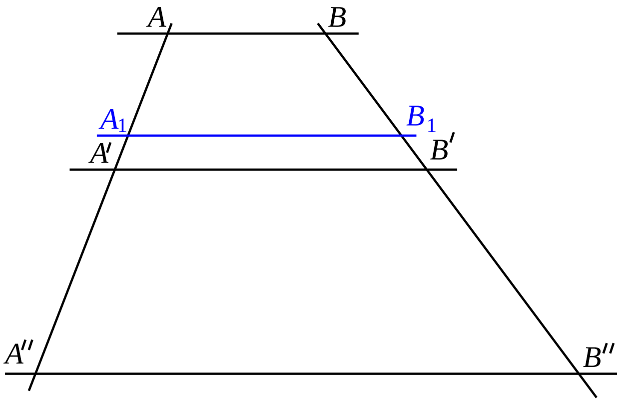

Figure 2.5: Proof of Thales' Theorem

$\Rightarrow$ Choose A_1B_1 such that

$$\frac{|A'A_1|}{|AA_1|} = \frac{1}{m-1} \text{ and } A_1B_1||A'B'.$$

By Case 1 of this theorem,

$$\frac{|B_1B'|}{|B_1B|} = \frac{1}{m-1} \text{ and } \frac{|B_1B'|}{|B'B''|} = \frac{1}{n}.$$

An algebraic computation shows that

$$\frac{|BB'|}{|B'B''|} = \frac{m}{n} = \frac{|AA'|}{|A'A''|},$$

as desired.

$\Leftarrow$ Choose now B_1 such that

$$\frac{|A'A_1|}{|AA_1|} = \frac{|B_1B'|}{|BB_1|} = \frac{1}{m-1}.$$

Then on the one hand, by Case 1 applied "bottom to top", $A_1B_1||A'B'$. On the other hand

$$\frac{|A_1A'|}{|A'A''|} = \frac{|B_1B'|}{|B'B''|} = \frac{1}{n}.$$

Applying again Case 1, we deduce that $A''B''||A'B'$.

Case 3. If $|AA'|/|A'A''| = x$, with x a real number, approximate x by rational numbers, then pass to the limit. In the process we use the axioms of continuity. Note that if $A_nB_n||AB$ for all n, and $A_n \to A_*$, $B_n \to B_*$, then $A_*B_*||AB$. □

Corollary 2.2.1. Let $\angle AOA'$ be an angle, $M \in |OA$ and $M' \in |OA'$. Then

$$AA'||MM' \quad \Leftrightarrow \quad \frac{|OM|}{|OA|} = \frac{|OM'|}{|OA'|}.$$

Proof. In Thales' theorem, choose $A = B$. □

2.2.2 Similar triangles

Definition. We say that ΔABC is similar to $\Delta A'B'C'$, and write $\Delta ABC \sim \Delta A'B'C'$, if

$$\angle A \equiv \angle A', \quad \angle B \equiv \angle B', \quad \angle C \equiv \angle C',$$

and

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|AC|}{|A'C'|}.$$

Theorem 2.2.3. If in ΔABC and $\Delta A'B'C'$ we have $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$ and $\angle C \equiv \angle C'$, then $\Delta ABC \sim \Delta A'B'C'$.

Proof. Let $B'' \in |AB$ and $C'' \in |AC$ such that $|AB''| \equiv |A'B'|$ and $|AC''| \equiv |A'C'|$ (Figure 2.6). Then by Theorem SAS, $\Delta AB''C'' \equiv \Delta A'B'C'$. Hence $\angle AB''C'' \equiv \angle A'B'C' \equiv \angle ABC$. By Theorem 2.1.1, $B''C''||BC$. Applying Thales' Theorem, we deduce that

$$\frac{|AB''|}{|B''B|} = \frac{|AC''|}{|C''C|}.$$

A little algebra gives

$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}. \tag{2.2.1}$$

Repeating the argument at vertex B, we deduce that

$$\frac{|A'B'|}{|AB|} = \frac{|B'C'|}{|BC|}. \tag{2.2.2}$$

Combining (2.2.1) and (2.2.2), we obtain

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|},$$

which together with the congruence of angles shows that $\Delta ABC \sim \Delta A'B'C'$. □

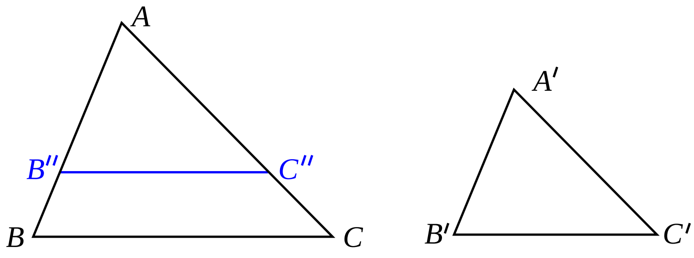

Figure 2.6: Similarity of triangles

Note that because of Theorem 2.1.2, in order for the triangles to be similar it suffices to check that two pairs of angles are respectively congruent.

Theorem 2.2.4. If in ΔABC and $\Delta A'B'C'$

$$\angle A \equiv \angle A' \text{ and } \frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|},$$

then $\Delta ABC \sim \Delta A'B'C'$.

Proof. Let $B'' \in |AB$ and $C'' \in |AC$ such that $|AB''| \equiv |A'B'|$ and $|AC''| \equiv |A'C'|$ (Figure 2.6). Then by Theorem SAS, $\Delta AB''C'' \equiv \Delta A'B'C'$. We have

$$\frac{|AB''|}{|AB|} = \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \frac{|AC''|}{|AC|},$$

so by Thales' theorem $B''C''||BC$. This implies that

$$\angle A'B'C' \equiv \angle AB''C'' \equiv \angle ABC$$
$$\angle A'C'B' \equiv \angle AC''B'' \equiv \angle ACB.$$

Because the triangles ΔABC and $\Delta A'B'C'$ have the angles respectively congruent, by Theorem 2.2.3, they are similar. □

Theorem 2.2.5. If in ΔABC and $\Delta A'B'C'$,

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|},$$

then $\Delta ABC \sim \Delta A'B'C'$.

Proof. Choose $B'' \in |AB$ and $C'' \in |AC$ such that $|AB''| \equiv |A'B'|$ and $|AC''| \equiv |A'C'|$ (Figure 2.6). Then

$$\frac{|AB''|}{|AB|} = \frac{|AC''|}{|AC|},$$

so by Theorem 2.2.4, $\Delta ABC \sim \Delta AB''C''$. Then

$$\frac{|B''C''|}{|BC|} = \frac{|AB''|}{|AB|} = \frac{|B'C'|}{|BC|},$$

which implies $|B''C''| \equiv |B'C'|$. We can apply Theorem SSS to conclude that $\Delta AB''C'' \equiv \Delta A'B'C'$. As the first of these two triangles is similar to ΔABC, so is the second. □

2.2.3 Problems

1. Let ΔABC be a triangle and let M, N, P be the midpoints of $|BC|$, $|AC|$, respectively $|AB|$. Let also M', N', P' be the midpoints of $|NP|, |MP|, |MN|$. Prove that the triangles ΔABC and $\Delta M'N'P'$ are similar.

2. Let ΔABC be an equilateral triangle. On the rays $|AB$, $|BC$, and $|CA$ choose the points M, N, P respectively, such that $|AM| \equiv |BN| \equiv |CP$. Prove that ΔMNP is equilateral.

3. Let ΔABC be an equilateral triangle and let $M \in AC$ and $N \in BC$ be such that $|AM|/|MC| = 1/2$ and $|BN|/|NC| = 1/3$. Let P be the intersection of AN and BM. Find $|BP|/|PM|$.

4. Let ΔABC be a triangle and let $|AD|$ and $|BE|$ be its altitudes from A and B. Prove that $\Delta CEB \sim \Delta CDA$ and that $\Delta CED \sim \Delta CBA$.

2.3 The four important points in a triangle

2.3.1 The incenter

Recall that Theorem 1.17 shows that the three angle bisectors of a triangle intersect at one point.

Definition. The point of intersection of the three angle bisectors of a triangle is called the incenter of the triangle.

The incenter is denoted by I.

2.3.2 The centroid

Definition. In a triangle, the segments that join the vertices with the midpoints of the opposite sides are called medians.

Theorem 2.3.1. In a triangle the three medians intersect at a point, called the *centroid* of the triangle. The centroid divides each median in the ration 2 : 1.

Proof. (The physical proof) Place three equal masses at the vertices A, B, C of the triangle. Combine the masses at B and C to a mass twice as large placed at the midpoint M of BC. Since M is the center of mass of the system formed by B and C, the old and the new system of masses have the same center of mass. The second system has its center of mass on $|AM|$, dividing $|AM|$ in the ration 2 : 1.

