Computational Methods and Function Theory
Volume 2 (2002), No. 1, 229–247

Minimal Harmonic Measure on Complementary Regions

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(Communicated by Doron Lubinsky)

Abstract. For any two points a_1 and a_2 in an open disk Δ on the complex sphere $\overline{\mathbb{C}}$, let L be a curve separating a_1 from a_2 on $\overline{\mathbb{C}}$, which splits $\overline{\mathbb{C}}$ into two complementary regions $B_1 \ni a_1$ and $B_2 \ni a_2$. Let l be the part of this curve lying in $\bar{\Delta}$. In this note we study how small the average harmonic measure

$$\frac{1}{2}(\omega(a_1, l, B_1) + \omega(a_2, l, B_2))$$

can be. This question can be interpreted as a problem on the minimal average temperature at two points of a long cylinder composed of two media separated by a heating membrane each of which contains a reference point.

Keywords. Harmonic measure, module of a quadrilateral, complete elliptic integral.

2000 MSC. 30C85, 33E05.

1. Introduction to the problem

Let a closed (non-Jordan in general) curve L split the complex sphere $\overline{\mathbb{C}}$ into two complementary regions B_1 and B_2 containing points a_1 and a_2 respectively. Let Δ be an open disk containing a_1 and a_2 and let l_1, l_2 be parts of boundaries ∂B_1 and ∂B_2 lying in $\bar{\Delta}$.

How small compared to L can the set $l = l_1 \cup l_2$ be? This is the main question studied in this note. For a measure of "smallness" it is natural to choose the average of harmonic measures of the sets l_1 and l_2 evaluated at the reference points a_1 and a_2. In this way we arrive at the following extremal problem.

Problem $\mathcal{P}_\Delta(a_1, a_2)$. *Find the minimization of the sum*

$$\frac{1}{2}(\omega(a_1, l_1, B_1) + \omega(a_2, l_2, B_2)) \to \inf . \tag{1.1}$$

Received May 13, 2002.
The research of the third author was supported in part by Russian Fund for Fundamental Research, grant no. 00-01-00118a.

ISSN 1617-9447/\$ 2.50

Here $\omega(z, E, D)$ denotes the harmonic measure of the set $E \subset \partial D$ with respect to the region D evaluated at the point z. That is, $\omega(z, E, D)$ is the bounded harmonic function having boundary values 1 a.e. on E and 0 a.e. on $\partial D \setminus E$. The infimum in (1.1) is taken among all curves L separating a_1 from a_2 on $\overline{\mathbb{C}}$ as it is illustrated in Figure 1. For properties of the harmonic measure we refer to [A1, Ch. 1.1] or [G, Ch. 8].

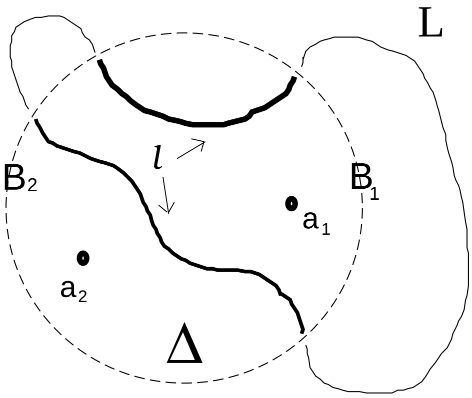

FIGURE 1. Competing domains of Problem $\mathcal{P}_\Delta(a_1, a_2)$.

Putting more data one can additionally require that L traces through a prescribed set of contact points $b_1, \dots, b_n$ lying in $\Delta' = \bar{\Delta} \setminus \{a_1, a_2\}$. In this paper we deal with the simplest case when the set of contact points is empty. A set of similar extremal problems on the harmonic measure was studied in [S1].

Since the harmonic measure is conformally invariant, the problem formulated above is invariant under Möbius maps. Therefore in what follows we may assume without loss of generality that Δ is the unit disk $\Delta = \{z : |z| < 1\}$ and $a_1 = r$, $a_2 = -r$, where r, $0 < r < 1$ is defined by

$$\log \frac{1+r}{1-r} = \rho_\Delta(a_1, a_2),$$

where $\rho_\Delta(a_1, a_2)$ stands for the hyperbolic distance between a_1 and a_2 with respect to the disk Δ, see [A1, G]. Thus the Problem $\mathcal{P}_\Delta(a_1, a_2)$ actually depends on one positive parameter $0 < r < 1$ only.

In Section 2, we use results from the theory of modules of families of curves to give a qualitative solution of Problem $\mathcal{P}_\Delta(a, b)$ by including the minimal harmonic measure into a one-parameter family of functions. Then, in Section 3 we show that for every value of the parameter r, $0 < r < 1$, except for one special value $r_0 = g - \sqrt{g} = .34601\ldots$, where $g = (1 + \sqrt{5})/2$ is the golden ratio, the corresponding one-parametric family of configurations contains a unique minimal configuration that is either symmetric or degenerate. In the last section we consider possible generalizations and discuss some open questions.

2. Problem $\mathcal{P}(r)$

2.1. Reduction of the problem. First we reduce the problem considered to a simpler one. Note that the assumptions of the problem imply the existence of a continuum (i.e. connected compact set) $E \subset \bar{\Delta} \cap L$, which splits Δ into two disjoint regions $D_1 \ni r$ and $D_2 \ni -r$. Thus, $\Delta \setminus E = D_1 \cup D_2$. By the Carleman Principle for the harmonic measure (or essentially by the Maximum Principle for harmonic functions),

$$\frac{1}{2}(\omega(r, l_1, B_1) + \omega(-r, l_2, B_2)) \geq \frac{1}{2}(\omega(r, E, D_1) + \omega(-r, E, D_2))$$

and therefore the infimum of Problem $\mathcal{P}_\Delta(a_1, a_2)$ (1.1) is greater or equal to the infimum

$$\Omega(r) = \inf \frac{1}{2}(\omega(r, E, D_1) + \omega(-r, E, D_2)) \tag{2.1}$$

taken among the set of all continua $E \subset \bar{\Delta}$ that split Δ into a pair of regions described above. Standard approximation arguments applied to the extremal configurations of Problem (2.1) described in Theorem 1, Section 3 show that the infima in (1.1) and (2.1) actually coincide. In what follows the minimization Problem (2.1) will be denoted by $\mathcal{P}(r)$.

One advantage of formulation (2.1) compared to (1.1) is that the infimum in (2.1) is realized for certain continuum E^* or equivalently for a certain pair of non-overlapping regions D_1^*, D_2^* one of which but not both may be degenerate.

