TWO PARAMETER FAMILIES OF CLOSE-TO-CONVEX FUNCTIONS AND CONVOLUTION THEOREMS

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Abstract. Using a recently obtained coefficient condition for functions with positive real part by M. Obradović and S. Ponnusamy, we construct four different useful examples of two parameter families of close-to-convex functions. As a consequence of another well-known coefficient condition due to M. Obradović and S. Ponnusamy, we obtain a number of results concerning the convolution of two univalent functions. Using these results, we obtain criteria for combinations of hypergeometric functions to be univalent.

Key words. Gaussian hypergeometric function, univalent function, starlike function, Hadamard product.

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1. Introduction

Let $\mathcal{H}$ denote the class of all functions which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. Set $\mathcal{A} = \{f \in \mathcal{H} : \; f(0) = f'(0) - 1 = 0\}$ and $\mathcal{S} = \{f \in \mathcal{A} : \; f \text{ is univalent in } \Delta\}$. Let $\mathcal{S}^*$ denote the class of all functions in $\mathcal{A}$ such that $f(\Delta)$ is starlike with respect to origin. A function $f \in \mathcal{A}$ will be said to be close-to-convex if there exists a convex function g (meaning that $g(\Delta)$ is convex) such that $\mathrm{Re}\,(f'(z)/g'(z)) > 0$ in Δ. Every close-to-convex function is known to be univalent in Δ and therefore, the set of all close-to-convex functions form a subset of $\mathcal{S}$. For $\beta < 1$, we let $\mathcal{P}_\eta(\beta)$ denote the subclass of functions $p \in \mathcal{H}$ which satisfies the condition $\mathrm{Re}\, e^{i\eta}(p(z) - \beta) > 0$ for all $z \in \Delta$ and for some $\eta \in \mathbb{R}$ with the normalization $p(0) = 1$. Set $\mathcal{P}_0(\beta) = \mathcal{P}(\beta)$ and $\mathcal{P}(0) = \mathcal{P}$. We are interested in the following

PROBLEM 1.1. Are there nice conditions on the Taylor's coefficients of $f \in \mathcal{A}$ which can be used to check the univalency, close-to-convex, starlikeness and convexity of f?

As indicated in Section 2, there exist few conditions on the Taylor's coefficient of $f \in \mathcal{A}$ so that f is close-to-convex on Δ, [14]. On the other hand, there is the following starlikeness condition due to Fejer [3].

LEMMA 1.2. *If $a_n \geq 0$, and the sequences $\{na_n\}$ and $\{na_n - (n+1)a_{n+1}\}$ are both decreasing, then the function $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{A}$ is starlike.*

This result is particularly useful in the study of starlikeness of hypergeometric functions, see [16]. In [8, Theorem 2.1] the following result has been obtained as a special case and as a consequence, a number of previous conditions for the starlikeness of hypergeometric functions are improved.

LEMMA 1.3. *If a function $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{A}$ satisfies any one of the following conditions*

$$\sum_{n=2}^{\infty} |(n+1)a_n - na_{n-1}| + |(n-1)a_n - (n-2)a_{n-1}| \leq 2,$$

$$\sum_{n=2}^{\infty} |(n+1)a_n - (n-1)a_{n-2}| + |(n-1)a_n - (n-3)a_{n-2}| \leq 2,$$

$$\sum_{n=2}^{\infty} |(n-2)(a_n - 2a_{n-1} + a_{n-2}) + a_n - a_{n-2}| + |n(a_n - 2a_{n-1} + a_{n-2}) + a_n - a_{n-2}| \leq 2,$$

then f is starlike.

We do not discuss all aspects of Problem 1.1, but in Section 2, we provide four different two parameter families of close-to-convex functions by applying recently known coefficient conditions due to Obradović and Ponnusamy [9]. Actually in the coefficient condition of [9] (see equation (1.5) below), the corresponding function $g \in \mathcal{A}$ was required to be locally univalent. On the other hand, in the construction of the four examples (see Examples 2.4, 2.6, 2.8, 2.9), we do not need to assume the local univalency because the associated coefficient conditions itself guarantee the local univalency of the corresponding function. We note that this is not true in general.

Moreover, the close-to-convexity of the functions involved in these four sets of the examples is not possible to obtain with the help of other known methods and in view of these observations, it is appropriate to say that they are in some sense "useful examples".

Let us start by comparing two different known coefficient inequalities for univalent functions. By the Noshiro-Warschawski-Wolff Theorem (see [1] for example) it follows that if $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and if

$$\sum_{n=1}^{\infty} |p_n| \leq 1 - \beta \quad \left(p_n = (n+1)b_{n+1} = \frac{g^{(n+1)}(0)}{n!},\ n \geq 1\right), \tag{1.4}$$

then it is clear that $|g'(z) - 1| < 1 - \beta$ and, in particular, $g'(z) \in \mathcal{P}(\beta)$ so that g is close-to-convex, hence univalent in Δ for $0 \leq \beta < 1$. From a result of Obradović and Ponnusamy [9, Theorem 3.2] it follows that if $g \in \mathcal{A}$ defined above is *locally univalent with real coefficients* and satisfies the coefficient condition

$$\sum_{n=2}^{\infty} n|p_{n+1} - p_{n-1}| \leq (1 - \beta) - |p_2 - (1 - \beta)| \tag{1.5}$$

then one also has $g' \in \mathcal{P}(\beta)$. Let us first start by showing by examples that (1.4) neither implies nor is implied by (1.5). In addition, in view of (1.5), we provide a number of examples to demonstrate the existence of a two parameter family of close-to-convex functions (and hence in $\mathcal{S}$) which cannot be shown to be in $\mathcal{S}$ by the conditions (1.4). The example

$$g(z) = -\log(1 - z)$$

shows that it is not possible to use (1.4) to conclude that g is univalent in Δ. On the other hand, according to (1.5) with $\beta = 1/2$, it follows that $g' \in \mathcal{P}(1/2)$ and the bound $\beta = 1/2$ is sharp. We remark that g is a convex function (of order 1/2). Next we consider the function

$$\phi(z) = 2\log\left(\frac{2}{2-z}\right), \quad z \in \Delta.$$

Then the corresponding p_n gives

$$p_n = \frac{\phi^{(n+1)}(0)}{n!} = 1/2^n$$

so that $\sum_{n=1}^{\infty} |p_n| = 1$ and hence, according to (1.4) with $\beta = 0$, we have $|\phi'(z) - 1| < 1$ for all $z \in \Delta$. On the other hand

$$\sum_{n=1}^{\infty} n|p_{n+1} - p_{n-1}| = 3\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 3$$

which shows that the condition (1.5) is not applicable in this case even to claim that $\operatorname{Re}\phi'(z) > 0$ in Δ. Thus, these two examples show that (1.5) is more appropriate to use when p_n's are big while (1.4) is good to use when p_n's are small enough.