Now combine the masses at A and C, respectively A and B, to conclude that the center of mass of the system lies on the medians from B and C as well, and divides these medians in the ration 2 : 1. □

Proof. (The mathematical proof) Let M be the midpoint of $|BC|$ and $G \in |AM|$ such that $|AG|/|GM| = 2$. Construct $A' \in |GM$ such that $|MG| \equiv |MA'|$ (Figure 2.8). The diagonals of $BGCA'$ intersect at their midpoint, so by Lemma 2.1.1, $BGCA'$ is a parallelogram. It follows that $BG||A'C$ and $CG||A'B$.

Let N be the intersection of BG and AC, and P the intersection of CG and AB. By Thales' Theorem

$$\frac{|AN|}{|NC|} = \frac{|AG|}{|GA'|} = \frac{|AP|}{|PB|}.$$

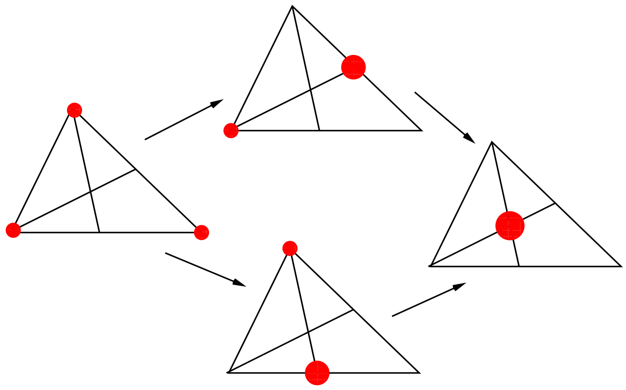

Figure 2.7: The medians of a triangle intersect at one point (physical proof)

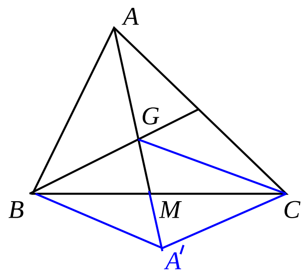

Figure 2.8: The medians of a triangle intersect at one point

It follows that N and P are the midpoints of the sides. So the three medians intersect at one point. Of course, as $|AG|/|GM| = 2$, the same must be true about $|BG|/|GN|$ and $|CG|/|GP|$, by just repeating the construction using the vertices B, respectively C instead of A. The theorem is proved. □

The centroid is denoted by G.

2.3.3 The orthocenter

Definition. In a triangle ΔABC, the *altitude* from A is the segment $|AD|$ with $D \in |BC|$ that is perpendicular to $|BC|$.

Note that the altitude from A is unique. A triangle has three altitudes, one for each vertex.

Theorem 2.3.2. In a triangle, the three altitudes intersect at one point, called the *orthocenter* of the triangle.

Proof. Let us consider first the case where ΔABC is acute. We argue on Figure 2.9. Let $|AD|$, $|BE|$, $|CF|$ be the three altitudes. Because $\angle C \equiv \angle C$ and $\angle ADC \equiv \angle BEC$, being right angles, it follows from Theorem 2.2.3 that $\Delta ADC \sim \Delta BEC$. Hence

$$\frac{|CE|}{|CD|} = \frac{|BC|}{|AC|}.$$

Using Theorem 2.2.4, we deduce that $\Delta CED \sim \Delta CBA$. It follows that $\angle EDC \equiv \angle BAC$.

A similar argument shows that $\Delta BDF \sim \Delta BAC$, so $\angle BDF \equiv \angle BAC$. It follows that $\angle EDC \equiv \angle BDF$ so $\angle FDA \equiv \angle ADE$, because they have congruent complements. We thus

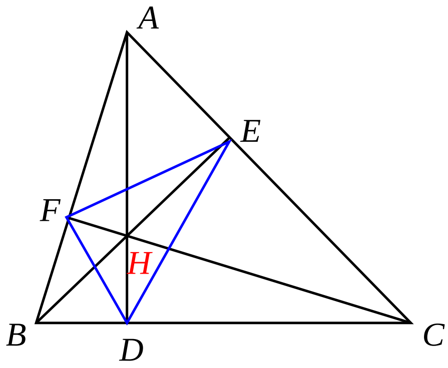

Figure 2.9: The altitudes of a triangle intersect at one point

found that $|DA$ is the angle bisector of $\angle FDE$. Similarly $|EB$ and $|FC$ are angle bisectors in ΔDEF. We conclude that $|AD|$, $|BE|$ and $|CF|$ intersect at the incenter of ΔDEF. The theorem is proved.

Now assume that $\angle BAC$ is obtuse. Let H be the intersection of BE and CF. Then BA and CA are altitudes in the acute triangle ΔHBC, so AH is also an altitude. This implies that AH is perpendicular to BC, and so $AH = AD$. Consequently AD passes through H and we are done. □

The orthocenter is denoted by H.

Definition. The points D, E, F are called the feet of the altitudes. Triangle ΔDEF is called the *orthic triangle* of ΔABC.

2.3.4 The circumcenter

Definition. Given two points in the plane, the *perpendicular bisector* of the segment $|AB|$ is the locus of the points P in the plane such that $|AP| \equiv |BP|$

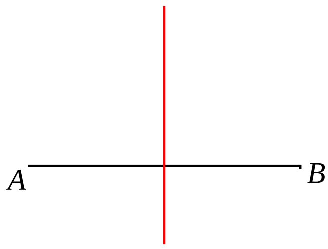

Figure 2.10: Perpendicular bisector of a segment

Proposition 2.3.1. The perpendicular bisector of a segment is the line perpendicular to the segment passing through its midpoint.

Proof. Let $|AB|$ be the segment, and M its midpoint. If P is a point on the perpendicular bisector, then ΔPAB is isosceles ($|PA| \equiv |PB|$), so $\angle PAB \equiv \angle PBA$. Given that also $|AM| \equiv |BM|$, we have that $\Delta PMA \equiv \Delta PMB$, by Theorem SAS. Thus $\angle PMA \equiv \angle PMB$, so both are right, showing that M is on the line perpendicular to AB passing through M.

Conversely, if P belongs to the line through M that is perpendicular to AB, the in the right triangles ΔMPA and ΔMPB, $|PM| \equiv |PM|$, and $|AM| \equiv |BM|$, so the triangles are congruent. It follows that $|PA| \equiv |PB|$, so P belongs to the perpendicular bisector. □

Theorem 2.3.3. In a triangle, the perpendicular bisectors of the sides intersect at one point, called the circumcenter of the triangle.

Proof. Let the triangle be ΔABC, and let M, N, P be the midpoints of $|BC|$, $|AC|$, and $|AB|$, respectively. Consider the perpendicular bisectors of $|AB|$ and $|AC|$. Because they are perpendicular to lines that are not parallel, they are not parallel themselves, so they intersect at a point O (see Figure 2.11). Then $|OA| \equiv |OB|$, and $|OA| \equiv |OC|$, which implies $|OB| \equiv |OC|$. It follows that O is on the perpendicular bisector of $|BC|$ as well, so the three perpendicular bisectors intersect at one point.

□

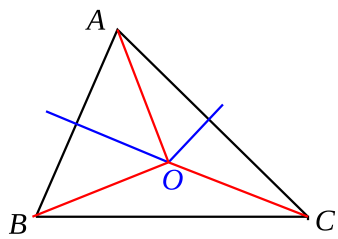

Figure 2.11: The perpendicular bisectors of the sides of a triangle intersect.

It is standard to denote the circumcenter by O.

2.3.5 The Euler line

Theorem 2.3.4. (L. Euler) In a triangle the circumcenter, centroid, and orthocenter are collinear. Moreover, the centroid lies between the circumcenter and the orthocenter, and divides the segment formed by the circumcenter and the orthocenter in the ratio 1 : 2.

Proof. Let M and N be the midpoints of $|BC|$ respectively $|AC|$. Then by Thales' Theorem $MN||AB$. Consequently, ΔCMN and ΔCBA have congruent angles, so they are similar. It follows that $|MN|/|AB| = |CM|/|CB| = 1/2$.

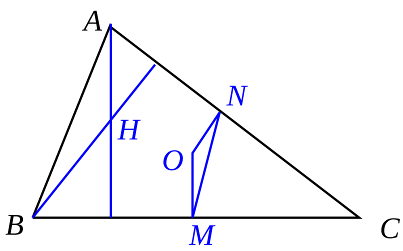

Figure 2.12: Proof of Euler's Theorem

On the other hand $OM||AH$ because both are perpendicular to BC, and $ON||BH$ because both are perpendicular to AC (see Figure 2.12). This combined with $MN||AB$ implies that triangles ΔHBA and ΔONM have parallel sides, hence they have congruent angles. It follows that $\Delta OMN \sim \Delta HAB$. We thus have

$$\frac{|OM|}{|AH|} = \frac{|ON|}{|BH|} = \frac{|MN|}{|AB|} = \frac{1}{2}. \tag{2.3.1}$$

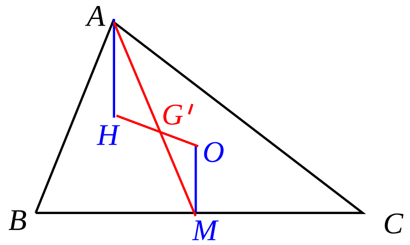

Figure 2.13: Proof of Euler's Theorem

Next, let G' be the intersection of the median $|AM|$ with $|OH|$ (Figure 2.13). Because $OM||AH$, we have that $\angle HAG' \equiv \angle OMG'$, $\angle AHG' \equiv \angle MOG'$. Also $\angle AG'H \equiv \angle OG'M$, being opposite angles. Hence $\Delta G'AH \sim \Delta G'MO$. We obtain

$$\frac{|G'H|}{|G'O|} = \frac{|AG'|}{|G'M|} = \frac{|OM|}{|AH|} = \frac{1}{2}.$$

Comparing this to (2.3.1), we deduce that $G = G'$, the centroid, and

$$\frac{|OG|}{|GH|} = \frac{1}{2}.$$

The theorem is proved. □

2.3.6 Problems

1. Show that in an equilateral triangle, the incenter, centroid, orthocenter, and circumcenter coincide.

2. Where does the circumcenter of a right triangle lie?

3. Show that in an isosceles triangle the Euler line passes through one of the vertices. Show conversely, that if in a triangle the Euler line passes through one of the vertices, then the triangle is isosceles.