2.2. Physical motivation. It might also be interesting to note that in formulation (2.1) our problem models a steady-state heat flow in a long cylindrical solid bar (one can have in mind, for instance, the plutonium bars in a nuclear reactor). Assume that the bar is composed of two media separated by a thin membrane having the same shape in each horizontal section as it is shown in Figure 2. Assume that we cool this bar along the cylindrical surface keeping the temperature at least t_0 at each of its points and assume that, in order to be operational, the membrane requires a temperature of at least $t_1 > t_0$ at any of its points.

Assume next that in controlling the temperature at two points, a_1 in the first medium and a_2 in the second medium, we are interested in keeping the average temperature at these points as small as possible. What is the optimal design of the membrane that corresponds to the minimal average temperature? This question is equivalent to our Problem (2.1).

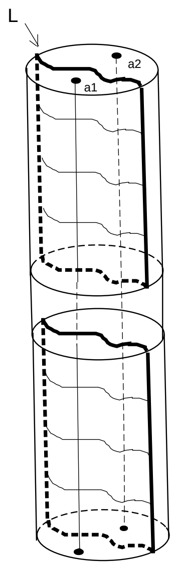

FIGURE 2. The minimal average temperature problem.

2.3. Extremal partition and module of a triad. Problem (2.1) is also connected with a problem on the extremal partition of Δ into a pair of disjoint quadrilaterals. Following J. Jenkins [J3], by a triad (a, α, D) we shall mean a simply connected region D on $\overline{\mathbb{C}}$ with a fixed closed arc α on its boundary and a fixed point a inside D. Let $\{Q\}$ be the set of all quadrilaterals Q in $D' = D \setminus \{a\}$ each of which has a distinguished pair of sides on the complementary boundary arc $\beta = \partial D \setminus \alpha$ such that curves joining the distinguished sides of Q separate a from α inside D. Let $\operatorname{mod} Q$ denote the module of Q with respect to the family of curves in D joining the distinguished sides of Q. Let $w = f(z)$ map Q conformally onto a rectangle $\{w : |\operatorname{Re} w| < a, |\operatorname{Im} w| < b\}$ such that the distinguished sides of Q map onto the vertical sides of the rectangle. Then, $\operatorname{mod} Q = b/a$. For properties of the module of a quadrilateral we refer to [A1, A2, D, J2, S2].

It is well known [J1, J3], that there is a unique quadrilateral $Q_0 = Q_0(a, \alpha, D)$ in the family $\{Q\}$ having the maximal module:

$$\operatorname{mod} Q_0 = \max_{\{Q\}} \{\operatorname{mod} Q\}.$$

Let s be a conformal mapping of D onto Δ normalized by the condition $s(a) = 0$, $s(\alpha) = \{e^{i\theta} : |\theta - \pi| \leq \psi_0\}$ with $\psi_0 = \pi\omega$, where $\omega = \omega(a, \alpha, D)$. Let $t = s^{-1}$. Then the extremal quadrilateral Q_0 is the t- image of the quadrilateral $Q(\psi_0) = \Delta \setminus [0, 1]$ with vertices at the points $1 + i0$, $-e^{-i\psi_0}$, $-e^{i\psi_0}$, and $1 - i0$.

By the module of a triad (a, α, D) we shall mean the module

$$m(Q_0) = m(Q_0(a, \alpha, D)).$$

The latter module and the harmonic measure $\omega = \omega(a, \alpha, D)$ are linked via a well known formula attributed to J. Hersch:

$$m(Q_0) = \frac{\mathbf{K}(\tan^2(\pi(1-\omega)/4))}{\mathbf{K}'(\tan^2(\pi(1-\omega)/4))}, \tag{2.2}$$

where $\mathbf{K}$ denotes the complete elliptic integral of the first kind and

$$\mathbf{K}'(\tau) = \mathbf{K}(\sqrt{1-\tau^2}).$$

Later on similar notation $\mathbf{E}$ and $\mathbf{E}'$ will be used for the complete elliptic integrals of the second kind. In what follows it will be convenient to employ the notation

$$\mu(s) = \frac{\pi}{2} \frac{\mathbf{K}'(s)}{\mathbf{K}(s)} \tag{2.3}$$

for the quotient of the complete elliptic integrals, see [AVV, Chapter 5], which contains numerous useful properties of $\mu(s)$ and is accompanied by a software package useful in numerical computations concerning $\mu(s)$. Equation (2.2) shows that $m = m(Q_0)$ decreases from $+\infty$ to 0 when ω runs from 0 to 1.

2.4. Harmonic measure and extremal quadrilaterals. Let

$$\omega_*(r) = \omega(-r, [r,1], \Delta).$$

Carleman's Principle and a symmetrization result of J. Krzyz [Kr], see also [B, Theorem 6.1], imply that

$$\omega(-r, E, D_2) \geq \omega_*(r),$$

with the sign of equality only for $E = [r,1]$. For a fixed ω_0, $\omega_*(r) < \omega_0 < 1$, consider Problem (2.1) under an additional constraint

$$\omega(-r, E, D_2) = \omega_0. \tag{2.4}$$

Our previous analysis shows that Problem (2.1), (2.4) is equivalent to the following problem on the modules. Let Q_1, Q_2 be a pair of disjoint quadrilaterals in $\Delta' = \Delta \setminus \{\pm r\}$ each of which has a pair of distinguished sides on $\mathbb{T}$. We assume in addition that curves $\gamma \subset Q_1$ joining the distinguished sides of Q_1 separate the point r from Q_2 and the point $-r$ inside Δ' and that similar curves $\gamma \subset Q_2$ separate the point $-r$ from Q_1 and r inside Δ'. The problem is to find

$$\operatorname{mod} Q_1 \to \max$$

among all pairs of quadrilaterals Q_1, Q_2 satisfying the above mentioned conditions and such that

$$\operatorname{mod} Q_2 = m_0 = \frac{\pi}{2}\,\mu\left(\tan^2 \frac{\pi(1-\omega_0)}{4}\right).$$

Let γ_1 denote the non-distinguished side of Q_1 that does not separate Q_1 from Q_2 and let γ_2 be a similar side of Q_2. Then we can consider a quadrilateral $Q_{1,2} = \Delta \setminus (\gamma_1 \cup \gamma_2)$ with γ_1 and γ_2 as its non-distinguished sides.

The quadrilateral $Q_{1,2}$ separates the points r and $-r$ in Δ'. Therefore as a very special case of the Jenkins Theorem in [J1] we get the inequality

$$\operatorname{mod} Q_{1,2} \leq \operatorname{mod} Q(r), \tag{2.5}$$

where $Q(r)$ denotes the quadrilateral $\Delta \setminus ([-1,-r] \cup [r,1])$ with the distinguished sides $\mathbb{T}^+ = \{z \in \mathbb{T} : \operatorname{Im} z > 0\}$ and $\mathbb{T}^- = \{z \in \mathbb{T} : \operatorname{Im} z < 0\}$. Equality occurs in (2.5) if and only if $Q_{1,2} = Q(r)$. Inequality (2.5) follows also from a symmetrization result of O. Teichmüller, see [A2].

