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Preservation of local dynamics when applying central difference methods: application to SIR model

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We apply the central difference method $(u_{t+1} - u_{t-1})/(2\Delta t) = f(u_t)$ to an epidemic SIR model and show how the local stability of the equilibria is changed after applying the numerical method. The above central difference scheme can be used as a numerical method to produce a discrete-time model that possesses interesting local dynamics which appears inconsistent with the continuous model. Any fixed point of a differential equation will become an unstable saddle node after applying this method. Two other implicitly defined central difference methods are also discussed here. These two methods are more efficient for preserving the local stability of the fixed points for the continuous models. We apply conformal mapping theory in complex analysis to verify the local stability results.

Keywords: Central difference method; Difference equations; Conformal mapping; Local dynamics

1. Introduction

In some areas of mathematical biology, for example, insect population, it is reasonable to consider discrete-time models (or difference equations) for the population growth. Recently, it has become popular to obtain discrete-time systems using either standard or nonstandard numerical methods. Also, local dynamics are compared between the continuous model and its discrete counterparts [2,7–10].

The following is a basic epidemic SIR model [4] that includes vital dynamics:

$$S'(t) = -\lambda SI + \mu - \mu S, \quad I'(t) = \lambda SI - \gamma I - \mu I, \quad S(0) = S_0 > 0, I(0) = I_0 \geq 0, \tag{1}$$

and $R(t) = 1 - S(t) - I(t) \geq 0$. This is an endemic model for which infection confers permanent immunity and the disease goes through a population in a relatively long time. $S(t)$, $I(t)$, and $R(t)$, are the fractions of the total population of the susceptible, the infective, and the removed classes. The results of this model are well-known [4]. If $\sigma = \lambda/(\gamma + \mu) \leq 1$, the equilibrium point $(S, I) = (1,0)$ is asymptotically stable. That is to say that the disease goes extinct. If $\sigma > 1$, then the positive equilibrium $(S^*, I^*) = (1/\sigma, \mu(\sigma - 1)/\lambda)$ is asymptotically stable. Therefore, the disease will stay in the population.

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If we apply the central difference method for a system of differential equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(\mathbf{x}), \tag{2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ is an n-vector, then we obtain

$$\frac{\mathbf{x}_{t+1} - \mathbf{x}_{t-1}}{2h} = f(\mathbf{x}_t). \tag{3}$$

Many examples were shown in Refs. [5,6,11,12] that the central difference method (3) produces a discrete-time system with local dynamics that are inconsistent with the continuous-time system, and even produces chaos when used as a numerical method. Mickens [6] applied the above method for the decay equation $\mathrm{d}x/\mathrm{d}t = -x$ and showed that the numerical scheme for the decay equation has numerical instabilities regardless of the chosen step-size. Ushiki [11] applied the central difference method to the logistic equation $\mathrm{d}x/\mathrm{d}t = x(1-x)$ and showed that the local dynamics of the logistic difference equation $x_{t+1} = x_{t-1} + 2hx_t(1 - x_t)$ are inconsistent with this continuous-time equation, including chaotic behaviors. He showed that the discrete dynamical system is chaotic. There exists chaotic ghost solutions for any non-zero time step h. Yamaguti and Ushiki [12] also studied the discretization of the logistic equation by the central difference scheme when the step size h is fixed. In their numerical computation, the scheme produces some ghost solutions for long-range calculations. They mentioned that one of the reasons for the ghost solution phenomenon is that the central difference scheme is a second order difference scheme and that the instability enters at $x = 1$ and $x = 0$, the two equilibria of the logistic equation.