Next we consider $f \in \mathcal{A}$ with $f(z)/z \neq 0$ for $z \in \Delta$. Let

$$\log\left(\frac{f(z)}{z}\right) = \sum_{n=1}^{\infty} p_n z^n.$$

Then

$$\frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} np_n z^n, \quad z \in \Delta.$$

According to (1.5) (with $\beta = 0$), the coefficient condition

$$\sum_{n=2}^{\infty} n|(n+1)p_{n+1} - (n-1)p_{n-1}| \leq 1 - |2p_2 - 1| \tag{1.6}$$

implies that $f \in \mathcal{S}^*$. For example, if $f(z) = z/(1-z)^2$ then

$$\log\left(\frac{f(z)}{z}\right) = -2\log(1-z) = 2\sum_{n=1}^{\infty} \frac{z^n}{n}$$

so that, with $p_n = 2/n$, the condition (1.6) is clearly satisfied. Similarly, the starlikeness of $f(z) = z/(1-z^2)$ is easy to see from the condition (1.6).

2. Two Parameter Families of Close-to-convex Functions

In this section we exploit the fact that if zg' is starlike then f'/g' being in $\mathcal{P}(\beta)$ for some $0 \leq \beta < 1$ implies that f is close-to-convex. Thus, by making careful choices of g we can obtain conditions on the coefficients, p_n, of f'/g' to relate recurrence properties of the p_n's to the close-to-convexity of f.

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we compute (with $a_0 = 0 = a_1 - 1$),

$$(1-z)f'(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (p_n = (n+1)a_{n+1} - na_n)$$

$$(1-z^2)f'(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (p_n = (n+1)a_{n+1} - (n-1)a_{n-1})$$

$$(1-z)^2 f'(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (p_n = (n+1)a_{n+1} - 2na_n + (n-1)a_{n-1})$$

$$(1-z+z^2)f'(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (p_n = (n+1)a_{n+1} - na_n + (n-1)a_{n-1})$$

and, in view of these equations, we have the following result whose extended form was shown in the work of Ponnusamy [14].

LEMMA 2.1. *If $f \in \mathcal{A}$ and p_n is one of the above forms, then $\sum_{n=1}^{\infty} |p_n| \leq 1$ implies that the corresponding f is close-to-convex in Δ.*

This lemma has been very useful in finding the range values of the parameters a, b, c for which the Gaussian hypergeometric functions $F(a, b; c; z) - 1$ and $zF(a, b; c; z)$ are close-to-convex. For example, if $f \in \mathcal{A}$ satisfies the condition $\mathrm{Re}\,\{(1-z^2)f'(z)\} > 0$ for all $z \in \Delta$, then f is close-to-convex and that

$$\sum_{n=1}^{\infty} |(n+1)a_{n+1} - (n-1)a_{n-1}| \leq 1 \quad (a_0 = 0,\ a_1 = 1) \tag{2.2}$$

is sufficient for that. But we can get an appropriate condition by using the coefficient condition (1.5) which seems to be more useful in the special situations that we consider in this section in the form of examples. Now,

$$(1-z^2)f'(z) = 1 + \sum_{n=1}^{\infty} ((n+1)a_{n+1} - (n-1)a_{n-1})z^n \quad (a_0 = 0,\ a_1 = 1)$$

so that, for $a_n \in \mathbb{R}$, the inequality (1.5) is equivalent to

$$\sum_{n=2}^{\infty} n|(n+2)a_{n+2} - 2na_n + (n-2)a_{n-2}| \leq 1 - |3a_3 - 2|. \tag{2.3}$$

Observe that if $f_0(z) = z/(1-z)$, then

$$(1-z^2)f_0'(z) = \frac{1+z}{1-z}$$

so that f_0 is close-to-convex with respect to $\frac{1}{2}\log((1+z)/(1-z))$. However, the (sufficient) condition (2.2) is not satisfied for $f_0(z) = z/(1-z)$ although it does satisfy the condition (1.5) with the condition (2.2) being equivalent to $\sum_{n=0}^{\infty} 2 \leq 1$! but the condition (2.3) is equal to $0 \leq 0$ so that (2.3) is applicable for $f_0(z)$. Similar situations arise when we consider the other three coefficient conditions addressed in Lemma 2.1.

In the next four examples, we use the condition (1.5) to obtain two parameter families of close-to-convex with respect to specific convex functions.

EXAMPLE 2.4. Consider a locally univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \in \mathbb{R}$. Then, we have

$$(1-z)f'(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

where $p_n = (n+1)a_{n+1} - na_n$. Suppose that $|p_2 - 1| \leq 1$ and, for convenience, we restrict our attention to the special case $p_{n+1} = p_{n-1}$ for $n \geq 2$. Then, it is easy to find the explicit form of $f(z)$ satisfying (1.5) for $\beta = 0$ which is equivalent to showing that $\mathrm{Re}\,\{(1-z)f'(z)\} > 0$ in Δ. Indeed, for $n \geq 2$, the condition $p_{n+1} = p_{n-1}$ is equivalent to the recurrence relation

$$(n+2)a_{n+2} - (n+1)a_{n+1} - na_n + (n-1)a_{n-1} = 0, \quad n \geq 2.$$

Solving this, one can easily see that for $k \geq 1$

$$a_{2k} = \frac{3(k-1)a_3 + 2a_2 - (k-1)}{2k} \quad \text{and} \quad a_{2k+1} = \frac{3ka_3 - (k-1)}{2k+1}.$$