By Grötzsch's Lemma, see [J2, Ch.2],

$$\operatorname{mod} Q_1 + \operatorname{mod} Q_2 \leq \operatorname{mod} Q_{1,2},$$

which together with 2.5 leads to the inequality

$$\operatorname{mod} Q_1 \leq \operatorname{mod} Q(r) - m_0. \tag{2.6}$$

Let f_r map $Q_{1,2}$ conformally onto a suitable rectangle

$$\Pi(a) = \{w : |\operatorname{Im} w| < 1, |\operatorname{Re} w| < a\}, \qquad a > 0,$$

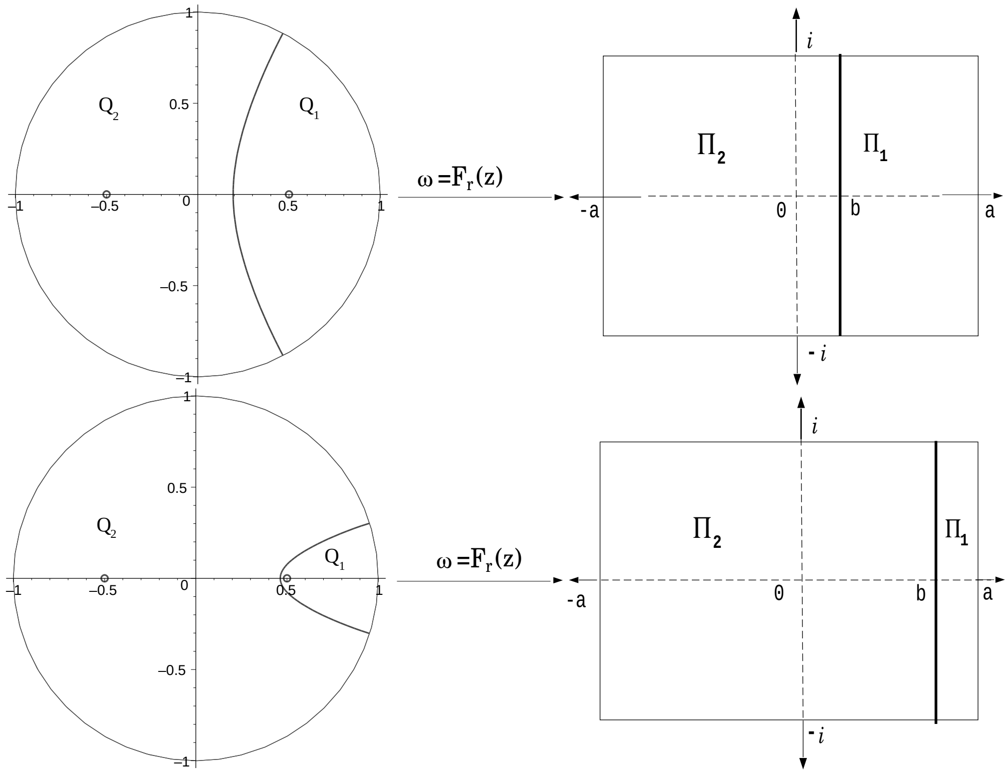

FIGURE 3. Extremal rectangles for $r = 0.5$.

such that the slits $(-1, -r]$ and $[r, 1)$ correspond to the left and right vertical sides of $\Pi(a)$, respectively, and let F_r be a function corresponding in a similar way to the extremal quadrilateral $Q(r)$. Equality in (2.6) occurs if and only if $Q_{1,2} = Q(r)$ and the sets $\Pi_1 = f_r(Q_1)$ and $\Pi_2 = f_r(Q_2)$ are two complementary rectangles in $\Pi(a)$, see Figure 3.

Combining all of these results we obtain Lemma 1, solving Problem (2.1), (2.4).

Lemma 1. *For a fixed ω_0, $\omega_*(r) < \omega_0 < 1$, there is a unique pair of domains D_1^*, D_2^* minimizing the sum in* (2.1) *under the additional constraint*

$$\omega(-r, E, D_2) = \omega_0.$$

Moreover,

$$\begin{aligned} D_1^* &= D_1(a,b) = G_r(\Pi(b,a)) \cup [r,1), \\ D_2^* &= D_2(a,b) = G_r(\Pi(-a,b)) \cup (-1,-r], \end{aligned}$$

where $G_r = F_r^{-1}$ and the parameters a and b, $-a < b < a < \infty$, are defined by the equations

$$a = \frac{\pi}{\mu(r^2)}, \qquad b = \frac{\pi}{\mu\Big(\tan^2 \frac{\pi(1-\omega_0)}{4}\Big)} - \frac{\pi}{\mu(r^2)}. \tag{2.7}$$

The minimum in (2.1) *is*

$$\Omega(a,b) = 1 - \frac{2}{\pi}\left(\tan^{-1}\sqrt{\tau_1} + \tan^{-1}\sqrt{\tau_2}\right) \tag{2.8}$$

with τ_1 *and* τ_2 *defined by*

$$\frac{\pi}{\mu(\tau_1)} = a - b, \qquad \frac{\pi}{\mu(\tau_2)} = a + b, \tag{2.9}$$

where $\mu(s)$ *is defined in* (2.3).

3. Minimal configurations

We have shown in Section 2 that the extremal configurations of Problem (2.1) are among the configurations $D_1(a,b)$, $D_2(a,b)$ described by Lemma 1 with a defined by (2.7) and some b such that $-a \le b \le a$. Now we will study how the minimal average harmonic measure $\Omega(a,b)$ depends on the parameter b. Equations (2.8) and (2.9) show that for a fixed a, $\Omega(a,b)$ is an even function of b. Therefore the configuration $D_1(a,-b)$, $D_2(a,-b)$ provides the extremal value for $\Omega(a,b)$ if and only if the configuration $D_1(a,b)$, $D_2(a,b)$ does. This allows us to deal in what follows with non negative values of b only. First we examine the limit cases.

Since $\Omega(a,b)$ is even in b, $b=0$ is always a critical point of $\Omega(a,b)$ and corresponds to the symmetric configuration $D_1(a,0) = \Delta^+$, $D_2(a,0) = \Delta^-$, where

$$\Delta^+ = \{z \in \Delta :\ \operatorname{Re} z > 0\}, \qquad \Delta^- = \Delta \setminus \bar{\Delta}^+.$$

Let $u_s(r) = \omega(r, I, D_1(a,0))$, where $I = [-i, i]$.