If we apply the central difference method (3) to the SIR model (1), then we have the following system of difference equations:

$$S_{t+1} = S_{t-1} + 2h(-\lambda S_t I_t + \mu - \mu S_t), \quad I_{t+1} = I_{t-1} + 2h(\lambda S_t I_t - \gamma I_t - \mu I_t). \tag{4}$$

We will choose the initial conditions for this model to be $S(0) = S_0$, $I(0) = I_0$ and $S(1) = S_0 + h(-\lambda S_0 I_0 + \mu - \mu S_0)$, $I(1) = I_0 + h(\lambda S_0 I_0 - \gamma I_0 - \mu I_0)$ from Euler's scheme. System (4) is a second-order difference equation SIR model. Since the model is derived from the continuous SIR model, we may also assume that $0 < S(t) + I(t) \leq 1$. If a solution $(I(t), S(t))$ does not satisfy $0 < S(t) + I(t) \leq 1$, it is invalid. On the other hand, we can always choose the step size h small enough so that the solutions are bounded.

The results we develop in Section 2 show that in the discrete-time SIR model (4), the positive equilibrium is no longer asymptotically stable when $\sigma > 1$. Instead, the solution oscillates near the equilibrium and never settles down since the positive equilibrium is now a saddle point (figure 1).

The central-difference formula (3) is widely used to estimate the derivative of a function [3]. The error in the central-difference method is smaller than that of Euler's method. Standard textbooks in numerical methods mention little about the local stability of the central-difference method (see for example, [1,3]). In this paper, we will look at three central difference schemes and show that the local numerical instability is a general property of method (3) but not of the other two.

The other two methods are the mixed implicit, central difference scheme

$$\frac{\mathbf{x}_{t+1} - \mathbf{x}_{t-1}}{2h} = f\left(\frac{\mathbf{x}_{t+1} + \mathbf{x}_{t-1}}{2}\right), \tag{5}$$