In view of these two expressions, f takes the form

$$f(z) = z + \sum_{k=1}^{\infty} \frac{3(k-1)a_3 + 2a_2 - (k-1)}{2k} z^{2k} + \sum_{k=1}^{\infty} \frac{3ka_3 - (k-1)}{2k+1} z^{2k+1}$$

which after a routine computation may be written equivalently as

$$f(z) = z + \left(\frac{3a_3 - 2a_2 - 1}{2}\right)\left(\frac{z^2}{1-z} + \log(1-z^2) - \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) + z\right)$$
$$+a_2\left(\frac{z^2}{1-z}\right) + (1-a_2)\left(\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) - z\right). \tag{2.5}$$

The condition $|p_2 - 1| \leq 1$ is equivalent to $|3a_3 - 2a_2 - 1| \leq 1$. Under this condition every locally univalent function $f(z)$ of the form (2.5) is always close-to-convex with respect to the convex function $-\log(1-z)$. It is a simple exercise to see that if f is given by (2.5), then

$$f'(z) = \frac{1 + (2a_2 - 1)z + (3a_3 - 2a_2 - 1)z^2}{(1-z)^2(1+z)}.$$

In particular, if $3a_3 = 2a_2 + 1$ then f takes the form

$$f(z) = a_2\left(\frac{z}{1-z}\right) + (1-a_2)\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

so that

$$f'(z) = \frac{1 + (2a_2 - 1)z}{(1-z)^2(1+z)}$$

which shows that f is locally univalent whenever a_2 is real and lies in the closed unit interval $[0, 1]$, and therefore, f is close-to-convex in Δ. Note that the well-known functions

$$f(z) = \frac{z}{1-z} \quad \text{and} \quad f(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

which correspond to the cases $a_2 = 1$ and $a_2 = 0$, respectively, satisfy the condition $\mathrm{Re}\,\{(1-z)f'(z)\} > 0$ in Δ, and in particular, they are close-to-convex with respect to $-\log(1-z)$.

EXAMPLE 2.6. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ($a_n \in \mathbb{R}$) is locally univalent. Then we compute

$$(1-z^2)f'(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

where $p_n = (n+1)a_{n+1} - (n-1)a_{n-1}$. Suppose that $|p_2 - 1| \leq 1$ and $p_{n+1} = p_{n-1}$ for $n \geq 2$. Then it is easy to find an explicit form of $f(z)$, involving the coefficients a_2 and a_3, which ensures that $\mathrm{Re}\,\{(1-z^2)f'(z)\} > 0$ in Δ. For $n \geq 2$, the condition $p_{n+1} = p_{n-1}$ is equivalent to the recurrence relation

$$(n+2)a_{n+2} - 2na_n + (n-2)a_{n-2} = 0, \quad n \geq 2.$$

Solving this, one can easily see that for $k \geq 1$

$$a_{2k} = a_2 \quad \text{and} \quad a_{2k+1} = \frac{3ka_3 - (k-1)}{2k+1}.$$

Using these two expressions, f takes the form

$$f(z) = z + \sum_{k=1}^{\infty} a_2 z^{2k} + \sum_{k=1}^{\infty} \frac{3ka_3 - (k-1)}{2k+1} z^{2k+1}$$

which after computation may be written equivalently as

$$\begin{aligned} f(z) = z + a_2\left(\frac{z^2}{1-z^2}\right) + \left(\frac{3a_3-2}{2}\right)\left(\frac{z^3}{1-z^2} - \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) + z\right) \\ +\frac{1}{2}\left(\frac{z^3}{1-z^2} + \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) - z\right). \end{aligned} \tag{2.7}$$

The condition $|p_2 - 1| \leq 1$ is equivalent to the inequality $|3a_3 - 2| \leq 1$. Under this condition, we conclude that every locally univalent function of the form f defined by (2.7) is always close-to-convex with respect to the convex function $\frac{1}{2}\log((1-z)/(1+z))$. To discuss the local univalency, we may compute

$$f'(z) = \frac{1 + 2a_2 z + (3a_3 - 2)z^2}{(1-z^2)^2}.$$

In particular, if $a_3 = 1$, then f takes the form

$$f(z) = \frac{z}{1-z^2} + a_2\left(\frac{z^2}{1-z^2}\right)$$

so that

$$f'(z) = \frac{1 + 2a_2 z + z^2}{(1-z^2)^2}$$

and therefore, for $a_2 \in \mathbb{R}$ with $|-a_2 + \sqrt{a_2^2 - 1}| = 1$, f in this case is seen to be locally univalent in Δ. The cases $a_2 = 0$ and $a_2 = 1$ give the functions $z/(1-z^2)$ and $z/(1-z)$, respectively. Moreover, if $a_3 = \dfrac{2}{3}$, then we have

$$f(z) = z + a_2\left(\frac{z^2}{1-z^2}\right) + \frac{1}{2}\left(\frac{z^3}{1-z^2} + \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) - z\right)$$

which is close-to-convex in Δ if $a_2 \in \mathbb{R}$ with $|-a_2+\sqrt{a_2^2-1}|=1$. For example, for $a_2=\frac{1}{2}$, we have the close-to-convex function

$$f(z)=\frac{1}{2}\left(\frac{z}{1-z}+\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right).$$

EXAMPLE 2.8. For $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{A}$, we write

$$(1-z)^2f'(z)=1+\sum_{n=1}^{\infty}p_nz^n$$

where

$$p_n=(n+1)a_{n+1}-2na_n+(n-1)a_{n-1}\quad (a_0=0,\ a_1=1).$$

Note that $|p_2-1|\leq 1$ if and only if $|3a_3-4a_2|\leq 1$. Again, we are interested in verifying the coefficient inequality (1.5) with $\beta=0$ and, for the case $p_{n+1}=p_{n-1}$. Thus, $p_{n+1}=p_{n-1}$ for $n\geq 2$ is equivalent to the relation

$$(n+2)a_{n+2}=2(n+1)a_{n+1}-2(n-1)a_{n-1}+(n-2)a_{n-2}.$$

Solving this recurrence relation, one can obtain that for $k\geq 1$

$$2ka_{2k}=k(k-1)(3a_3-1)-k(k-2)2a_2$$

and

$$(2k+1)a_{2k+1}=k^2(3a_3)-k(k-1)(2a_2)-(k^2-1)$$

which may be rewritten as

$$a_{2k}=\frac{k(3a_3-2a_2-1)-(3a_3-4a_2-1)}{2}$$

and

$$a_{2k+1}=\left(\frac{3a_3-2a_2-1}{2}\right)k-\left(\frac{3a_3-6a_2-1}{4}\right)+\left(\frac{3a_3-6a_2+3}{4(2k+1)}\right),$$

respectively. Thus, a simple calculation shows that f has the form

$$\begin{aligned} f(z) &= z+\left(\frac{3a_3-2a_2-1}{2}\right)\frac{z^2(1+z)}{(1-z^2)^2}-\left(\frac{3a_3-4a_2-1}{2}\right)\frac{z^2}{1-z^2} \\ &\quad -\left(\frac{3a_3-6a_2-1}{4}\right)\frac{z^3}{1-z^2}+\left(\frac{3a_3-6a_2+3}{4}\right)\left\{\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)-z\right\}. \end{aligned}$$