2.4 Quadrilaterals

2.4.1 The centroid of a quadrilateral

The following results apply to all quadrilaterals, including the skew ones.

Lemma 2.4.1. The midpoints of the four sides of a quadrilateral form a parallelogram.

Proof. Let the quadrilateral be $ABCD$, and let M, N, P, Q be the midpoints of $|AB|$, $|BC|$, $|CD|$ and $|DA|$ respectively. Because

$$\frac{|AQ|}{|AD|} = \frac{|AM|}{|AB|} = \frac{1}{2},$$

by Thales' Theorem $QM||BD$. Similarly $PN||BD$. Therefore $QM||PN$.

A similar argument shows that $QP||PN$. The lemma is proved. □

Theorem 2.4.1. The two segments joining the midpoints of the opposite sides of a quadrilateral have the same midpoint.

Proof. (The physical proof) Let the quadrilateral be $ABCD$. Place at each vertex a weight of one pound. Then the system consisting of A and B has the center of mass at the midpoint of $|AB|$ and a combined weight of 2 pounds, and the system consisting of C and D has the center of mass at the midpoint of $|CD|$ and a total weight of 2 pounds. The center of mass of the system $ABCD$ is the midpoint of the segment that joins the midpoints of $|AB|$ and $|CD|$. A similar argument shows that the center of mass of this system is the segment that joins the midpoints of $|AD|$ and $|BC|$, and it is also the midpoint of the segment that joints the midpoints of $|AC|$ and $|BD|$. □

Proof. (The mathematical proof) Let the quadrilateral be $ABCD$, and let M, N, P, Q be the midpoints of $|AB|$, $|BC|$, $|CD|$, $|DA|$, respectively. By Lemma 2.4.1, $MNPQ$ is a parallelogram. By Proposition 2.1.1, the diagonals $|MP|$ and $|NQ|$ intersect at their midpoint. □

Definition. The common midpoint of the two segments that join the midpoints of opposite sides in a quadrilateral is called the centroid of the quadrilateral.

2.4.2 Problems

1. Show that the segment joining the midpoints of the diagonals of a quadrilateral passes through the centroid and is divided by it into two equal parts.

2. Let $ABCD$ be a quadrilateral. Show that the segments joining A with the centroid of ΔBCD, B with the centroid of ΔCDA, C with the centroid of ΔDAB, and D with the centroid of ΔABC intersect at one point.

2.5 Measurements

2.5.1 Measuring segments and angles

Euclidean geometry can be scaled, so there is no a priori unit of length for segments. To measure segments we start by fixing a segment $|OX|$ and declare it to have length 1.

We define the length of a segment $|AB|$ to be

$$\|AB\| = \frac{|AB|}{|OX|}.$$

Note that congruent segments have equal lenghts, and that length is additive, meaning that the length of the sum of two segments is the sum of the lengths of the segments. The axioms of continuity imply that any positive real number can be the length of a segment.

We want to measure angles so that congruent angles have equal measures and the measure of the sum of two angles is equal to the sum of the measures of the angles.

There are two standard ways of measuring angles. One was introduced by the Babylonians, in which a straight angle was declared to have $180°$. Then a right angle has $90°$, and the angles of an equilateral triangle have all $60°$.

There is a modern way of measuring angles, in which the straight angle is declared to have π radians (where π is the length of a semicircle of radius 1- we will talk later about this). Then the right angle has $\pi/2$ radians, and the angles of an equilateral triangle have $\pi/3$ radians.

2.5.2 Areas

To be able to talk about areas we need a unit of length. Thus we start by declaring a certain segment $|OX|$ to be of length 1.

Definition. The *interior of a triangle* consists of all points P with the property that there are two points M and N on the sides of the triangle such that P is between M and N.

Definition. A polygonal surface is the union of several triangles and their interiors.

We define a function

$$\Sigma \mapsto A(\Sigma),$$

called *area*, which associates to each finite union of polygonal surfaces a number, such that the following four properties are satisfied:

A1. The area of the square whose sides have length 1 is equal to 1.

A2. $A(\Sigma) > 0$ for all unions of poligonal surfaces Σ.

A3. Congruent triangles have equal areas.

A4. If Σ_1 and Σ_2 are disjoint, or if they share only a finite union of segments (on the boundary), then

$$A(\Sigma_1 \cup \Sigma_2) = A(\Sigma_1) + A(\Sigma_2).$$

As a corollary of A2 and A4, if $\Sigma \subset \Sigma'$, then $A(\Sigma) \leq A(\Sigma')$.

Theorem 2.5.1. The area of a rectangle $ABCD$ is equal to $\|AB\| \cdot \|BC\|$.

Proof. Case 1. The side-lengths of the rectangle are integer numbers.

Let the sides have lengths m respectively n. Divide the rectangle into unit squares, then count the squares. There are mn squares, and by condition A4, their total area is mn.

Case 2. The side-lengths of the rectangle are rational numbers.

Let the side-lengths be m_1/n and m_2/n, with m_1, m_2, n integers (use the common denominator!). Divide the unit square into n^2 congruent squares. By A4, the area of each square is $1/n^2$. Next, divide the rectangle into $m_1 \times m_2$ squares, each of size $\frac{1}{n} \times \frac{1}{n}$. Using A4 again, we conclude that the total area is

$$m_1 m_2 \cdot \frac{1}{n^2} = \frac{m_1}{n} \cdot \frac{m_2}{n},$$

as desired.

Case 3. The side-lengths are arbitrary numbers.

Note that A4 implies that the area is an increasing function, namely that if $\Sigma_1 \subset \Sigma_2$, then $A(\Sigma_1) < A(\Sigma_2)$ (because Σ_2 is the union of Σ_1 and the "piece" inbetween). We can approximate the rectangle from above and below by rectangles with rational side-lengths, and pass to the limit to obtain the conclusion. $\square$

Theorem 2.5.2. The area of a triangle is equal to half the product of the lengths of a side and of the altitude from the opposite angle.

Proof. We argue on Figure 2.14. The idea is to double the triangle to a rectangle. For that, let ΔABC be the triangle and consider the altitude $|AD|$. Let E and F be such that FB and EC are both perpendicular to BC, and EF is parallel to BC. Then $\angle FAB \equiv \angle DBA$, and because $|AB| \equiv |AB|$, by Theorem 1.4.6 $\Delta AFB \equiv \Delta BDA$. For a similar reason $\Delta AEC \equiv \Delta CDA$. Using A3 and A4, we conclude that

$$\begin{aligned} A(\Delta ABC) &= \frac{1}{2}[A(\Delta ABD) + A(\Delta AFB) + A(\Delta ADC) + A(\Delta AEC)] = \frac{1}{2}A(BCEF) \\ &= \frac{1}{2}\|AD\| \cdot \|BC\|. \end{aligned}$$

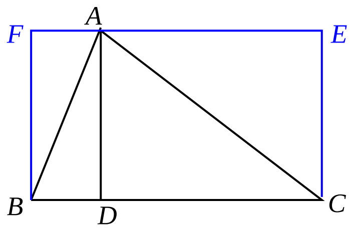

Figure 2.14: The area of a triangle

□

Corollary 2.5.1. The area of a right triangle is equal to the half the product of the lengths of the sides adjacent to the right angle.

Theorem 2.5.3. Let $ABCD$ be a trapezoid, $AD||BC$. Let M be a point on BC such that AM is perpendicular to BC. Then

$$A(ABCD) = \frac{\|AM\|}{2}(||AD|| + ||BC||).$$

Proof. Divide the trapezoid into the triangles ΔACD and ΔABC as shown in Figure 2.15, then apply Theorem 2.5.2 to these triangles. □

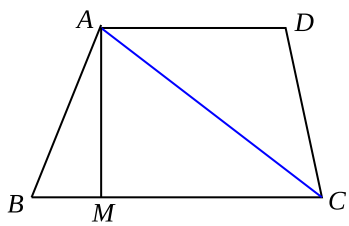

Figure 2.15: The area of a trapezoid

Corollary 2.5.2. The area of a parallelogram is equal to the product of the length of the base and the height.