To find $\omega(r, I, D_1(a,0))$, we note that the function $\varphi(z) = iz(1-z^2)^{-1}$ maps Δ^+ conformally onto the upper half-plane $\mathbb{H}$ such that $\varphi(I) = [-1/2, 1/2]$ and $\varphi(r) = ir(1-r^2)^{-1}$. Since harmonic measure is invariant under conformal mapping,

$$\omega(r, I, D_1(a,0)) = \omega(\varphi(r), [-1/2, 1/2], \mathbb{H}).$$

Let $2\alpha\pi$ denote the angle formed by the segments $[-1/2, \varphi(r)]$ and $[1/2, \varphi(r)]$ at the point $\varphi(r)$. Then $2\alpha = (2/\pi)\tan^{-1}((1-r^2)/2r)$. It is well known [A1] that $\omega(\varphi(r), [-1/2, 1/2], \mathbb{H}) = 2\alpha$. Therefore,

$$u_s(r) = \frac{2}{\pi}\tan^{-1}\frac{1-r^2}{2r} = 1 - \frac{4}{\pi}\tan^{-1} r. \tag{3.1}$$

In the second limit case, $b = a$. In this case $D_1(a,a) = \emptyset$, $D_2(a,a) = \Delta(r)$, where $\Delta(r) = \Delta \setminus I(r)$, $I(r) = [r, 1]$, where a and r are related via (2.7). Using a conformal mapping from $\Delta(r)$ onto $\mathbb{H}$,

$$\varphi_1(z) = \varphi\left(\left(\frac{r-z}{1-rz}\right)^{1/2}\right),$$

with φ defined above, we can compute as in the symmetric case,

$$u_d(r) = \frac{1}{2}(1+\omega(r, I(r), D_2(a,a))) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{(1-r)^2}{2\sqrt{2r(1+r^2)}}. \tag{3.2}$$

The graphs of $u_s(r)$, $u_d(r)$, and the graph of the function $u_n(r)$, which corresponds to a possible nondegenerate nonsymmetric configuration, as it will be explained later, are plotted in Figure 4.

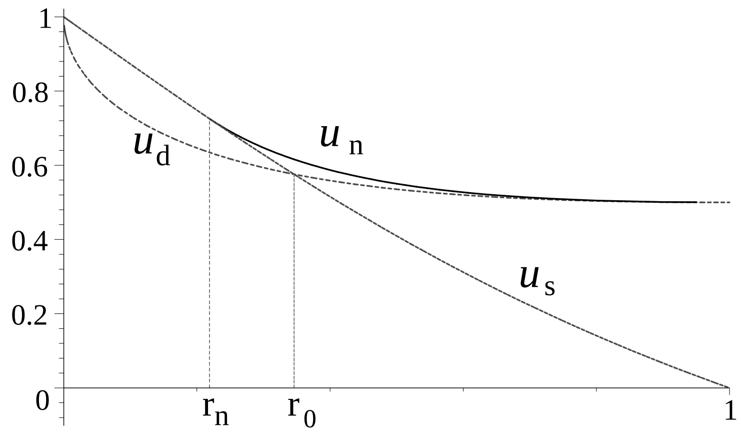

FIGURE 4. Graphs of u_s, u_d, and u_n.

Lemma 2. *For $0 < r < 1$, the equation $u_s(r) = u_d(r)$ has a unique solution $r_0 = g - \sqrt{g} = .34601\ldots$, where $g = (1+\sqrt{5})/2$ is the golden ratio and*

$$u_d(r) < u_s(r) \quad \textit{for} \quad 0 < r < r_0,$$
$$u_d(r) > u_s(r) \quad \textit{for} \quad r_0 < r < 1.$$

Proof. By (3.1), (3.2), the equation $u_s(r) = u_d(r)$ is equivalent to

$$\tan^{-1}\frac{1-r^2}{2r} = \frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{(r-1)^2}{2\sqrt{2r(r^2+1)}}. \tag{3.3}$$

Taking the tangent of the both sides in (3.3), we obtain

$$\frac{1-r^2}{2r} = \frac{2r^2+2-2\sqrt{2r^3+2r}}{-4r+2\sqrt{2r^3+2r}}.$$

After some algebra this leads to the equation

$$r^4 - 2r^3 - 2r^2 - 2r + 1 = 0. \tag{3.4}$$

Since

$$\frac{d}{dr}(r^4 - 2r^3 - 2r^2 - 2r + 1) = 2[2r(r^2-1) - (3r^2+1)]$$

is clearly negative on $0 < r < 1$, (3.4) and therefore the equation $u_s(r) = u_d(r)$ has a unique solution r_0 in $0 < r < 1$. Solving equation (3.4) by radicals we obtain

$$r_0 = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}} = .34601\ldots$$

This solution of (3.4) can be found also with MATHEMATICA. ■

Lemma 2 shows, in particular, that the symmetric configuration is not extremal for $0 < r < r_0$ and the degenerate configuration is not extremal for $r_0 < r < 1$.

To find values τ_1 and τ_2 minimizing the function (2.8), we consider a minimization problem

$$m(x_1, x_2) = 1 - \frac{2}{\pi}\left(\tan^{-1}\sqrt{x_1} + \tan^{-1}\sqrt{x_2}\right) \to \min \tag{3.5}$$

under the constraint

$$\frac{1}{\mu(x_1)} + \frac{1}{\mu(x_2)} = \frac{2}{\mu(r^2)}. \tag{3.6}$$

Differentiating the Lagrange function

$$\Phi(x_1, x_2) = 1 - \frac{2}{\pi}\left(\tan^{-1}\sqrt{x_1} + \tan^{-1}\sqrt{x_2}\right) - \lambda\left(\frac{1}{\mu(x_1)} + \frac{1}{\mu(x_2)}\right)$$

of problem (3.5), (3.6) and using the differentiation formula

$$\mu'(s) = \frac{-\pi^2}{4s(1-s^2)\mathbf{K}'^2(s)}, \tag{3.7}$$

see [AVV, p. 82], we find

$$\begin{aligned} -\frac{\pi}{2}\Phi_{x_1} &= \frac{1}{2\sqrt{x_1}(1+x_1)} + \lambda\frac{\pi}{2x_1(1-x_1{}^2)\mathbf{K}'^2(x_1)}, \\ -\frac{\pi}{2}\Phi_{x_2} &= \frac{1}{2\sqrt{x_2}(1+x_2)} + \lambda\frac{\pi}{2x_2(1-x_2{}^2)\mathbf{K}'^2(x_2)}. \end{aligned}$$

Therefore the critical values of x_1, x_2 of problem (3.5), (3.6) can be found from the equation

$$\sqrt{x}(1-x)\mathbf{K}'^2(x) = t \tag{3.8}$$

with $t = -\pi\lambda$.