Hence, every locally univalent function $f(z)$ of the above form with $|3a_3-4a_2|\leq 1$ is close-to-convex with respect to the convex function $z/(1-z)$. To check the local univalency of f, we rewrite f in the form

$$f(z)=z+A\frac{z^2(1+z)}{(1-z^2)^2}-B\frac{z^2}{1-z^2}-C\frac{z^3}{1-z^2}+D\left\{\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)-z\right\}$$

and obtain that

$$f'(z)=\frac{M(z)}{(1-z^2)^3},$$

where

$$\begin{aligned}M(z) &= 1+(2A-2B)z+(3A-3C+D-3)z^2+(2A+2B)z^3+(A+4C-2D+3)z^4\\ &\quad +(-C+D-1)z^6.\end{aligned}$$

Using the values of A,B,C,D, it is easy to see that

$$f'(z)=\frac{(1+z)^2[(3a_3-4a_2)z^2+(2a_2-2)z+1]}{(1-z^2)^3}.$$

By a computation, we can find the range for a_2 and a_3 for local univalency of $f(z)$. In particular, for $a_2=1$, one has

$$f'(z)=\frac{(1+z)^2[(3a_3-4)z^2+1]}{(1-z^2)^3}$$

which shows that, if $a_3\in[1,5/3]$ and $a_2=1$, then f is locally univalent in Δ and therefore, in this case f must be close-to-convex in Δ.

Similarly, if $a_3=(4/3)a_2$ with $a_2\in[1/2,3/2]$ then f becomes close-to-convex in Δ.

EXAMPLE 2.9. For $f\in\mathcal{A}$, consider

$$(1-z+z^2)f'(z)=1+\sum_{n=1}^{\infty}p_nz^n,$$

where, for $n\geq 1$,

$$p_n=(n+1)a_{n+1}-na_n+(n-1)a_{n-1}\qquad\left(a_n=\frac{f^{(n)}(0)}{n!}\in\mathbb{R}\right)$$

with $a_0=0=a_1-1$. As before, $|p_2-1|\leq 1$ if and only if $|3a_3-2a_2|\leq 1$ and we assume that f is locally univalent in Δ with real coefficients. Note that $p_{n+1}=p_{n-1}$ for $n\geq 2$ is equivalent to the recurrence relation

$$(n+2)a_{n+2}=(n+1)a_{n+1}-(n-1)a_{n-1}+(n-2)a_{n-2},\quad n\geq 2.$$

By a calculation, it is easy to see that, for $n \geq 4$

$$na_n = \begin{cases} 3a_3 - 1 & \text{for} \quad n = 6k+4 \\ 3a_3 - 2a_2 & \text{for} \quad n = 6k+5 \\ 0 & \text{for} \quad n = 6k+6 \\ 1 & \text{for} \quad n = 6k+7 \\ 2a_2 & \text{for} \quad n = 6k+8 \\ 3a_3 & \text{for} \quad n = 6k+9 \end{cases}, \quad k \in \mathbb{N} \cup \{0\}.$$

Thus a simple calculation shows that f has the form

$$\begin{aligned} f(z) &= z + a_2z^2 + a_3z^3 + \left(\frac{3a_3-1}{4}\right) z^4\phi_1(z^6) + \left(\frac{3a_3-2a_2}{5}\right) z^5\phi_2(z^6) \\ &\quad + \left(\frac{1}{7}\right) z^7\phi_3(z^6) + \left(\frac{2a_2}{8}\right) z^8\phi_4(z^6) + \left(\frac{3a_3}{9}\right) z^9\phi_5(z^6), \end{aligned} \tag{2.10}$$

where,

$$\begin{aligned} \phi_1(z^6) &= F\left(1, \frac{2}{3}; \frac{5}{3}; z^6\right) \\ \phi_2(z^6) &= F\left(1, \frac{5}{6}; \frac{11}{6}; z^6\right) \\ \phi_3(z^6) &= F\left(1, \frac{7}{6}; \frac{13}{6}; z^6\right) \\ \phi_4(z^6) &= F\left(1, \frac{4}{3}; \frac{7}{3}; z^6\right) \\ \phi_5(z^6) &= F\left(1, \frac{3}{2}; \frac{5}{2}; z^6\right). \end{aligned}$$

Here $F(a,b;c;z)$ denotes the classical/Gaussian hypergeometric function usually defined by

$$F(a,b;c;z) := {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} z^n, \quad a, b \in \mathbb{C}, \; c \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$$

with $(a,0) = 1$ for $a \neq 0$ and for $n \geq 1$, $(a,n) = a(a+1)\cdots(a+n-1)$. To verify the local univalency, by a calculation, we obtain that

$$(1-z^6)f'(z) = 1 + 2a_2z + 3a_3z^2 + (3a_3-1)z^3 + (3a_3-2a_2)z^4.$$

Thus, for all those real values of a_2, a_3 such that $|3a_3 - 2a_2| \leq 1$ and

$$\Phi(z) = 1 + 2a_2z + 3a_3z^2 + (3a_3-1)z^3 + (3a_3-2a_2)z^4 \tag{2.11}$$

a non-vanishing function in Δ, the function f is close-to-convex in Δ. In particular, if $3a_3 = 2a_2$ then, in this choice, we have

$$f'(z) = \frac{(1+z+z^2)(1+z(2a_2-1))}{1-z^6}$$

which clearly shows the corresponding f given by (2.10) is locally univalent in Δ whenever a_2 is real and lies in the closed unit interval $[0,1]$, and therefore, f is close-to-convex in Δ. Similarly, if $3a_3 = 2a_2 + 1$ then Φ defined by (2.11) becomes

$$\Phi(z) = 1 + 2a_2z + (2a_2+1)z^2 + 2a_2z^3 + z^4$$

which may be rewritten as

$$\Phi(z) = (1+z+z^2)(1+(2a_2-1)z+z^2).$$

It is easy to see that, $\Phi(z) \neq 0$ in Δ provided $a_2 \in \mathbb{R}$ is such that

$$\left| a_2 - \frac{1}{2} \pm \sqrt{\left(a_2 - \frac{1}{2}\right)^2 - 1} \right| = 1. \tag{2.12}$$