2.5.3 The Pythagorean Theorem

Theorem 2.5.4. Let ΔABC be a right triangle, with $\angle A$ right. Then

$$\|AB\|^2 + \|AC\|^2 = \|BC\|^2.$$

2.5.4 Problems

1. What is the area of a regular poligon of side-length 1?
2. Prove the Pythagorean theorem.
3. Prove that the area of a parallelogram with side-lengths 2 and 3 does not exceed 6.

2.6 The circle

Definition. The circle of center O and radius $|AB|$ is the locus of points M such that $|OM| \equiv |AB|$.

The radius is usually specified by a segment, or by a length. In the latter case it is denoted by one letter.

Given two circles of centers O_1 and O_2 and radii R_1 and R_2, their relative position can be:

1. one interior to the other, if $\|O_1O_2\| < |R_1 - R_2|$;
2. interior tangent, if $\|O_1O_2\| = |R_1 - R_2|$;
3. intersecting, if $|R_1 - R_2| < \|O_1O_2\| < R_1 + R_2$;
4. exterior tangent, if $\|O_1O_2\| = R_1 + R_2$;
5. exterior to each other, if $\|O_1O_2| > R_1 + R_2$.

These five situations are shown in Figure 2.16.

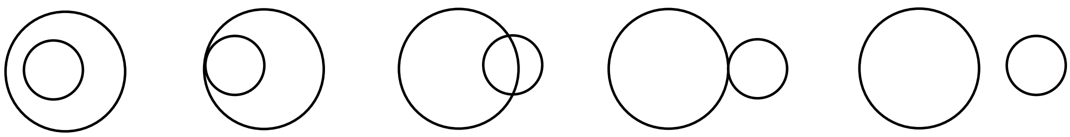

Figure 2.16: The relative position of two circles

The segment determined by two points on the circle is called *chord*. If the center of the circle belongs to the chord, the chord is called *diameter*. A line that intersects a circle at exactly one point is called *tangent*.

2.6.1 Measuring angles using arcs

Definition. Given a circle of center O and two points A and B on this circle, the arc $\overset{\frown}{AB}$ is the set of points on the circle that lie inside the angle $\angle AOB$ together with A and B.

Definition. We define the measure of the arc $\overset{\frown}{AB}$ by the equality $m(\overset{\frown}{AB}) = m(\angle AOB)$.

Theorem 2.6.1. Let A, B, C be points on a cricle. Then

$$m(\angle BAC) = \frac{1}{2} m(\overset{\frown}{BC}),$$

where $\overset{\frown}{BC}$ is the arc of the circle that lies inside the angle.

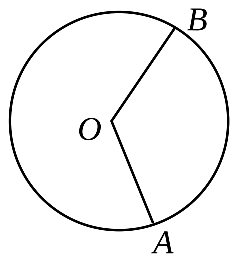

Figure 2.17: Measuring angles by arcs

Proof. Case 1. $O \in |AC|$ (Figure 2.18). The triangle ΔOAB is isosceles, so $\angle OAB \equiv \angle OBA$. By the Exterior Angle Theorem, $\angle BOC \equiv \angle OAB + \angle OBA$. It follows that

$$m(\angle BAC) = \frac{1}{2}m(\angle BOC) = \frac{1}{2}m(\overset{\frown}{BC}).$$

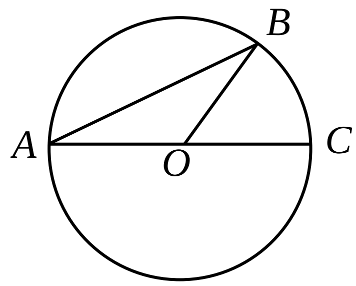

Figure 2.18: Inscribed angles, case 1

Case 2. O is in the interior of the angle $\angle BAC$ (Figure 2.19). Let M be on the circle such that $O \in |AM|$. Then

$$\begin{aligned} m(\angle BAC) &= m(\angle BAM) + m(\angle MAC) = \frac{1}{2}m(\overset{\frown}{BM}) + \frac{1}{2}m(\overset{\frown}{MC}) \\ &= \frac{1}{2}m(\overset{\frown}{BC}). \end{aligned}$$

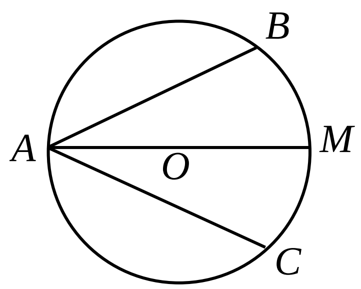

Figure 2.19: Inscribed angles, case 2

Case 3. O is outside the angle BAC (Figure 2.20). Let M be on the circle such that $O \in |AM|$.

Assume without loss of generality that C is in the interior of the angle $\angle BAM$. Then

$$m(\angle BAC) = m(\angle BAM) - m(\angle CAM) = \frac{1}{2}m(\widehat{BM}) - \frac{1}{2}m(\widehat{CM})$$
$$= \frac{1}{2}m(\widehat{BC}).$$

The theorem is proved.

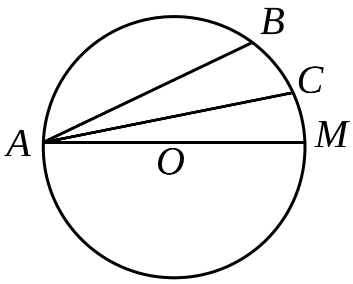

Figure 2.20: Inscribed angles, case 3

□

Theorem 2.6.2. Assume that the chords $|AB|$ and $|CD|$ of a circle intersect at M. Then

$$m(\angle BMD) = \frac{1}{2}m(\widehat{BD}) + \frac{1}{2}m(\widehat{AC}),$$

where the two arcs lie inside the angle and its opposite.

Proof. We argue on Figure 2.21. By the euclidean version of the Exterior Angle Theorem, $\angle BMD \equiv \angle MAD + \angle MDA$. It follows that

$$m(\angle BMD) = m(\angle MAD) + m(\angle MDA) = \frac{1}{2}m(\widehat{AC}) + \frac{1}{2}m(\widehat{BD}).$$

We are done. □

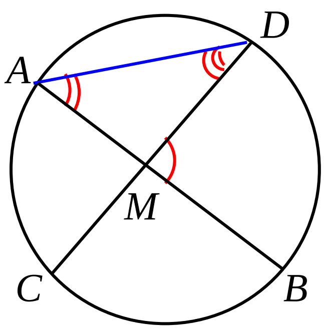

Figure 2.21:

Theorem 2.6.3. Assume that the lines of support of the chords AB and CD of a circle intersect outside the circle at M such that A is between M and B and C is between M and D. Then

$$m(\angle AMC) = \frac{1}{2}m(\widehat{BD}) - \frac{1}{2}m(\widehat{AC}),$$

where the arcs are defined as to lie inside the angle.

Proof. We argue on Figure 2.22. Again we apply the Exterior Angle Theorem to conclude that $\angle BCD =\equiv \angle MBC + \angle BMC$. We have

$$m(\angle AMC) = m(\angle BCD) - m(\angle MBC) = \frac{1}{2}m(\widehat{BD}) - \frac{1}{2}m(\widehat{AC}),$$

as desired. □

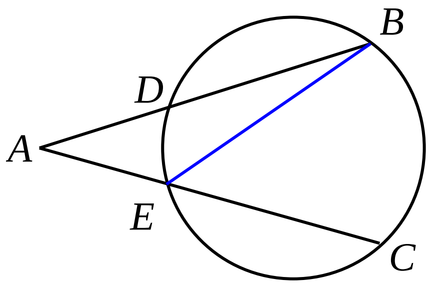

Figure 2.22:

Proposition 2.6.1. Let $\angle BAC$ be an angle such that $|AC|$ is a chord in a circle and AB is tangent to the circle. Then

$$m(\angle BAC) = \frac{1}{2}m(\widehat{AC}),$$

where the arc is taken as to lie inside the angle.

Proof. Consider the case where $|AB|$ is a chord, then rotate the chord until it becomes tangent. □

Corollary 2.6.1. The tangent is perpendicular to the radius at the point of contact.

2.6.2 The circumcircle and the incircle of a triangle

Theorem 2.6.4. Given a triangle ΔABC, there is a unique circle containing the vertices A, B, C.

Proof. Let us recall the proof of Theorem 2.3.3. There, we showed that the three perpendicular bisectors of the sides $|AB|$, $|BC|$, and $|CA|$ intersect at a point O which lie at equal distance from the vertices. Hence O is the center of a circle passing through A, B, C.

If there were a second circle containing A, B, C, then using the characterization of the perpendicular bisector of a segment given by Proposition 2.3.1, we deduce that the center of that circle is also at the intersection of the perpendicular bisectors of the sides, hence it equals O. And so the circle is the same, which proves uniqueness. □

Definition. The unique circle passing through the vertices of a triangle is called the circumcircle.