Lemma 3. *Let $f(x) = \sqrt{x}(1-x)\mathbf{K}'^2(x)$. There is $x_0 = .04771\ldots$ such that $f(x)$ increases from 0 to $t_0 = f(x_0) = 4.08366\ldots$ on $0 < x < x_0$ and decreases from t_0 to 0 on $x_0 < x < 1$.*

Proof. Let $g = \sqrt{f}$. Since

$$\frac{d}{dx}\mathbf{K}(x) = \frac{\mathbf{E}(x) - (1-x^2)\mathbf{K}(x)}{x(1-x^2)},$$

see [AVV, p. 50], differentiating we obtain

$$g'(x) = \frac{u(x)}{v(x)},$$

where

$$u(x) = (1-x)^2\mathbf{K}'(x) - 4\mathbf{E}'(x), \qquad v(x) = 4x^{3/4}(1+x)\sqrt{1-x}. \tag{3.9}$$

Since $v(x)$ is positive on $0 < x < 1$, we need to show that $u(x)$ changes its sign only once in the interval $0 < x < 1$.

First we show that $u(x)$ is convex on $0 < x < 1$. Differentiating twice we find

$$u''(x) = \frac{u_1(x)}{v_1(x)},$$

where

$$u_1(x) = q(x)\mathbf{E}'(x)\left(\frac{\mathbf{K}'(x)}{\mathbf{E}'(x)} + \frac{p(x)}{q(x)}\right) \tag{3.10}$$

with

$$\begin{aligned} p(x) &= \frac{1}{4}(1+x)(1-x^2)(1+2x-5x^2), \\ q(x) &= \frac{1}{2}x^2(1+x)(1-x^2)(1-x+x^2), \end{aligned}$$

and

$$v_1(x) = x^2(1-x^2)^2.$$

Since $v_1(x) > 0$ for $0 < x < 1$, the convexity of u will follow if we show that $u_1(x) > 0$ for $0 < x < 1$. Since $q(x) > 0$ and $\mathbf{E}'(x) > 0$ for $0 < x < 1$, it follows from (3.10) that $u_1(x) > 0$ if and only if,

$$\frac{\mathbf{K}'(x)}{\mathbf{E}'(x)} + \frac{p(x)}{q(x)} > 0 \tag{3.11}$$

for $0 < x < 1$. To prove (3.11), we apply the inequality

$$\frac{\mathbf{K}'(x)}{\mathbf{E}'(x)} > \frac{2}{1+x^2} \tag{3.12}$$

which is equivalent to the known inequality

$$\mathbf{K}(x) - \mathbf{E}(x) > \mathbf{E}(x) - (1-x^2)\mathbf{K}(x),$$

see [AVV, Theorem 3.21(6)]. Thus, (3.11) follows from the inequality

$$\frac{2}{1+x^2} + \frac{p(x)}{q(x)} = \frac{2+2x-2x^3-x^4}{x^2(1+x^2)(1-x+x^2)} > 0,$$

which obviously holds for all $0 < x < 1$. Therefore $u_1(x) > 0$ and $u(x)$ is convex on $0 < x < 1$.

Using the well known limit values of $\mathbf{K}'(x)$ and $\mathbf{E}'(x)$, see [AVV, p. 49], we find

$$\lim_{x\to+0} u(x) = +\infty, \qquad u(1) = -2\pi,$$

which combined with the convexity property of $u(x)$ implies the desired monotonicity of f.

Solving the equation $u(x) = 0$ with MAPLE, we find $x_0 = .04771\ldots$ and $f(x_0) = t_0 = 4.08366\ldots$. ■

According to Lemma 3, for $0 < t \le t_0$, equation (3.8) has two solutions. Let $x = x_1(t)$ and $x = x_2(t)$, where $0 < x_1(t) < x_0 < x_2(t) < 1$ for $t \ne t_0$. The graphs of x_1, x_2, and f, as defined in Lemma 3, are shown in Figure 5.

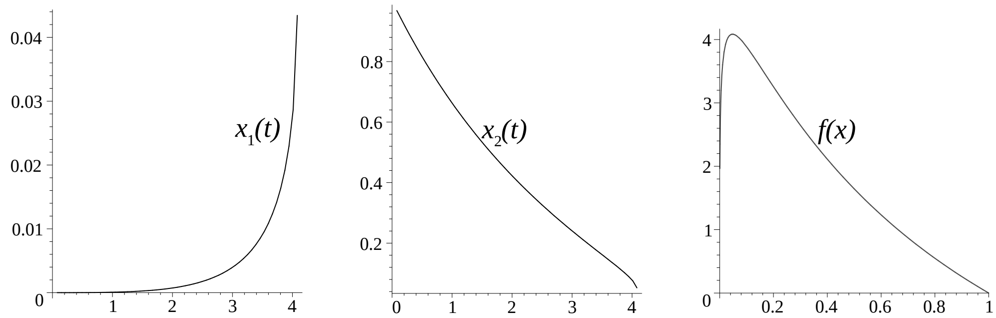

FIGURE 5. Graphs of x_1, x_2, and f.

The analysis made above leads to the following three possibilities for the extremal configurations of Problem (2.1):

1. Extremal configuration is symmetric. In this case the minimal harmonic measure is defined by (3.1).
2. Extremal configuration is degenerate. In this case the minimal harmonic measure equals $u_d(r)$, where $u_d(r)$ is defined by (3.2).
3. In the third possible case the minimal harmonic measure is $m(\tau_1, \tau_2)$ with $m(\cdot,\cdot)$ defined by (3.5) and $\tau_1 = x_1(t)$, $\tau_2 = x_2(t)$ for some t, $0 < t < t_0$, such that

$$\frac{1}{\mu(x_1(t)} + \frac{1}{\mu(x_2(t)} = \frac{2}{\mu(r^2)}. \tag{3.13}$$

Let $G(t)$ denote the left hand side of (3.13). The graphs of $G(t)$ and $2/\mu(r^2)$ are shown in Figure 6.

Now we prove our main result.