Hence, under this condition f defined by (2.10) is close-to-convex in Δ whenever $a_3 = \dfrac{2a_2+1}{3}$ and $a_2 \in \mathbb{R}$ is given by (2.12).

REMARK 2.13. In all the four examples, we have chosen p_n's such that $p_{n+1} = p_{n-1}$. This is just a convenient choice to demonstrate the usefulness of the coefficient inequality (1.5) in obtaining different types of two parameter families of close-to-convex functions which cannot be generated by other means.

However, other choices of p_n's can be also made for a similar purpose, for example to obtain three parameter families of close-to-convex functions and so on.

3. Convolution Theorems

THEOREM 3.1. $[5, Theorem\ 11, p.193]$ *If $f \in \mathcal{S}$ with the representation*

$$\left(\frac{z}{f(z)}\right)^{\alpha/2} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad \alpha > 0,$$

then

$$\sum_{n=1}^{\infty} \left(\frac{2n-\alpha}{\alpha}\right) |b_n|^2 \leq 1. \tag{3.2}$$

If $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ belong to $\mathcal{H}$, then the Hadamard product or convolution, denoted by $f*g$, is defined by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n$$

and that $f*g\in\mathcal{H}$. Next, we consider the class $\mathcal{U}(\lambda)$ $(0<\lambda\leq 1)$ which is defined as follows:

$$\mathcal{U}(\lambda)=\left\{f\in\mathcal{A}:\left|\left(\frac{z}{f(z)}\right)^2f'(z)-1\right|<\lambda,\quad z\in\Delta\right\}.$$

The class $\mathcal{U}(1)\equiv\mathcal{U}$ together with its various generalizations and subclasses have been discussed by M. Obradović and S. Ponnusamy [9] and later by a number of authors (see [10, 11, 15]). In fact, Krzyz [6] has shown that function in $\mathcal{U}(\lambda)$ admits a Q- quasiconformal extension to the whole complex plane with $Q=\dfrac{1+\lambda}{1-\lambda}$ whenever $0<\lambda<1$. However, until our recent work not much attention has been paid to this class. We have shown that, just like "every normalized (univalent) convex function is starlike of order 1/2", the following analogous result from [9, 11] holds,

$$P(2\lambda)\subseteq\mathcal{U}(\lambda),$$

where

$$P(2\lambda)=\left\{f\in\mathcal{A}:\left|\left(\frac{z}{f(z)}\right)''\right|\leq\lambda,\quad z\in\Delta\right\}.$$

Conditions on λ for which each function in $\mathcal{U}(\lambda)$ is starlike or close-to-convex (with respect to $g(z)=z$) respectively, has been obtained in [10]. This was the first step towards achieving the geometric properties of $\mathcal{U}(\lambda)$. Such information has been considered important in the study of certain integral operators in function theory.

EXAMPLE 3.3. Define $f(z)=(z-cz^2)(1-z)^{-2}$. Then

$$f'(z)=\frac{1-z(2c-1)}{(1-z)^3}$$

so that $f'(z)\neq 0$ in Δ if $|2c-1|\leq 1$. It can be easily seen that this is also a sufficient condition for f to be in $\mathcal{S}$. Further, a simple calculation shows that

$$\left(\frac{z}{f(z)}\right)^2f'(z)-1=-\left(\frac{(1-c)z}{1-cz}\right)^2$$

and it is a simple exercise to see that f is in $\mathcal{U}$ if $0\leq c\leq 1$. Note $c=1$ gives $f(z)=z/(1-z)$ so that $(z/f(z))^2f'(z)-1=0$. This observation shows that there will be no $\lambda>0$ such that

$\mathcal{U}(\lambda) \subset \mathcal{S}^*(\beta)$ with $\beta > 1/2$. More generally, there can be no $\lambda > 0$ such that $\mathcal{U}(\lambda) \subset \mathcal{M}(\beta)$ with $\beta > 1/2$, where

$$\mathcal{M}(\beta) = \{f \in \mathcal{A} : f(z)/z \in \mathcal{P}(\beta)\}.$$

Further the functions $z/(1-z)$ and $z/(1-z^2)$ show that the class $\mathcal{U}$ is not preserved under the square root transform. This fact may be verified by a simple computation.

At first we are interested in finding conditions on a, b, c so that $z/F(a, b; c; z)$ is in $\mathcal{U}$. Recall the following result from [9].

LEMMA 3.4. [9] *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function in Δ that satisfies the coefficient condition $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$. Then $\dfrac{z}{\phi(z)} \in \mathcal{U}$.*

LEMMA 3.5. *Assume that a, b,c are nonzero real numbers such that $0 < b \leq c$ and $a \in [-1, 1) \cup [c-1, c+1]$. Then $F(a, b; c; z) \neq 0$ for $z \in \Delta$.*

Proof. This is a consequence of [7, Lemma 2] and the fact that

$$\frac{C}{AB} F'(A, B; C; z) = F(A+1, B+1; C+1; z). \qquad \square$$

THEOREM 3.6. *Assume that a, b, c satisfy any one of the following conditions:*

(i) $0 < a \leq \frac{c}{b+1} - 1$, $b > 0$

(ii) $-1 < a < 0$, $b > 0$, $c > a + b + 1$ *and*

$$\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}[c - (a+1)(b+1)] \leq 2.$$

Then $f(z) = \dfrac{z}{F(a, b; c; z)} \in \mathcal{U}$.