Theorem 2.6.5. Given a triangle ΔABC, there is a unique circle tangent to the segments $|AB|$, $|BC|$, $|AC|$.

Proof. In Theorem 1.17 we showed that the angle bisectors of a triangle intersect at one point by showing that this point, denoted by I, is at equal distance from the sides. Let D, E, F be the projections of I onto BC, AC, and AB. Because because in triangle ΔIBC, $\angle IBC$ and $\angle ICB$ are acute, D is between B and C. Similarly we deduce that the other projections are on the sides.

Because $|ID| \equiv |IE| \equiv |IF|$, I is the circumcenter of ΔDEF. And because IE is perpendicular to BC, BC is tangent to the circle. For the same reason AC and AB are tangent, which shows that the circumcircle of ΔDEF has the desired property.

Uniqueness follows from the fact that if another circle is tangent to the sides, and I' is its center, then I' is equally distanced from the sides, and hence it lies on the angle bisector of each angle of ΔABC. It follows that $I' = I$, and consequently the circle is unique. □

Definition. The unique circle tangent to the three sides of the triangle is called the *incircle*.

Remark 2.6.1. If we require the circle to be only tangent to the lines AB, AC, and BC, there are *four* such circles, the other three are called the excircles.

2.6.3 Cyclic quadrilaterals

Theorem 2.6.6. A quadrilateral is cyclic if and only if two opposite angles add up to a straight angle.

Proof. Let the quadrilateral be $ABCD$ (see Figure 2.23). Assume that it is cyclic. Then

$$m(\angle DAC) + m(\angle DCB) = \frac{1}{2}m(\overset{\frown}{DCB}) + \frac{1}{2}m(\overset{\frown}{DAB}) = \frac{1}{2}2\pi = \pi.$$

Hence $\angle DAC$ and $\angle DCB$ add up to a straight angle.

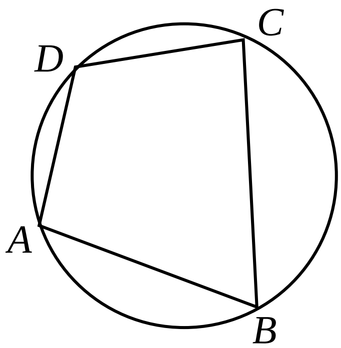

Figure 2.23:

Conversely, assume that $m(\angle DAB) + m(\angle DCB) = \pi$. Arguing by contradiction, assume that $ABCD$ is not cyclic.

Choose $C' \in BC$ such that A, B, C', D lie on a circle. We only discuss the case where $C' \in |BC|$, the other two cases ($B \in |CC'|$ and $C \in |BC'|$) being analogous (see Figure 2.24). Then $\angle DCB \equiv \angle DC'B$, since both have $\angle DAB$ as their supplement. But this contradicts the Exterior Angle Theorem in $\Delta DCC'$. Hence the conclusion. □

Theorem 2.6.7. A quadrilateral is cyclic if and only if the angle formed by one side with a diagonal is congruent to the angle formed by the opposite side with the other diagonal.

Proof. If $ABCD$ is cyclic then, as can be seen in Figure 2.25,

$$m(\angle BAC) = \frac{1}{2}m(\overset{\frown}{BC}) = m(\angle BDC).$$

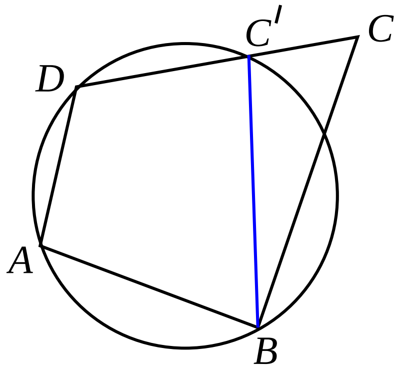

Figure 2.24:

Conversely, if $\angle BAC \equiv \angle BDC$ but $ABCD$ is not ciclic, let A' be the second intersection of AB with the circumcircle of ΔBCD. We discuss only the case where $A' \in |AB|$ (Figure 2.26), the other cases being similar. We have

$$\angle BAC \equiv \angle BDC \equiv \angle BA'C$$

which contradicts the exterior angle theorem for $\Delta DAA'$. Thus our assumption was false, showing that $ABCD$ is cyclic.

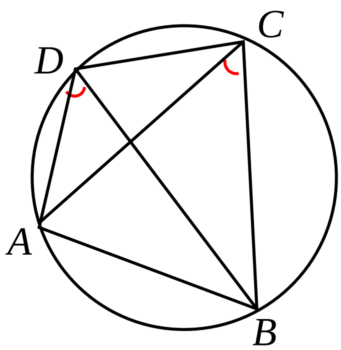

Figure 2.25:

□

Corollary 2.6.2. A quadrilateral with two opposite right angles is cyclic.

Corollary 2.6.3. The rectangle is the only cyclic parallelogram.

Corollary 2.6.4. The isosceles trapezoid is the only cyclic trapezoid.

Theorem 2.6.8. (Simson's line) Let ΔABC be a triangle and let P be a point on the circumcircle of ΔABC. Then the projections of P onto the lines AB, AC, and BC are collinear.

Proof. Without loss of generality, let us assume that P is on the arc $\overset{\frown}{AC}$. Let L, M, N be the projections onto BC, AC, respectively AB. Let us also assume that L is between B and C, M between A and C, and A between B and N, as in Figure 2.27, the proofs for the other possible configurations being similar.

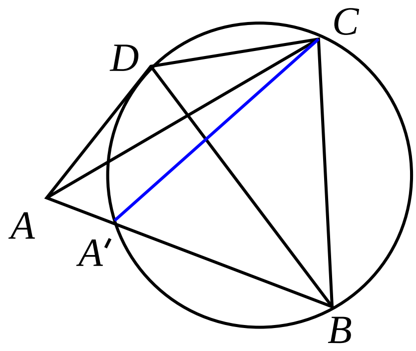

Figure 2.26:

The quadrilateral $AMPN$ is cyclic. Hence

$$\angle PNM \equiv \angle PAM = \angle PAC.$$

The quadrilateral $PNBL$ is cyclic, so

$$\angle PNL \equiv \angle PBL = \angle PBC.$$

The quadrilateral $ABCP$ is cyclic, hence

$$\angle PAC \equiv \angle PBC.$$

It follows that

$$\angle PNM \equiv \angle PNL,$$

so $|NM = |NL$, which means that the points L, M, N are collinear.

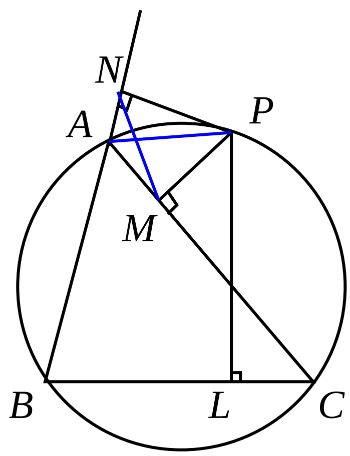

Figure 2.27: Simson's line

□

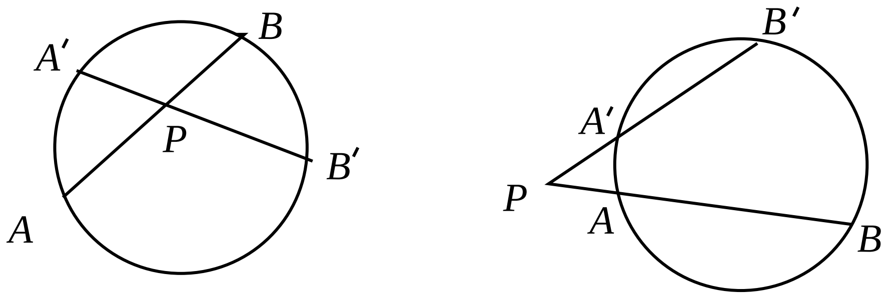

Figure 2.28: Power of a point

2.6.4 Power of a point with respect to a circle

Theorem 2.6.9. If P is a point in the plane of a circle, and AB and $A'B'$ are two lines passing through P such that A, B, A', B' are on the circle, then

$$\|PA\| \cdot \|PB\| = \|PA'\| \cdot \|PB'\|.$$

Proof. If P is on the circle, then the product is zero in both cases.

If P is outside the circle, suppose that A is between P and B and that A' is between P' and B (Figure 2.29). Then $ABB'A'$ is cyclic, hence $\angle A'AB$ and $\angle BB'A'$ add up to a straight angle. It follows that $\angle PAA' \equiv \angle PB'B$. This combined with the fact that $\angle APA' = \angle BPB'$ implies that $\Delta PAA' \sim \Delta PB'B$. Thus

$$\frac{|PA'|}{|PB|} = \frac{|PB'|}{|PA|}.$$

Multiplying out the denominators we obtain the equality from the statement.