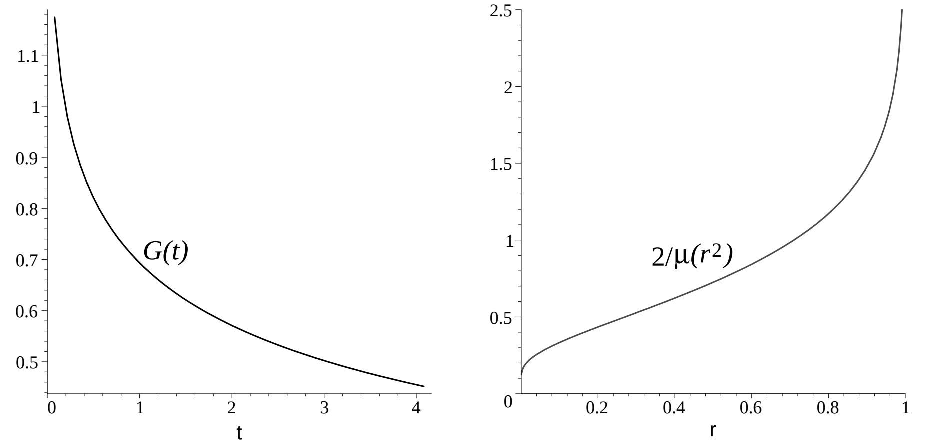

FIGURE 6. Graphs of $G(t)$ and $2/\mu(r^2)$.

Theorem 1. *Let $\Omega(r)$ denote the minimal harmonic measure of Problem $\mathcal{P}(r)$. Then $\Omega(r) = u_d(r)$ for $0 < r \le r_0$ and $\Omega(r) = u_s(r)$ for $r_0 \le r < 1$.*

For $r \ne r_0$, the extremal configuration is unique up to symmetry with respect to the imaginary axis. It is degenerate for $0 < r < r_0$ and symmetric for $r_0 < r < 1$. For $r = r_0$, the degenerate and symmetric configurations are the only extremal configurations of Problem $\mathcal{P}(r)$.

Proof. *Part 1:* First we show that there is no nonsymmetric nondegenerate minimal configuration with the corresponding value of t in the interval $\hat{t} \le t \le t_0$, where $\hat{t} = 3.85609\ldots$ is a solution to the equation

$$1 - \frac{2}{\pi}\left(\tan^{-1}\sqrt{x_0} + \tan^{-1}\sqrt{x_2(t)}\right) = u_d(r) \tag{3.14}$$

with $r = r(t)$ defined by

$$\frac{1}{\mu(x_1(t)} + \frac{1}{\mu(x_0)} = \frac{2}{\mu(r^2)}. \tag{3.15}$$

Since $\mu'(s) < 0$, see (3.7), the function $1/\mu(s)$ is strictly increasing. Therefore $r(t)$ is an increasing function uniquely determined by (3.15). Now since the left hand side of (3.14) strictly increases in t and the right hand side strictly decreases, $\hat{t}$ is uniquely determined.

Assume that for some $0 < r' < 1$ there is a nonsymmetric nondegenerate minimal configuration, which corresponds to t', $\hat{t} \le t' < t_0$. Let $x_1(t')$ and $x_2(t')$ be the corresponding solutions of (3.8). Since these solutions are monotone, we have

$$\begin{gathered} x_1(\hat{t}) \le x_1(t') < x_1(t_0) < x_2(t') \le x_2(\hat{t}), \\ x_1(\hat{t}) = .01836\ldots, \quad x_1(t_0) = .04772\ldots, \quad x_2(\hat{t}) = .10536\ldots. \end{gathered} \tag{3.16}$$

Therefore, it follows from (3.13) and (3.16) that

$$r' > r_-, \qquad \text{where } r_- = r(\hat{t}) = .17610\ldots$$

The corresponding average harmonic measure is defined by (2.8):

$$\Omega(r') = 1 - \frac{2}{\pi}\left(\tan^{-1}\sqrt{x_1(t')} + \tan^{-1}\sqrt{x_2(t')}\right).$$

This combined with (3.16) leads to the lower bound

$$\Omega(r') > 1 - \frac{2}{\pi}\left(\tan^{-1}\sqrt{x_0} + \tan^{-1}\sqrt{x_2(\hat{t})}\right) = u_d(r_-) > u_d(r')$$

contradicting the conjectured extremality of the corresponding configuration.

Part 2: Next we will show that the function $G(t)$ strictly decreases from $+\infty$ to $G_* = G(t_*) = .45204\ldots$, when t runs from 0 to t_*, where $t_* = 4.07389\ldots$ is defined by the equation

$$\frac{1}{\mu(x_0)} + \frac{1}{\mu(x_2(t_*))} = G(\hat{t}).$$

Since $x_1(t)$ and $x_2(t)$ are solutions to (3.8), implicit differentiation gives

$$G'(t) = \frac{\pi}{t}\frac{f_1(x_1) + f_1(x_2)}{f_1(x_1)f_1(x_2)},$$

where $x_1 = x_1(t)$, $x_2 = x_2(t)$ and

$$f_1(x) = 2\sqrt{x}(1+x)f'(x) = \mathbf{K}'(x)u(x),$$

where f and u are defined in Lemma 3. Note that f_1 strictly decreases in $0 < t < 1$. Indeed,

$$f_1'(x) = -\frac{2\mathbf{E}'^2(x)f_2(x)}{x(1-x^2)},$$

where

$$f_2(x) = x(1+x^2)\left(\frac{\mathbf{K}'(x)}{\mathbf{E}'(x)}\right)^2 + (x-1)^2\frac{\mathbf{K}'(x)}{\mathbf{E}'(x)} - 2.$$

Since $\mathbf{K}'(x)/\mathbf{E}'(x) > 2/(1+x^2)$, by (3.12), it follows that

$$f_2(x) > \frac{4x}{1+x^2} + \frac{2(1-x)^2}{1+x^2} - 2 = 0.$$

Therefore, $f_1'(x) < 0$ for $0 < x < 1$. Hence, f_1 is strictly decreasing on this interval.