Proof. Let $\phi(z) = F(a, b; c; z)$. For $a = 0$ or $b = 0$, then $\phi(z) = 1$. Further, if $a = -1$ then

$$F(a, b; c; z) = 1 - (b/c)z$$

and the conclusion holds whenever $|b| \leq |c|$. If $c = a$, then we see that

$$F(a, b; c; z) = (1-z)^{-b}.$$

As $-1 < a$ and $b > 0$, we note that $c > a+b+1$ implies that $c > \max\{b, a+1\}$. From the hypothesis and Lemma 3.5, we find that

$$\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \quad \left(b_n = \frac{(a,n)(b,n)}{(c,n)(1,n)} \right),$$

is a non-vanishing analytic function in Δ. In view of the coefficient condition in Lemma 3.4, it suffices to show that

$$S := \sum_{n=2}^{\infty} (n-1)|b_n| \leq 1.$$

Now, for $c > a+b+1$, we find that

$$\begin{aligned} S &= \sum_{n=2}^{\infty} (n-1) \left| \frac{(a,n)(b,n)}{(c,n)(1,n)} \right| \\ &= \frac{|a|b}{c} \left[\sum_{n=2}^{\infty} \frac{(a+1,n-1)(b+1,n-1)}{(c+1,n-1)(1,n-1)} - \sum_{n=2}^{\infty} \frac{(a+1,n-1)(b+1,n-1)}{(c+1,n-1)(1,n)} \right] \\ &= \frac{|a|b}{c} [F(a+1,b+1;c+1;1) - 1] - |a| \sum_{n=2}^{\infty} \frac{(a+1,n-1)(b,n)}{(c,n)(1,n)}. \end{aligned}$$

We will need the well-known result

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c > a+b). \tag{3.7}$$

Case (i): Suppose that $a, b > 0$. Clearly, $c \geq (a+1)(b+1)$ implies that $c > a+b+1$ and therefore,

$$S = \frac{ab}{c} [F(a+1,b+1;c+1;1) - 1] - F(a,b;c;1) + 1 + \frac{ab}{c}$$

so that, by (3.7), it follows that

$$S = 1 - \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} [c - (a+1)(b+1)]$$

which is ≤ 1 whenever a, b, c are related by $0 < a \leq c/(b+1) - 1$.

Case (ii): Suppose that $-1 < a < 0$, $b > 0$ and $c > a+b+1$. Then we obtain that

$$\begin{aligned} S &= \frac{|a|b}{c} [F(a+1,b+1;c+1;1) - 1] + \sum_{n=2}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} \\ &= |a|b \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \\ &= \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} [|a|b + c - a - b - 1] - 1 \\ &\leq 1 \quad \text{whenever } \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} [|a|b + c - a - b - 1] \leq 2 \end{aligned}$$

which concludes the lemma. □

THEOREM 3.8. *Assume that $a, b, c > 0$ and satisfy any one of the following conditions:*

(i) $0 < a \leq 1$, $b > 0$ *and* $c \geq (a+1)(b+1)$

(ii) $a > 1$, $0 < b \leq (a+1)/(a-1)$, $c \geq (a+1)(b+1)$

(iii) $a > 1$, $b \geq (a+1)/(a-1)$, $c \geq 2ab$.

Then $f(z) = \dfrac{z}{F(a,b;c;z)} \in \mathcal{U}$.

Proof. From [16], we recall if $a, b, c > 0$ and

$$c \geq \max\{a+b, a+b+(ab-1)/2, 2ab\},$$

then $zF(a,b;c;z)$ is close-to-convex with respect to the convex function $-\log(1-z)$. Therefore, under this condition $F(a,b;c;z)$ is non-vanishing in Δ. This observation together with the discussion for Case (i) of Lemma 3.6 complete the proof. □

THEOREM 3.9. *Let $f, g \in \mathcal{S}$ with the representations*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

If

$$\Phi(z) = \frac{z}{f(z)} * \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} b_n c_n z^n \neq 0$$

for every $z \in \Delta$, then $F(z) = \dfrac{z}{\Phi(z)} \in \mathcal{U}$, and, in particular, F is univalent in Δ.

Proof. From (3.2) for $\alpha = 2$, we have

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1 \quad \text{and} \quad \sum_{n=2}^{\infty}(n-1)|c_n|^2 \leq 1. \tag{3.10}$$

Since

$$\begin{aligned} \sum_{n=2}^{\infty}(n-1)|b_n c_n| &= \sum_{n=2}^{\infty}(\sqrt{n-1}|b_n|)(\sqrt{n-1}|c_n|) \\ &\leq \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty}(n-1)|c_n|^2\right)^{1/2} \\ &\leq 1 \quad (\text{ from } (3.10)), \end{aligned}$$

then, by the hypothesis, we have

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} b_n c_n z^n = \frac{z}{f(z)} * \frac{z}{g(z)} \neq 0$$

for every $z \in \Delta$ and therefore, by Lemma 3.4, we conclude that $\dfrac{z}{\Phi(z)} \in \mathcal{U}$. □

COROLLARY 3.11. *If $f \in \mathcal{S}$ with $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ such that $1 + \sum_{n=1}^{\infty} b_n^2 z^n$ is non-vanishing in the unit disc Δ, then*

$$\frac{z}{\sum_{n=1}^{\infty} b_n^2 z^n} \in \mathcal{S}.$$

EXAMPLE 3.12. Define $A*B = \{f*g : \; f \in A, \; g \in B\}$. As a motivation to our next result which deals with modified convolution of f and g, i.e. we define $f \bigoplus g = z/f(z) * z/g(z)$, then $A' \bigoplus B' = \{f \bigoplus g : f \bigoplus g \neq 0, f \in A', g \in B'\}$ with an understanding that $A' = \{f \in A : \; z/f(z) \neq 0 \;\text{ for } z \in \Delta\}$. Recall that the set $\mathcal{F}$ of all univalent analytic functions [4]

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

with the property that $a_n \in \mathbb{Z}$ is finite, namely the set

$$\mathcal{F} = \left\{ z, \frac{z}{1 \mp z}, \frac{z}{(1 \mp z)^2}, \frac{z}{1 \mp z^2}, \frac{z}{1 \mp z + z^2} \right\}.$$