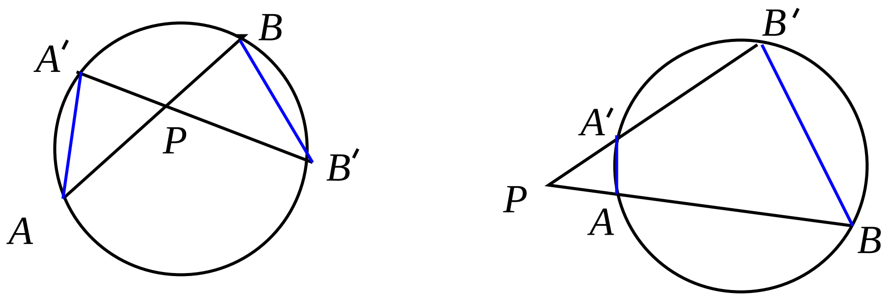

Figure 2.29: Proof of power of a point property

The case where P is inside the circle is similar (look carefully at Figure 2.29. $\square$

Definition. Given a point P and a circle, let AB be a line through P, with A and B on the circle. The power of P with respect to the circle is equal to $\|PA\| \cdot \|PB\|$ if P is outside of the circle, to 0 if P is on the circle, and to $-\|PA\| \cdot \|PB\|$ if P is inside the circle.

Proposition 2.6.2. Given a point P and a circle of center O and radius R, the power of P with respect to the circle is equal to $\|PO\|^2 - R^2$.

Proof. The property is true if P is on the circle. Let A and B be the intersections of line PO with the circle. If P is outside of the circle, then

$$\|PA\| \cdot \|PB\| = (\|PO\| + R)(\|PO\| - R) = \|PO\|^2 - R^2.$$

If P is inside the circle, then

$$-\|PA\| \cdot \|PB\| = -(R + \|PO\|)(R - \|PO\|) = -(R^2 - \|PO\|^2) = \|PO\|^2 - R^2.$$

□

Theorem 2.6.10. The locus of points that have equal powers with respect to two circles is a line.

Proof. Assume the circles have centers and radii O_1, O_2 respectively R_1, R_2. Let P be a point on the locus, and consider Q on O_1O_2 such that PQ is orthogonal to O_1O_2 (see Figure 2.30).

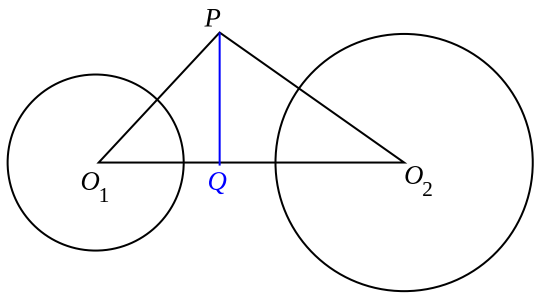

Figure 2.30: The radical axis

By the Pythagorean theorem,

$$\|PO_1\|^2 = \|PQ\|^2 + \|QO_1\|^2 \text{ and } \|PO_2\|^2 = \|PQ\|^2 + \|QO_2\|^2.$$

Subtracting we obtain

$$\|QO_1\|^2 - \|QO_2\|^2 = \|PO_1\|^2 - \|PO_2\|^2,$$

and the latter is equal to $R_1^2 - R_2^2$, by Proposition 2.6.2. This completely determines the position of Q on the line O_1O_2, and consequently P must belong to a line that passes through this point Q and is perpendicular to O_1O_2. Conversely, if P belongs to this line, then the same application of the Pythagorean theorem implies that P has equal powers with respect to the two circles. □

Definition. The set of points that have equal powers with respect to two circles is called the *radical axis* of the two circles.

Theorem 2.6.11. Given three circles with noncollinear centers, there is a unique point in the plane, called *radical center*, that has equal powers with respect the three circles.

Proof. Let the circles by $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$. Let P be the intersection of the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_2$ with the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_3$. Then P has equal power with respect to $\mathcal{C}_1$ and $\mathcal{C}_2$, and equal power with respect to $\mathcal{C}_1$ and $\mathcal{C}_3$. Consequently, it has equal power with respect to all three circles. □

Corollary 2.6.5. Given three circles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, whose centers are noncollinear, the radical axes of the pairs $\{\mathcal{C}_1, \mathcal{C}_2\}$, $\{\mathcal{C}_1, \mathcal{C}_3\}$, $\{\mathcal{C}_2, \mathcal{C}_3\}$, intersect at the radical center of the three circles.

2.6.5 Problems

1. Given two circles that are exterior tangent at M, consider a line AB that passes through M such that A is on the first circle and B is on the second circle. Prove that the measure of one of the arcs determined by A and M is equal to the measure of one of the arcs determined by B and M.

2. Let ΔABC be a triangle, and let $|AA'|$, $|BB'|$, $|CC'|$ be diameters in the circumcircle. Prove that $\Delta ABC \equiv \Delta A'B'C'$.

3. Let $ABCD$ be a quadrilateral with the property that AC and BD are orthogonal. Let M, N, P, Q be the midpoints of the sides $|AB|$, $|BC|$, $|CD|$, and $|DA|$ respectively. Prove that $MNPQ$ is cyclic.

4. Let ΔABC be an acute triangle, and let $D \in |BC|$ and $E \in |AC|$ be the feet of the altitudes. Prove that $\angle ABE \equiv \angle ADE$.

5. In the triangle ΔABC, $\angle A = 60°$ and the angle bisectors $|BB'$ and $|CC'$ intersect at I. Prove that $|IB'| \equiv |IC'|$.

6. Let $\angle AOB$ be a right angle, M and N points on the rays $|OA$ respectively $|OB$ and let $MNPQ$ be a square such that MN separates points O and P. Find the locus of the center of the square when M and N vary.

7.* (Euler's circle) Show that in a triangle the feet of the altitudes, the midpoints of the sides, and the midpoints of the segments connecting the orthocenter to the vertices are on a circle.

8. Assume that the circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ have noncollinear centers and assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B, $\mathcal{C}_1$ and $\mathcal{C}_3$ intersect at C and D, and $\mathcal{C}_2$ and $\mathcal{C}_3$ intersect at E and F. Prove that AB, CD, and EF intersect at one point.

9. Let P be a point inside a circle such that there exist three chords through P of equal length. Prove that P is the center of the circle.

10.* (Ţiţeica's Five Lei Coin Problem) Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ be three circles of equal radii that pass through a common point P and intersect pairwise at A, B, C. Prove that the circumcircle of ΔABC has the same radius as $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$.

Chapter 3

Geometric Transformations

3.1 Isometries

3.1.1 Translations

A vector is an oriented segment. More precisely, a vector is an equivalence class of oriented segments, all parallel, congruent, and pointing in the same direction. A vector is denoted by a lower case letter with an arrow on top of it, or by two upper case letters, the endpoints, with an arrow on top.

Definition. Given a vector $\vec{v}$, the translation of a point A by $\vec{v}$ is a point A' such that $\vec{AA'} = \vec{v}$.

Theorem 3.1.1. 1. The translation of a segment is a segment parallel and congruent to it.
2. The translation of an angle is an angle congruent to it.
3. The translation of a triangle, is a triangle congruent to it.

Proof. For the proof of 1, let $|AB|$ be a segment, and let A' and B' be the translates of A and B. Then $|AA'|$ and $|BB'|$ are parallel and congruent, so $ABB'A'$ is a parallelogram. It follows that $|AB|$ and $|A'B'|$ are parallel and congruent.

That the translation of a triangle is a triangle congruent to it follows from 1 and Theorem SSS. Finally, we get 2 by including the angle in a triangle. □

The addition of vectors is defined using the parallelogram rule. This turns the set of vectors into a group, with the identity element being the zero vector, and the negative of a vector being the vector of same length pointing in the oposite direction.

The set of translations endowed with composition is a group which is isomorphic to the group of vectors.

Example 3.1.1. *Two cities lie on the opposite sides of a river, at some distance from the river. The river has non-negligible width. Construct a road and a bridge that minimize the distance traveled between the cities. (Note: The bridge should be perpendicular to the river.)*

Proof. Let the river be defined by two parallel lines l and l', with city A on the shore l and the city B on the shore l'. Let $\vec{v}$ be a vector orthogonal to l and l', of length equal to the distance between l and l' and pointing from l to l'. Let A' be the image of A through the translation. Consider the line $A'B$ and let M' be the intersection of l' and $A'B$. Of course, M' is the image through the translation of a point M on l. We claim that the bridge should be $|MM'|$.

Indeed, for another location of the bridge, say $|NN'|$, the total distance

$$|AN| + |NN'| + |N'B| \equiv |A'N'| + |MN| + |N'B| < |AB| + |MN| = |AM| + |MM'| + |M'B|,$$

where the inequality is follows from the triangle inequality. The problem is solved. □

3.1.2 Reflections

Definition. Given a line l in the plane, the *reflection* of a point A over the line l is a point A' such that $AA' \perp l$, A and A' are in different half-planes determined by l and the distances from A and A' to l are equal.