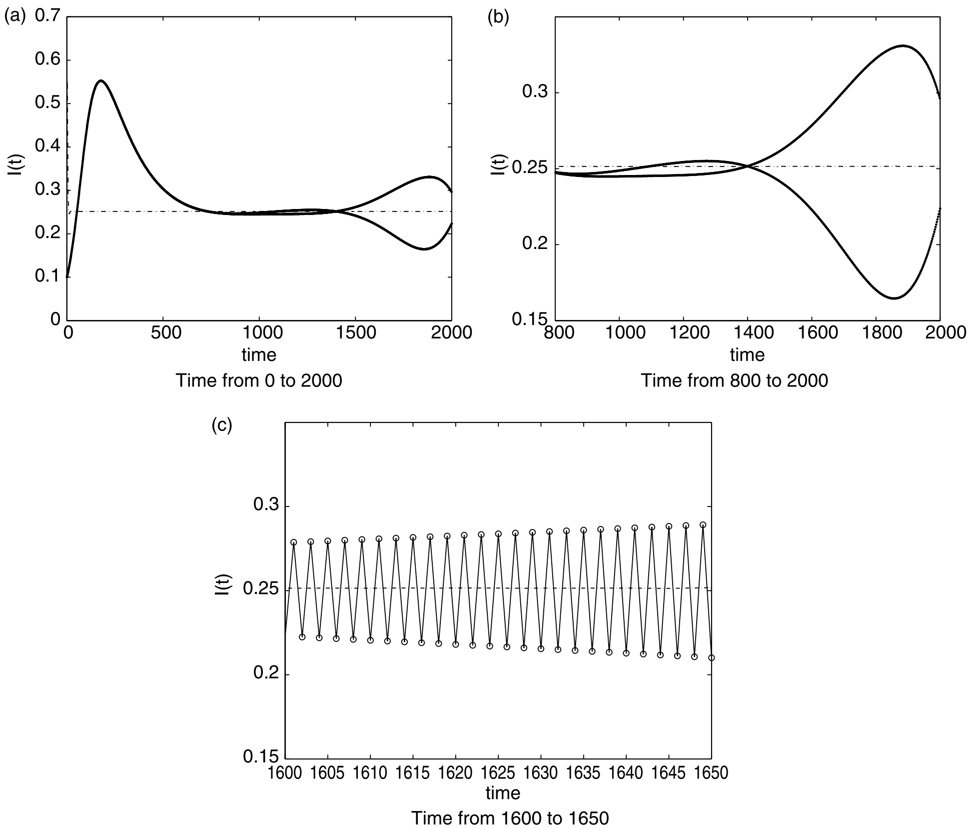

Figure 1. The SIR model (1) when $\sigma = 4.7727 > 1$. In these figures, the dotted line is the stable equilibrium of $I^* = \mu(\sigma - 1)/\lambda = 0.252$ for the SIR model (1). The two darker curves describe the oscillation of the solution of $I(t)$ using the central difference method (3). The time evaluated is from 0 to 2000. (b) Is close-up of (a). (c) Is a closer look of the results when the time is between 1600 and 1650. The parameter values are $\lambda = 2.1$, $\mu = 0.14$, $\gamma = 0.3$. The step-size is $h = 0.014$ and the initial values, $S_0 = 0.9$, $I_0 = 0.1$.

and another implicit central difference scheme

$$\frac{\mathbf{x}_{t+1} - \mathbf{x}_{t-1}}{2h} = f\left(\frac{\mathbf{x}_{t+1} + \mathbf{x}_t + \mathbf{x}_{t-1}}{3}\right). \tag{6}$$

Methods (5) and (6) are briefly mentioned in Mickens [5] as possible ways to discretize a differential equation.

In section 2, we explain the changing behavior of the local dynamics using the central difference method (3): the stability of any fixed point will be unstable no matter what step-size is chosen. This is the reason that the solution of the SIR model oscillates near the equilibrium point. We discuss the other two implicit central difference schemes (5) and (6) in section 3. The two implicit central difference schemes preserve the stability of the fixed points.

2. The central difference scheme

If we apply the central difference scheme to the differential equation (2), we have the second order difference equation (3). We can rewrite equation (3) as

$$\mathbf{x}_{t+2} = \mathbf{x}_t + 2hf(\mathbf{x}_{t+1}). \tag{7}$$

Let $(\mathbf{u}_t, \mathbf{v}_t) = (\mathbf{x}_t, \mathbf{x}_{t+1})$. Then the equation (7) is a system of first-order difference equations:

$$\mathbf{u}_{t+1} = \mathbf{v}_t, \quad \mathbf{v}_{t+1} = \mathbf{u}_t + 2hf(\mathbf{v}_t). \tag{8}$$

Let g denote the right hand side function of equation (8). If $\mathbf{p}_0$ is a fixed point of the differential equation (2), then $(\mathbf{u}, \mathbf{v})^t = (\mathbf{p}_0, \mathbf{p}_0)^t$ is a fixed point of the difference equation (8). The Jacobian matrix of the system (8) evaluated at $(\mathbf{p}_0, \mathbf{p}_0)^t$ is

$$Dg(\mathbf{p}_0) = \begin{pmatrix} 0 & I \\ I & 2hDf(\mathbf{p}_0) \end{pmatrix}.$$

The eigenvalue w of $Dg(\mathbf{p}_0)$ and the eigenvalue z of $Df(\mathbf{p}_0)$ are related. Let $(\mathbf{x}_1, \mathbf{x}_2)^t$ be an eigenvector associated to w. Then we have

$$Dg(\mathbf{p}_0)\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 2hDf(\mathbf{p}_0) \end{pmatrix}\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = w\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}.$$

This is the same as

$$\mathbf{x}_2 = w\mathbf{x}_1, \quad \mathbf{x}_1 + 2hDf(\mathbf{p}_0)\mathbf{x}_2 = w\mathbf{x}_2.$$

If we multiply the second equation by w, replace $w\mathbf{x}_1$ by $\mathbf{x}_2$, and rearrange the equation, we have

$$Df(\mathbf{p}_0)\mathbf{x}_2 = \frac{1}{2h}\left(w - \frac{1}{w}\right)\mathbf{x}_2.$$

Therefore $z = 1/2h(w - (1/w))$ is an eigenvalue of $Df(\mathbf{p}_0)$.