If $f \in \mathcal{F}$, then we have

$$\frac{z}{f(z)} \in \{1, 1 \mp z, (1 \mp z)^2, 1 \mp z^2, 1 \mp z + z^2\} = \mathcal{F}'$$

and for $f, g \in \mathcal{F}$,

$$\frac{z}{f(z)} * \frac{z}{g(z)} \in \{1, 1 \mp z, (1 \mp z)^2, 1 \mp z^2, 1 \mp z + z^2\} \cup \{1 \mp 4z + z^2, 1 \mp 2z\}$$

so that

$$\mathcal{F}' \oplus \mathcal{F}' = \{1, 1 \mp z, (1 \mp z)^2, 1 \mp z^2, 1 \mp z + z^2\}.$$

Observe that each of the functions $1 \mp 4z + z^2$ and $1 \mp 2z$ has a zero inside the unit disc and therefore, we can not conclude that the functions $z/(1 \mp 4z + z^2)$ and $z/(1 \mp 2z)$ are univalent in Δ.

As an application of Theorem 3.9, we have the following results.

THEOREM 3.13. *Let $f, g \in \mathcal{S}$ be such that $z/f(z) \in \mathcal{P}(\beta)$ and $z/g(z) \in \mathcal{P}(\gamma)$, where $1 \geq 2(1-\beta)(1-\gamma)$. Then $F(z) = \dfrac{z}{(z/f(z)) * (z/g(z))}$ belongs to $\mathcal{U}$.*

Proof. Recall that if $\operatorname{Re} p(z) > \beta$ and $\operatorname{Re} q(z) > \gamma$, then (see for example [14])

$$\operatorname{Re}\,(p(z) * q(z)) > 1 - 2(1-\beta)(1-\gamma) \quad \text{for} \quad z \in \Delta.$$

In view of this result, the hypothesis implies that

$$\operatorname{Re}\left(\frac{z}{f(z)} * \frac{z}{g(z)}\right) > 0, \quad z \in \Delta.$$

In particular, $\phi(z) = \dfrac{z}{f(z)} * \dfrac{z}{g(z)}$ is non-vanishing analytic function in the unit disc Δ. By Theorem 3.9, $F(z) = z/\phi(z)$ belongs to $\mathcal{U}$ and in particular, F is univalent in Δ. □

EXAMPLE 3.14.

(i) If $f, g \in \mathcal{S}$ are such that

$$\operatorname{Re}\,(f(z)/z) > 0 \quad \text{and} \quad \left|\frac{g(z)}{z} - 1\right| < 1,$$

then $F(z) = \dfrac{z}{(z/f(z)) * (z/g(z))}$ belongs to $\mathcal{U}$ and in particular, F is univalent in Δ.

(ii) Let $f \in \mathcal{U}$ with $f''(0) = 0$. Then, according to a recent general result from [10], it follows that $\operatorname{Re}\,(f(z)/z) > 1/2$, $z \in \Delta$. In particular, $\operatorname{Re}\,(z/f(z)) > 0$. In view of this observation, we have the following: if

$$f \in \mathcal{U} \text{ with } f''(0) = 0 \text{ and } g \in \mathcal{S} \text{ with } \left|\frac{g(z)}{z} - 1\right| < 1,$$

then

$$F(z) = \frac{z}{(z/f(z)) * (z/g(z))} \in \mathcal{U}.$$

Although a bound for the radius of starlikeness has been established for functions $f \in \mathcal{U}$ with $f''(0) = 0$, it is not yet known whether the bound is sharp. Note that every starlike function f of order $1/2$ satisfies the condition $\operatorname{Re}\,(f(z)/z) > 1/2$.

(iii) If $f \in \mathcal{S}^*(1/2)$ and $g \in \mathcal{S}$ with $\left|\dfrac{g(z)}{z} - 1\right| < 1$, then $F(z) = \dfrac{z}{(z/f(z)) * (z/g(z))}$ belongs to $\mathcal{U}$. In particular, choosing $g(z) = z - \alpha z^2$ ($|\alpha| \leq 1/2$), it follows that if

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k,$$

then

$$F(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n \alpha^n z^n} \in \mathcal{U}$$

whenever $|\alpha| \leq 1/2$.

THEOREM 3.15. *Let* $f, g \in S$ *with* $\dfrac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$, $\dfrac{z}{g(z)} = 1 + c_1 z + c_2 z^2 + \cdots$.

(i) *If* $\sum_{n=1}^{\infty} |b_n c_n| \leq 1$, *then the function* $F(z) = \dfrac{z}{(z/f(z)) * (z/g(z))}$ *belongs to* $\mathcal{U}$.

(ii) *If* $\sum_{n=1}^{\infty} n|b_{n+1}c_{n+1} - b_{n-1}c_{n-1}| \leq 1$ $(b_0 = c_0 = 1)$, *then the function* $F(z)$ *belongs to* $\mathcal{U}$ *whenever* $(z/f(z)) * (z/g(z))$ *is non-vanishing in the unit disc* Δ.

Proof. Note that

$$\phi(z) = \frac{z}{f(z)} * \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} b_n c_n z^n$$

and therefore, by the first condition, we have

$$\mathrm{Re}\,\phi(z) > 1 - \sum_{n=1}^{\infty} |b_n c_n| \geq 0 \quad \text{for} \quad z \in \Delta$$

showing that $\phi(z)$ is a non-vanishing analytic function in the unit disc. By Theorem 3.9, $F(z) = z/\phi(z) \in \mathcal{U}$.

For the proof of the second part, apply (1.5) with $\beta = 0$. □

4. Integral Transform

LEMMA 4.1. *Let* $\mathrm{Re}\, c > -1$ *and* G *denote the Bernardi transform of* $f \in \mathcal{A}$ *defined by*

$$G(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t)dt. \tag{4.2}$$

Let β_c *be defined by*

$$\frac{\beta_c}{1-\beta_c} = -(\mathrm{Re}\, c + 1) \int_0^1 t^{\mathrm{Re}\, c} \frac{1-t}{1+t} dt.$$

We have the following:

(1) $f' \in \mathcal{P}_\eta(\beta_c)$ *implies that* $G' \in \mathcal{P}_\eta(0)$

(2) *If* c *is real and* $-1 < c \leq 2$, *then* $f' \in \mathcal{P}_\eta(\beta_c)$ *implies that* $G \in \mathcal{S}^*$.