Theorem 3.1.2. 1. The reflection of a segment over a line is a segment congruent to it. 2. The reflection of an angle over a line is an angle congruent to it. 3. The reflection of a triangle over a line is a triangle congruent to it.

Proof. 1. Let l be the line over which we reflect. Consider a segment $|AB|$ and let A' and B' be the reflections of A and B over l. Let AA' intersect l at M and BB' intersect l at N. Then $|AM| \equiv |A'M|$ and $|MN| \equiv |MN|$, so the right triangle ΔAMN and $\Delta A'MN$ are congruent. It follows that $|AN| \equiv |A'N|$ and $\angle ANM \equiv \angle A'NM$. Hence $\angle ANB \equiv \angle A'NB'$, and Theorem SAS implies $\Delta ABN \equiv \Delta A'B'N$. It follows that $|AB| \equiv |A'B'|$.

By the same argument, for every $P \in |AB|$, with P' its image under the reflection, $|PA| \equiv |PA'|$ and $|PB| \equiv |P'B'|$, and hence $|A'B'| \equiv |A'P'| + |P'B'|$. This implies that $P' \in |AB|$. It is not hard to see that every point on $|A'B'|$ is the image of a point on $|AB|$. This proves the first part.

3. This follows from 1, by using Theorem SSS.

2. By 1, the reflection of an angle is an angle, namely the two rays that determine the angle are mapped to two rays. To show that the reflected angle is congruent to the original one, place the original angle in a triangle, then reflect the triangle and use 3. □

Proposition 3.1.1. If σ is the reflection with respect to a line l, then σ^2 is the identity map. Consequently, the two-element set formed by a reflection and the identity map is a group.

Remark 3.1.1. This is a geometric realization of the group $\mathbb{Z}_2$.

3.1.3 Rotations

Definition. The rotation about O by angle α maps a point A to a point A' such that $m(\angle AOA') = \alpha$ and $|OA| \equiv |OA'|$.

We agree to measure α in degrees. One distinguishes between positive (counter-clockwise) rotations, and negative (clockwise) rotations.

Theorem 3.1.3. 1. The rotation of a segment is a segment congruent to the original one.
2. The rotation of an angle is an angle congruent to the original one.
3. The rotation of a triangle is a triangle congruent to the original one.

Proof. Exercise. □

Proposition 3.1.2. 1. The inverse of the counter-clockwise rotation about O by α is the counter-clockwise rotation by $360° - \alpha$ about the same point.
2. The composition of the rotation of angle α_1 about O and the rotation of angle α_2 about O is the rotation of angle $\alpha_1 + \alpha_2$ about O.

Using addition of angles, and composition of rotations we can define rotations for every real value of α.

Corollary 3.1.1. The set of all rotations about a point is a group.

Theorem 3.1.4. (D. Pompeiu) Given an equilateral triangle ΔABC and a point P that does not lie on the circumcircle of ΔABC, one can construct a triangle of side-lengths $\|PA\|, \|PB\|, \|PC\|$.

Proof. Rotate ΔABC about C by $60°$. Then $|AP'| \equiv |B'P'| \equiv |BP|$. Also $|CP| \equiv |CP'|$, which means that $\Delta CPP'$ is an isosceles triangle with one angle of $60°$. It follows that $\Delta CPP'$ is equilateral (prove it!). Then in $\Delta APP'$, $|AP|$, $|AP'| \equiv |BP|$, and $|PP'| \equiv |CP|$, so this triangle has the desired property. □

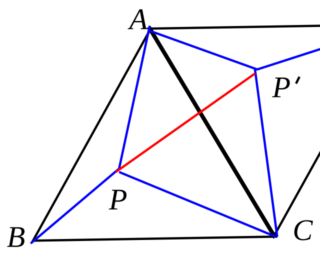

Figure 3.1: Proof of Pompeiu's Theorem

Example 3.1.2. *Given a closed polygonal line, show that it contains 3 points which form an equilateral triangle.*

Solution: To prove this fact, rotate the polygonal line by $60°$ around a point A on a side. Let B be a point where the new line intersects the old. Point C which is the preimage of B together with A and B form an equilateral triangle.

3.1.4 Isometries

Definition. A transformation of the plane which preserves lengths is called *isometry*.

Rotations, reflections, and translations are isometries.

Remark: An isometry is a one-to-one transformation of the plane. The next result will also prove that it is onto.

Theorem 3.1.5. Every isometry of the plane is of the form $r \circ t$ or $s \circ r \circ t$, where s is a reflection, r is a rotation, and t is a translation.

Proof. Let f be the isometry. Consider a segment $|AB|$, and let $A' = f(A)$ and $B' = f(B)$. If M is on $|AB|$, and $M' = f(M)$, then $|A'M'| + |M'B'| \equiv |AM| + |MB| \equiv |AB| \equiv |A'B'|$, so M' is on $|A'B'|$, and moreover, any M' on $|A'B'|$ can be obtained as the image of a point M. This shows that $f(|AB|) = |A'B'|$. In fact by slightly modifying the argument we deduce that $f(AB) = A'B'$. Let r and t be a rotation and a translation such that $r \circ t(|AB|) = |A'B'|$. Then rt restricted to AB is equal to f.

Consider a point P which does not belong to AB. Then because $|AP| \equiv |A'P'|$ and $|BP| \equiv |B'P'|$, there are only two locations that $P' = f(P)$ can have, namely either $P' = r \circ t(P)$, or $P' = s \circ r \circ t(P)$, where s is the reflection over AB.

Now let us show that an isometry is completely determined by the image of 3 non-collinear points. Let A, B, P be the three noncollinear points mapped to A', B', P'. We saw above that the restriction of the isometry is completely determined when restricted to AB, AP, and BP. Let Q be a point that does not lie on any of these lines. Then Q is mapped to one of two points lying on one side or the other of AB. If Q is not separated by AB from P, then $|PQ|$ does not intersect $A'B'$, so $|P'Q'|$ does not intersect AB (or else the isometry wouldn't be one-to-one). This shows that there is a unique choice for the location of Q'. Similarly, if $|PQ|$ intersects AB, then $|P'Q'|$ intersects $A'B'$, so again the location of Q' is unique. This proves our claim.

We conclude that the isometry can only be either $r \circ t$ or $s \circ r \circ t$, and the theorem is proved. □

3.1.5 Problems

1. What is the composition of two reflections over parallel lines?
2. What is the composition of two reflections over non-parallel lines?
3. Show that every isometry is a composition of reflections.
4. Given a polygon, and a point P in its interior, show that there are two points A and B on the polygon such that P is the midpoint of $|AB|$.
5. Two towns are on the same side of the river, at some distance from the river. The want to build a common water pump that would supply both with water. In what location should the pump be build in order to minimize the total length of the two pipes that connect it to the cities?

3.2 Homothety and Inversion

3.2.1 Homothety

Let O be a point in the plane and r a real number.

Definition. The homothety of center O and ratio r sends a point P to a point P' such that $\vec{OP'} = r\vec{OP}$.

Theorem 3.2.1. Homothety maps a maps a segment to a segment parallel to it whose length is $|r|$ times the length of the original segment.

Corollary 3.2.1. Homothety maps an angle to an angle congruent to it. Homothety maps a triangle to a triangle similar to it, with similarity ration $|r|$.

3.2.2 Inversion

Definition. Given a circle of center O and radius $r > 0$, the inverse of a point $P \neq O$ with respect to this circle is a point P' on the ray $|OP$ such that

$$\|OP\| \cdot \|OP'\| = r^2.$$

By abuse of language, we map O to the "point at infinity". With this convention, the square of the inversion is the identity map.

Theorem 3.2.2. Inversion maps circles through O into lines not passing through O, and lines not passing through O into circles through O. Inversion maps lines through O into themselves, and circles that do not pass through O into circles that do not pass through O.

Theorem 3.2.3. Inversion distorts distances according to the following formula

$$\|A'B'\| = \frac{r^2}{\|OA\| \cdot \|OB\|}\|AB\|.$$

Chapter 4

Non-Euclidean Geometry

4.1 The negation of Euclid's fifth postulate

Postulate: There is a line l, and a point A that does not belong to l, such that through A pass two lines that do not intersect l.

Theorem 4.1.1. For the line l and the point A from the above postulate, there exist infinitely many lines passing through A that do not intersect l.

Proof. Let l_1 and l_2 be the lines through A that do not intersect l. There is a ray of l_2 that is separated from l by l_1. Pick P on this ray and Q on l. Let R be the intersection of $|PQ|$ with l_1. Choose one of the infinitely many points of the segment $|PR|$, call this point M.

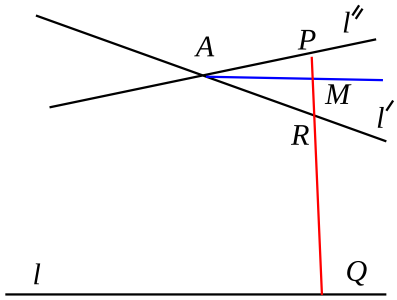

Figure 4.1: Proof that there are infinitely many parallels

Assume that AM intersects l at some point S. Then l_1 intersects side $|MQ|$ of ΔMQS, but does not intersect side $|QS|$, and it cannot intersect $|MS|$ since it already intersected MA at A. This contradicts the Axiom of Pasch, proving that AM does not intersect l. Since each point $M \in |PR|$ determines a different line, there are infinitely many lines that pass through A and don't intersect l. $\square$

4.2 Euclid's fifth postulate and the sum of the angles of a triangle

Theorem 4.2.1. The sum of the angles of a triangle is less than or equal to a straight angle.