In other words, if z is an eigenvalue for $Df(\mathbf{p}_0)$, then $w_{1,2} = hz \pm \sqrt{h^2z^2 + 1}$ are eigenvalues for the matrix $Dg(\mathbf{p}_0)$. We have $|w_1w_2| = 1$, and neither $|w_1| = 1$ nor $|w_2| = 1$. It follows that unless $z = 0$, either $|w_1| > 1$ and $|w_2| < 1$ or $|w_1| < 1$ and $|w_2| > 1$. We conclude that the fixed point $(\mathbf{p}_0, \mathbf{p}_0)$ of the difference equation (7) is a saddle point and unstable. Therefore, we have the following theorem.

THEOREM 1. *For $h > 0$, all of the fixed points of the differential equation $d\boldsymbol{x}/dt = f(\boldsymbol{x})$ become saddle points and unstable after applying the central difference scheme* (3).

3. Implicit central difference schemes

3.1 Mixed implicit, central difference scheme

The second central difference scheme is described as in the equation (5). Equation (5) is a system of second order difference equations. We can rewrite equation (5) as the following

$$\mathbf{x}_{t+2} = \mathbf{x}_t + 2hf\left(\frac{\mathbf{x}_{t+2} + \mathbf{x}_t}{2}\right),$$

or as a system of first-order difference equations:

$$\mathbf{u}_{t+1} = \mathbf{v}_t, \quad \mathbf{v}_{t+1} = \mathbf{u}_t + 2hf\left(\frac{\mathbf{u}_t + \mathbf{v}_{t+1}}{2}\right). \tag{9}$$

Let g denote the right hand side function of equation (9). If $\mathbf{p}_0$ is a fixed point for the differential equation (2), then $(\mathbf{p}_0, \mathbf{p}_0)$ is a fixed point of equation (9). At the fixed point $\mathbf{p}_0$, implicit differentiation and solving for the partial derivative give us

$$\frac{\partial \mathbf{v}_{t+1}}{\partial \mathbf{u}_t}(\mathbf{p}_0) = \big(I - hDf(\mathbf{p}_0)\big)\big(I + hDf(\mathbf{p}_0)\big)$$

provided h is small enough so that $I - hDf(\mathbf{p}_0)$ is invertible. Then the Jacobian matrix $Dg(\mathbf{p}_0)$ of the difference equations (9) at $(\mathbf{p}_0, \mathbf{p}_0)$ is

$$Dg(\mathbf{p}_0) = \begin{pmatrix} 0 & I \\ (I - hDf(\mathbf{p}_0))^{-1}(I + hDf(\mathbf{p}_0)) & 0 \end{pmatrix}.$$

We have similar results relating the eigenvalues z of $Df(\mathbf{p}_0)$ and w of $Dg(\mathbf{p}_0)$. Let $(\mathbf{x}_1, \mathbf{x}_2)^t$ be an eigenvector associated to w. Then we have

$$Dg(\mathbf{p}_0)\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ (I - hDf(\mathbf{p}_0))^{-1}(I + hDf(\mathbf{p}_0)) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = w\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}.$$

After some manipulation we are able to obtain

$$Df(\mathbf{p}_0)\mathbf{x}_2 = \frac{w^2 - 1}{h(w^2 + 1)}\mathbf{x}_2.$$

Therefore, if w is an eigenvalue of $Dg(\mathbf{p}_0)$, then $z = (w^2 - 1)/(hw^2 + h)$ is an eigenvalue of $Df(\mathbf{p}_0)$. Also, if z is an eigenvalue of $Df(\mathbf{p}_0)$, then $w = \pm\sqrt{(1 + hz)/(1 - hz)}$ are eigenvalues of $Dg(\mathbf{p}_0)$. We will show that all of the values of z on the left half plane $\Re(z) < 0$ will eventually be mapped into the unit disk $|w| < 1$. Similar arguments imply that all values of z on the right half plane $\Re(z) > 0$ will be mapped to $|w| > 1$. That is, stable (unstable) fixed points will remain stable (unstable) after the mixed implicit central difference scheme (5). We have the following theorem.