Finding the limit values of $f_1(x)$, we obtain

$$\lim_{x\to+0} f_1(x) = +\infty, \qquad \lim_{x\to 1-0} f_1(x) = -\pi^2.$$

Lemma 3 implies that $f_1(x_1)f_1(x_2) < 0$ for all t, $0 < t < t_0$. Therefore, $G'(t) < 0$ if and only if

$$f_1(x_1(t)) + f_1(x_2(t)) > 0. \tag{3.17}$$

To prove that (3.17) holds for $0 < t \le t_*$, we consider iterations t_n, where $t_1 = 0$ and t_{n+1} is defined inductively by the following chain:

$$\begin{aligned} t_n \ &\mapsto\ x_{2,n} = x_2(t_n) \\ &\mapsto\ f_{1,n} = -f_1(x_{2,n}) \\ &\mapsto\ x_{1,n} = f_1^{-1}(f_{1,n}) \\ &\mapsto\ t_{n+1} = x_1^{-1}(x_{1,n}). \end{aligned}$$

The results of 12 iterations are shown in Table 1.

n	1	2	3	4	5	6
t_n	0	3.64142	3.90656	3.98852	4.02433	4.04314
$x_{2,n}$	1	0.14002	0.09653	0.08070	0.07258	0.06765
$f_{1,n}$	π^2	5.46528	3.78714	2.89890	2.34856	1.97399
$x_{1,n}$	0.01174	0.02079	0.02638	0.03009	0.03271	0.03466
t_{n+1}	3.64142	3.90656	3.98852	4.02433	4.04314	4.05424

n	7	8	9	10	11	12
t_n	4.05424	4.06133	4.06613	4.06954	4.07204	4.07393
$x_{2,n}$	0.06435	0.06198	0.06020	0.05881	0.05770	0.05680
$f_{1,n}$	1.70253	1.49674	1.33535	1.20539	1.09848	1.00900
$x_{1,n}$	0.03616	0.03736	0.03833	0.03913	0.03981	0.04039
t_{n+1}	4.06133	4.06613	4.06954	4.07204	4.07393	4.07539

TABLE 1. Iterations t_n, $x_{2,n}$, $f_{1,n}$, $x_{1,n}$.

We should emphasize that all explicitly shown digits of all entries in Table 1 are exact. The computed functions $x_1(t)$, $x_2(t)$, $f_1(x)$ and their inverses are real analytic. The known computational methods allow us to compute their values at a finite number of points (here 48) with the desired accuracy. Throughout the paper, we keep 5 exact digits, but the computations were actually done up to 10 digits. All computations were made with MAPLE and then doublechecked with MATHEMATICA.

Note that

$$t_{12} = 4.07394\ldots > t_* = 4.07389\ldots.$$

Let t', $0 < t' \le t_*$ satisfy $t_n < t' \le t_{n+1}$. Then, since the functions x_1, x_2, f_1 are monotone, we have:

$$x_{1,n-1} < x_1(t') \le x_{1,n}, \qquad x_{2,n+1} \le x_2(t') < x_{2,n},$$

$$-f_{1,n} < f_1(x_2(t')) \le -f_{1,n+1}, \quad f_{1,n} \le f_1(x_1(t')) < f_{1,n-1}$$

for $n = 1, ..., 16$, where $x_{1,0} = 0$ and $f_{1,0} = +\infty$. Hence, $f_1(x_1(t')) + f_1(x_2(t')) > 0$. Therefore, $G'(t) < 0$ and $G(t)$ strictly decreases in $0 < t \le t_*$.

Part 3: Now we show that for a fixed r, $0 < r < 1$, and a defined by (2.7), the limit point $b = a$ provides a local minimum for $\Omega(a, b)$. We put $b = a - \delta$ with $\delta > 0$ small enough and then estimate the arctangents in (2.8). The well known upper bound

$$\mu(s) < \log \frac{4}{s}$$

see [AVV, p. 80] leads to the following estimate for $\tau_1 = \tau_1(a, a - \delta)$:

$$0 < \tau_1 < 4e^{-\pi/\delta},$$

which shows that

$$0 \le \tan^{-1} \sqrt{\tau_1(a, a - \delta)} < 2e^{-\pi/(2\delta)} = o(\delta^n)$$

for any positive integer n as $\delta \to +0$.

Since the second arctangent in (2.8) is differentiable at $b = a$, we obtain

$$\Omega(a, a - \delta) = \Omega(a, a) - \frac{\delta \mu^2(\tau_2)}{\pi^2 \mu'(\tau_2)} \frac{1}{\tau_2^{1/2}(1 + \tau_2)} + o(\delta) \tag{3.18}$$

as $\delta \to +0$. Since $\mu'(\tau) < 0$ for $0 < \tau < 1$, (3.18) shows that the degenerate configuration provides a local minimum in the problem under consideration.

Part 4: Since $G(t)$ and $\mu(r^2)$ both are monotone, it follows that (3.13) defines a decreasing function $t = t(r)$ from $r_* \le r < 1$ onto $0 < t \le t_*$, where $r_* = .21886\ldots$ is a solution of the equation

$$\frac{2}{\mu(r^2)} = \frac{1}{\mu(x_1(t_*))} + \frac{1}{\mu(x_2(t_*))}.$$

Assume now that for some r', $0 < r' < 1$ there is a nonsymmetric nondegenerate extremal configuration. Let t', $0 < t' < t_0$ be a corresponding value of t. Since we have shown in Part 1 that for $\hat{t} \le t \le t_0$ there is no non-symmetric non-degenerate configuration, it follows that $0 < t' < \hat{t}$.

Note that

$$G(t) \le G(\hat{t}) \qquad \text{for } \hat{t} \le t \le t_0. \tag{3.19}$$

Indeed, $G(t)$ strictly decreases in $\hat{t} < t \le t_*$ and

$$G(t) \le \frac{1}{\mu(x_0)} + \frac{1}{\mu(x_2(t_*))} = G(\hat{t})$$

for $t_* \le t \le t_0$.

Since $G(t') = 2/\mu(r'^2) > G(\hat{t})$, (3.19) and the monotonicity property of $G(t)$ for $0 < t \le t_*$ imply that for fixed $r = r'$, equation (3.13) has a unique solution $t = t'$. Therefore, for $r = r'$ there is only one critical point $b = b'$, $0 < b' < a$ of the constrained minimization problem (2.7), (2.8).

Since we have shown in Part 3 that $b = a$ provides a local minimum, it follows that $b = b'$ provides a local maximum for the average harmonic measure considered as a function of b. Hence the symmetric and degenerate configurations are the only possible ones minimizing the average harmonic measure. Now the required assertion follows from Lemma 2. ∎

The idea used to show that $G'(t) < 0$ for $0 < t \leq t_*$ can be applied to prove monotonicity of G for the whole range $0 < t < t_0$. For this we first estimate $f_1(x_1(t)) + f_1(x_2(t))$ using asymptotic expansions of $x_1(t)$ and $x_2(t)$ in a vicinity of $t = t_0$, then apply iterations described above. However, this requires more iterations and is not necessary for the proof of Theorem 1.