Proof. We will need two results.

Lemma 4.2.1. The sum of two interior angles of a triangle is less than a flat angle.

Proof. Let ΔABC be a triangle. By the Exterior Angle Theorem

$$\angle A < \text{ supplement of } \angle C.$$

Adding $\angle C$ to both sides we obtain that $\angle A + \angle C$ is strictly less than a straight angle. $\square$

Lemma 4.2.2. For any triangle, there exists a triangle with the same sum of angles and with one of the angles as small as we want.

Proof. In ΔABC, let M be the midpoint of $|AC|$. Either $\angle ABM$ or $\angle MBC$ is less than or congruent to $\angle B/2$. Say $\angle MBC \leq \angle B/2$. Let C' be such that M is the midpoint of $|BC'|$ (Figure 4.2). Consider the triangle $\Delta ABC'$. By Theorem SAS, $\Delta MAC' \equiv \Delta MBC$. We have

$$\angle BAC' + \angle AC'B + \angle ABC' =\equiv \angle BAC + \angle CAC' + \angle AC'B + \angle ABC'$$
$$\equiv \angle BAC + \angle ACB + \angle ABC' + \angle C'BC \equiv \angle BAC + \angle ACB + \angle CBA.$$

Triangle $\Delta ABC'$ has the same sum of angles as ΔABC and $\angle C' \leq \angle B/2$. Repeating the construction n times we find a triangle with the same sum of angles as ΔABC and with one angle less than or congruent to $\angle B/2^n$.

$\square$

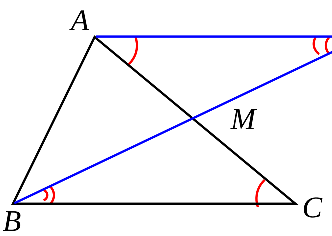

Figure 4.2: Sum of angles of a triangle

Let us return to the proof of the theorem. Assume that for some triangle ΔABC,

$$\angle A + \angle B + \angle C \equiv \text{straight angle } + \angle\alpha, \quad \angle\alpha > 0.$$

Costruct a triangle $\Delta A'B'C'$ with the same sum of angles and with $\angle C' < \angle\alpha/2$. Then, by the first lemma, $\angle A' + \angle B'$ is less than a straight angle. Adding $\angle C'$ we obtain $\angle A' + \angle B' + \angle C'$ less than a straight angle plus $\angle\alpha/2$. This contradicts the fact that $\angle A' + \angle B' + \angle C'$ equal to a straight angle plus $\angle\alpha$. Hence the conclusion.

$\square$

Theorem 4.2.2. If the sum of the angles of a certain triangle is *strictly* less than a straight angle, then the same is true for any triangle.

Proof. Let M be a point on $|BC|$. If the sum of the angles of ΔABC is less than a straight angle, (recall Theorem 4.2.1) then the same must be true for one of the triangles ΔABM and ΔAMC, otherwise by adding we would obtain equality for ΔABC. Dividing further we can obtain an arbitrarily small triangle with the sum of angles less than a straight angle.

Consider now some triangle $\Delta A'B'C'$ in the plane. Place inside a small triangle with the sum of the angles less than a straight angle. Divide $\Delta A'B'C'$ into triangles one of which is the one with sum of angles less than a straight angle. By adding the angles of the triangles in the decomposition and removing the straight angles that are formed we deduce that the sum of the angles of $\Delta A'B'C'$ is less than a straight angle. The theorem is proved. $\square$

Corollary 4.2.1. Either in every triangle the sum of the angles is a straight angle, or in every triangle the sum of the angles is strictly less than a straight angle.

Proposition 4.2.1. If the fifth postulate holds for one point and one line, then there exists a triangle with the sum of the angles equal to a straight angle.

Proof. Let A and l be the point and line for which the postulate holds. Choose B and C on l. Pick l' such that the alternate angles formed by AC with l and l' are congruent. The Exterior Angle Theorem implies that l and l' do not intersect.

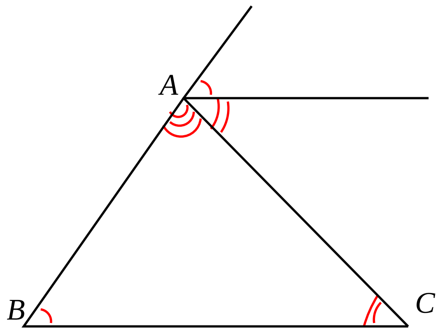

Figure 4.3: Fifth postulate and the sum of angles of a triangle

Pick a line l'' so that the alternate angles formed by AB with l and l'' are congruent. Again l and l'' do not intersect. Uniqueness implies $l' = l''$. Examining the Figure 4.3 we see that $\angle A + \angle B + \angle C$ is congruent to a straight angle. □

Theorem 4.2.3. If the sum of the angles of every triangle is congruent to a straight angle, then Euclid's fifth postulate holds.

Proof. Let l be a line and A a point, $A \notin l$. Let AB be the perpendicular to l, $B \in l$. The line l' perpendicular to AB at A does not intersect l. We will show that any other line through A intersects l. We argue on Figure 4.4.

Let l'' be a line passing throught A and $\angle\alpha$ the acute angle this line makes with $|AB|$. Consider the point $B_1, B_2, \ldots, B_n, \ldots$ on l that lie on the same side of AB as $\angle\alpha$, and such that

$$|BB_1| \equiv |AB|, \quad |B_1B_2| \equiv |AB_1|, \quad \ldots, |B_{n-1}B_n| \equiv |AB_{n-1}|.$$

In ΔABB_1,

$$m(\angle A) = m(\angle B_1) = \frac{\pi}{4}.$$

Using the fact that the sum of the angles of a triangle is π, we further compute, in ΔAB_1B_2,

$$m(\angle A) = m(\angle B_2) = \frac{\pi}{8}, \ldots$$

in $\Delta AB_{n-1}B_n$,

$$m(\angle A) = m(\angle B_n) = \frac{\pi}{2^{n+1}}.$$

For sufficiently large n,

$$m(\angle BAB_n) = \frac{\pi}{4} + \frac{\pi}{8} + \cdots + \frac{\pi}{2^{n+1}}$$

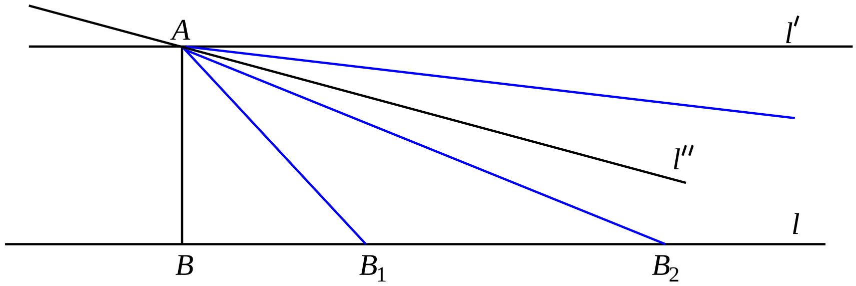

Figure 4.4: Euclid's fifth postulate and the sum of the angle of triangles

will be very close to $\pi/2$, hence greater than $\angle\alpha$. Then l'' will run inside $\angle BAB_n$, so it will have to cross $|BB_n|$. Hence l'' intersects l.

□

Theorem 4.2.4. If Euclid's fifth postulate fails for a point and a line, then it fails for every other point and every other line not containing the point.

Proof. If fifth postulate holds for a point and a line, then the sum of the angles of some triangle is a straight angle. But then the sum of the angles of any other triangle is congruent to a straight angle. In that case the fifth postulate must hold for every point and every line. □

Corollary 4.2.2. Either Euclid's fifth postulate holds, or for every line l and every point A, $A \notin l$, there are infinitely many lines passing through A which do not intersect l.

4.3 The area of a triangle in non-euclidean geometry

Since in non-euclidean geometry the area of a triangle is strictly less than π, we can associate to each triangle ΔABC the positive number

$$\epsilon(\Delta ABC) = \pi - m(\angle A) - m(\angle B) - m(\angle C).$$

Proposition 4.3.1. If the triangular surface T is the union of the triangular surfaces T_1 and T_2, then

$$\epsilon(T) = \epsilon(T_1) + \epsilon(T_2).$$

Because of this we can make the following definition.

Definition. The area of a triangle ΔABC is equal to $\epsilon(\Delta ABC)$.

As a corollary, we have that the area of a polygon with n sides is $(n-2)\pi$ minus the sum of the angles of the polygon. This satisfies the four properties that the area should satisfy.