THEOREM 2. *The implicit mixed, central difference scheme* (5) *preserves the local stability of the fixed points of the differential equation* $d\boldsymbol{x}/dt = f(\boldsymbol{x})$.

Proof. We only need to show that the real part, $\Re(z)$, of the map $z = (w^2 - 1)/h(w^2 + 1)$ is less than (greater than) zero if $|w| < 1$ ($|w| > 1$). Note that z is an analytic function inside the unit disk $|w| < 1$ (and outside the unit disk $|w| > 1$). Therefore, $\Re(z)$ is a harmonic function inside and outside the unit disk and its behavior is determined by its boundary values. Let $w = re^{i\theta}$. Then we have

$$\Re(z) = \frac{1}{h(|w^2 + 1|^2)}\Re((w^2 - 1)(\bar{w}^2 + 1)) = \frac{1}{h(|w^2 + 1|^2)}(r^4 - 1)$$
$$= \frac{1}{h(|w^2 + 1|^2)}(r^2 + 1)(r + 1)(r - 1).$$

Since $h > 0$, $|w^2 + 1| > 0$, and $r > 0$, we have $\Re(z) < 0$ if $r < 1$ and $\Re(z) > 0$ if $r > 1$, where $r = |w|$. □

3.2 Another implicit central difference scheme

The other implicit central difference scheme is (6). We can rewrite the implicit central difference scheme (6) as

$$\mathbf{x}_{t+2} = \mathbf{x}_t + 2hf\left(\frac{\mathbf{x}_{t+2} + \mathbf{x}_{t+1} + \mathbf{x}_t}{3}\right).$$

Let

$$\begin{pmatrix} \mathbf{u}_t \\ \mathbf{v}_t \end{pmatrix} = \begin{pmatrix} \mathbf{x}_t \\ \mathbf{x}_{t+1} \end{pmatrix}.$$

Then we can rewrite this system into a system of first-order difference equations as follows

$$\mathbf{u}_{t+1} = \mathbf{v}_t, \quad \mathbf{v}_{t+1} = \mathbf{u}_t + 2hf\left(\frac{\mathbf{v}_{t+1} + \mathbf{u}_t + \mathbf{v}_t}{3}\right). \tag{10}$$

Since

$$\frac{\partial \mathbf{v}_{t+1}}{\partial \mathbf{u}_t} = I + 2hDf\cdot\left(\frac{1}{3}\frac{\partial \mathbf{v}_{t+1}}{\partial \mathbf{u}_t} + \frac{1}{3}I\right), \quad \frac{\partial \mathbf{v}_{t+1}}{\partial \mathbf{v}_t} = 2hDf\cdot\left(\frac{1}{3}\frac{\partial \mathbf{v}_{t+1}}{\partial \mathbf{v}_t} + \frac{1}{3}I\right),$$

we have

$$\frac{\partial \mathbf{v}_{t+1}}{\partial \mathbf{u}_t} = \left(I - \frac{2h}{3}Df\right)^{-1}\left(I + \frac{2h}{3}Df\right), \quad \frac{\partial \mathbf{v}_{t+1}}{\partial \mathbf{v}_t} = \left(I - \frac{2h}{3}Df\right)^{-1}\left(\frac{2h}{3}Df\right).$$

Then at the fixed points $\mathbf{p}_0$ and $(\mathbf{p}_0, \mathbf{p}_0)$, we obtain the Jacobian matrix $Dg(\mathbf{p}_0)$ of the system of difference equation (10) to be

$$Dg(\mathbf{p}_0) = \begin{pmatrix} 0 & I \\ \left(I - \frac{2h}{3}Df(\mathbf{p}_0)\right)^{-1}\left(I + \frac{2h}{3}Df(\mathbf{p}_0)\right) & \left(I - \frac{2h}{3}Df(\mathbf{p}_0)\right)^{-1}\left(\frac{2h}{3}Df(\mathbf{p}_0)\right) \end{pmatrix}.$$

We can show that if w is an eigenvalue of $Dg(\mathbf{p}_0)$, then $3(w^2 - 1)/2h(w^2 + w + 1)$ is an eigenvalue of $Df(\mathbf{p}_0)$.