Some remarks concerning the symmetric case might be interesting. As we noticed in the begining of this section, $b = 0$ is a critical point of the minimization problem under consideration. To decide whether it provides a minimum or maximum, we apply the standard second derivative test. From (2.7), (2.8), we find

$$\pi \frac{\partial^2 \Omega}{\partial b^2} = \sum_{i=1}^{2} \left(\frac{1}{2} \frac{1+3\tau_i}{\tau_i^{3/2}(1+\tau_i)^2} \left(\frac{\partial \tau_i}{\partial b} \right)^2 - \frac{1}{\tau_i^{1/2}(1+\tau_i)} \frac{\partial^2 \tau_i}{\partial b^2} \right), \tag{3.20}$$

$$\begin{aligned} \frac{\partial \tau_i}{\partial b} &= \frac{(-1)^{i+1}\mu^2(\tau_i)}{\pi \mu'(\tau_i)}, \\ \frac{\partial^2 \tau_i}{\partial b^2} &= -\frac{\mu''(\tau_i)\mu(\tau_i) - 2\mu'^2(\tau_i)}{\mu'(\tau_i)\mu(\tau_i)} \left(\frac{\partial \tau_i}{\partial b} \right)^2. \end{aligned} \tag{3.21}$$

Let $\tau = \tau_1(a,0) = \tau_2(a,0) = r^2$. Then (3.20), (3.21) lead to

$$\begin{aligned} \pi \frac{\partial^2 \Omega}{\partial b^2}\bigg|_{b=0} &= \frac{\mu^3(\tau)}{\pi^2 \tau^{3/2}(1+\tau)^2 \mu'^3(\tau)} \\ &\quad \times \left[(1+3\tau)\mu'(\tau)\mu(\tau) + 2\tau(1+\tau)(\mu''(\tau)\mu(\tau) - 2\mu'^2(\tau)) \right] \end{aligned} \tag{3.22}$$

Using the differentiation formulas for $\mu(\tau)$, see [AVV, p. 82], and Legendre's relation

$$\mathbf{E}(\tau)\mathbf{K}'(\tau) + \mathbf{E}'(\tau)\mathbf{K}(\tau) - \mathbf{K}(\tau)\mathbf{K}'(\tau) = \frac{\pi}{2},$$

we obtain from (3.22),

$$\pi \frac{\partial^2 \Omega}{\partial b^2}\bigg|_{b=0} = \frac{1}{\pi^2} \tau^{1/2}(1-\tau)\mathbf{K}'^3(\tau)\, u(\tau)$$

with $u(x)$ defined by (3.9). Since $u(x)$ has only one zero $x = x_0$ in $0 < x < 1$, see Lemma 3, it follows that

$$\left.\frac{\partial^2\Omega}{\partial b^2}\right|_{b=0} < 0 \quad \text{if} \quad 0 < r = \sqrt{\tau} < r_0,$$

$$\left.\frac{\partial^2\Omega}{\partial b^2}\right|_{b=0} > 0 \quad \text{if} \quad r_0 < r = \sqrt{\tau} < 1,$$

which shows that for $0 < r < r_0$ the symmetric configuration provides a local maximum and for $r_0 < r < 1$ it provides a local minimum. Figure 7 demonstrates typical behavior of the average harmonic measure $\Omega(a, b)$ as a function of b for some values of a or, equivalently, for some values of the main parameter r.

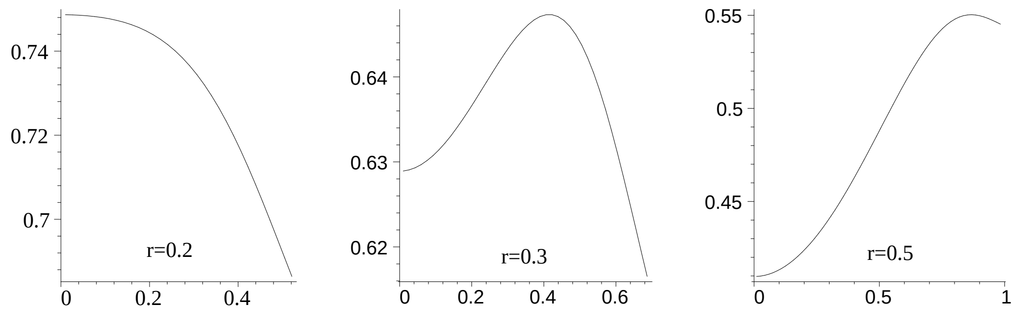

FIGURE 7. Typical behavior of $\Omega(a, b)$.

4. Some remarks and questions

Lemma 1, which actually describes the range of all possible pairs of the harmonic measures $\omega(r, E, D_1)$, $\omega(-r, E, D_2)$, can be used to extremize some other characteristics, besides the average harmonic measure, of the studied configurations. In particular, a solution to the problem

$$\min_E \max\{\omega(r, E, D_1),\ \omega(-r, E, D_2)\} \tag{4.1}$$

is an immediate consequence of Lemma 1 and the easy observation that the equality $\omega(r, E, D_1) = \omega(-r, E, D_2)$ must hold for any extremal configuration. This implies that the symmetric configuration is the unique minimizer of (4.1).

It is interesting to note that the n-dimensional counterparts of Problems (2.1) and (4.1) are much harder. They are open even in the case $n = 3$, where both questions sound very natural. The difficulty here is the lack of an analogous theory to Jenkins' theory of the extremal decompositions into nonoverlapping regions, see [J1, S2].

A symmetrization result of A. Baernstein [B, BT] that is still applicable, shows that extremal regions D_1, D_2 are bounded by surfaces of revolution about the axis containing both reference points $a_1 = r$ and $a_2 = -r$. Of course, for the extremal regions D_1, D_2 of the n-dimensional version of problem (4.1) the equality $\omega(a_1, E, D_1) = \omega(a_2, E, D_2)$ is satisfied in all dimensions.

Another interesting open question concerns a similar control for the average or maximal harmonic measure at $n \geq 3$ points $a_k = re^{2\pi i(k-1)/n}$, $k = 1, \dots, n$, equally distributed on the circle $\{z : |z| = r\}$, $0 < r < 1$, in the unit disk Δ. More precisely, the following problem would be of some interest.

Let a continuum E split Δ into n simply connected (or more general) regions $D_k \ni a_k$, $k = 1, \dots, n$. Find

$$\inf_E \frac{1}{n} \sum \omega(a_k, E, D_k) \qquad \text{and} \qquad \inf_E \max_{k=1,\dots,n} \omega(a_k, E, D_k)$$

and describe possible extremal configurations.

The conjectured extremal configuration of the minmax problem consists of n circular sectors

$$S_k = \left\{ z : |z| < 1, \left| \arg z - 2\pi \frac{k-1}{n} \right| < \frac{\pi}{n} \right\}, \qquad k = 1, \dots, n.$$

Under an additional assumption that the closure of $\partial D_k \cap \mathbb{U}$ is connected for all $k = 1, \dots, n$, this can be proved applying Jenkins's theory on the extremal decomposition [J1, S2]. This result will be published elsewhere.

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