Let $(\mathbf{x}_1, \mathbf{x}_2)^t$ be an eigenvector associated to the eigenvalue w. Then we have

$$\mathbf{x}_2 = w\mathbf{x}_1, \quad \left(I - \frac{2h}{3}Df(\mathbf{p}_0)\right)^{-1}\left[\left(I + \frac{2h}{3}Df(\mathbf{p}_0)\right)\mathbf{x}_1 + \left(\frac{2h}{3}Df(\mathbf{p}_0)\right)\mathbf{x}_2\right] = w\mathbf{x}_2.$$

After some manipulation, we are able to obtain

$$Df(\mathbf{p}_0)\mathbf{x}_2 = \frac{3(w^2 - 1)}{2h(w^2 + w + 1)}\mathbf{x}_2.$$

Also, if z is an eigenvalue for $Df(\mathbf{p}_0)$, then $w = hz \pm \sqrt{9 - 3h^2z^2}/(3 - 2hz)$ are eigenvalues for the matrix $Dg(\mathbf{p}_0)$. Similarly, we have the following theorem.

THEOREM 3. *The implicit central difference scheme* (6) *preserves the local stability of the fixed points of the differential equation* $d\boldsymbol{x}/dt = f(\boldsymbol{x})$.

Proof. We only need to show that the real part, $\Re(z)$, of the map $z = 3(w^2 - 1)/2h(w^2 + w + 1)$ is less than (greater than) zero if $|w| < 1$ ($|w| > 1$). Since the roots of $w^2 + w + 1 = 0$ lie on $|w| = 1$ we have that z is an analytic function inside the unit disk, $|w| < 1$ (and outside the unit disk, $|w| > 1$). Therefore, $\Re(z)$ is a harmonic function inside and outside the unit disk. Let $w = re^{i\theta}$. Then we have

$$\begin{aligned}\Re(z) &= \frac{3}{2h(|w^2+w+1|^2)}\Re((w^2-1)(\bar{w}^2+\bar{w}+1)) \\ &= \frac{3}{2h(|w^2+w+1|^2)}(r^4 + r^3\cos\theta - r\cos\theta - 1) \\ &= \frac{3}{2h(|w^2+w+1|^2)}(r^2 + r\cos\theta + 1)(r+1)(r-1).\end{aligned}$$

Since $h > 0$, $|w^2 + w + 1| > 0$, $r > 0$, and $r^2 + r\cos\theta + 1 = (r + (\cos\theta/2))^2 + (1 - (\cos^2\theta/4)) \geq 1 - (\cos^2\theta/4) > 0$, we have $\Re(z) < 0$ if $r < 1$ and $\Re(z) > 0$ if $r > 1$, where $r = |w|$. □

4. Conclusions

We have applied the central difference method to the SIR model and showed in Theorem 1 that the positive equilibrium is now a saddle point instead of a stable node as in the continuous SIR model. The central difference method (3) does not preserve local stability of the fixed points of differential equations. Applying the central difference scheme (3) to differential equations will produce difference equations with interesting results that show local dynamics inconsistent with the continuous-time systems. Three different central difference methods are discussed with two of them preserving the local stability of the equilibria of differential equations.

Most of the standard numerical methods, for example, the Euler's method and the Runge–Kutta method, preserve the local stability of fixed points if the step-size is sufficiently small. However, the central difference scheme (3) alters the local stability no matter how small the step size. The two implicit schemes (5) and (6) seem to neutralize the bad behavior of the central difference scheme. They preserve local stability of fixed points. In the literature [1,5,11,12], examples are shown that the central difference scheme (3) does not preserve local stability. However, we have not seen similar arguments on the other two implicit schemes (5) and (6).

There is no unique way to discretize a differential equation. So, there are many ways to construct discrete-time models. If we want to build more traditional discrete-time epidemic models, we can choose the numerical methods that preserve local stability. If we want something different, we might want to try the central difference method.

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