For $\beta < \beta_c$, *the function* G *need not even be in* $\mathcal{S}$.

Case 1 of Lemma 4.1 is due to Ponnusamy [12] but in a somewhat different notation, whereas Case 2 of Lemma 4.1 is due to Fournier and Ruscheweyh [2]. We ask whether the

Bernardi transform G of f could still be univalent under a weaker/new condition on f for certain values of c. The answer is yes. To do this, we will need the following lemma.

LEMMA 4.3. [17]. *If $f, g \in \mathcal{H}$ and ϕ, ψ are convex (need not be normalized) functions in Δ such that $f \prec \phi$, $g \prec \psi$, then $f * g \prec \phi * \psi$.*

Here $\prec$ denotes the usual subordination [1].

THEOREM 4.4. *Let $-1 < c \leq 0$ and G be defined by (4.2). Suppose that $f(z)/z \in \mathcal{P}_\eta(\beta)$. Then we have $G'(z) \in \mathcal{P}_\eta(\gamma)$ where $\gamma = 1 - 2(1-\beta)(1-\beta')$ with*

$$\beta' = \frac{1+c}{2} - c\int_0^1 \frac{dt}{1+t^{1/(1+c)}} = \frac{1+c}{2} - cF(1, 1+c; 2+c; -1). \tag{4.5}$$

In particular, we have

(1) $\operatorname{Re} e^{i\eta}\left(\dfrac{f(z)}{z} - \dfrac{1-2\beta'}{2(1-\beta')}\right) > 0$ *implies* $\operatorname{Re} e^{i\eta} G'(z) > 0$.

(2) $\operatorname{Re} e^{i\eta}\left(\dfrac{f(z)}{z} - \dfrac{1}{2}\right) > 0$ *implies* $\operatorname{Re} e^{i\eta}(G'(z) - \beta') > 0$.

Proof. We have

$$G(z) = f(z) * zF(1, 1+c; 2+c; z) = f(z) * \left(\sum_{n=0}^{\infty} \frac{(1,n)(1+c,n)}{(2+c,n)(1,n)} z^{n+1}\right)$$

and therefore,

$$G'(z) = \frac{f(z)}{z} * F(2, 1+c; 2+c; z). \tag{4.6}$$

It is a simple exercise to see that

$$\begin{aligned} F(2, 1+c; 2+c; z) &= (1+c)\left(\sum_{n=0}^{\infty} z^n - c\sum_{n=0}^{\infty} \frac{z^n}{n+1+c}\right) \\ &= (1+c)\left(\frac{1}{1-z} - c\int_0^1 \frac{t^c}{1-zt}\,dt\right) \\ &= \frac{1+c}{1-z} - c\int_0^1 \frac{dt}{1-zt^{1/(1+c)}}. \end{aligned}$$

If we let $\alpha = 1/(1+c)$ and $W = 1/(1-zt^\alpha)$ then, for $|z| = r$ $(0 < r < 1)$ and $0 \leq t \leq 1$, we have

$$\left|1 - \frac{1}{W}\right| \leq rt^\alpha$$

so that

$$\left|W - \frac{1}{1-r^2t^{2\alpha}}\right| \leq \frac{rt^\alpha}{1-r^2t^{2\alpha}}$$

which gives

$$\frac{1}{1+rt^{\alpha}} \leq \operatorname{Re} W \leq \frac{1}{1-rt^{\alpha}}.$$

Therefore, for $\alpha > 0$, we have

$$\operatorname{Re} \int_0^1 \frac{dt}{1-zt^{\alpha}} \geq \int_0^1 \frac{dt}{1+rt^{\alpha}} > \int_0^1 \frac{dt}{1+t^{\alpha}}, \quad z \in \Delta.$$

Thus for $c \in (-1, 0]$, one has

$$\operatorname{Re} F(2, 1+c; 2+c; z) > \frac{1+c}{2} - c\int_0^1 \frac{dt}{1+t^{1/(1+c)}} = \beta', \quad z \in \Delta,$$

and the estimate here is clearly sharp. It is easy to see that β' is $\geq 1/2$. Assume that $f(z)/z \in \mathcal{P}_{\eta}(\beta)$. Now, we choose

$$\phi(z) = 1 + 2(1-\beta)\frac{z}{1-z} \quad \text{and} \quad \psi(z) = 1 + 2(1-\beta')\frac{z}{1-z}.$$

Both $\phi(z)$ and $\psi(z)$ are known to be convex in Δ. Further,

$$(\phi * \psi)(z) = 1 + 2(1-\beta)(1-\beta')\frac{z}{1-z} = 1 + 2(1-\gamma)\frac{z}{1-z}$$

and the desired conclusion follows from Lemma 4.3 and (4.6). □

THEOREM 4.7. *Let $-1 < c \leq 0$, G denote the Bernardi transform of $f \in \mathcal{A}$ defined by* (4.2) *and β' is given by* (4.5). *Suppose that*

$$\left|\frac{f(z)}{z} - \alpha\right| < \beta \quad \text{for } z \in \Delta$$

and for some real α and β with $|1-\alpha| < \beta$. Then we have

$$|G'(z) - 1 + 2(1-\beta')(1-\alpha)| < 2\beta(1-\beta').$$

In particular $\operatorname{Re} G'(z) > 0$ *(and hence G is univalent) whenever α and β is related by the condition* $1 \geq 2(1-\beta')(1-\alpha+2\beta)$.

Proof. Define

$$H(z) = F(2, 1+c; 2+c; z) = 1 + \sum_{n=1}^{\infty} H_n z^n$$

and assume that $-1 < c \leq 0$. From the proof Theorem 4.4, we have $\operatorname{Re} H(z) > \beta'$. Thus, then condition on f and the last condition on H may be rewritten as

$$\left|1 + \sum_{n=1}^{\infty} a_{n+1} z^n - \alpha\right| < \beta \quad \text{and} \quad 1 + \frac{1}{2(1-\beta')}\sum_{n=1}^{\infty} H_n z^n \prec \frac{1}{1-z}.$$

By Lemma 4.3, we have

$$\left|1+\frac{1}{2(1-\beta')}\sum_{n=1}^{\infty}a_{n+1}H_nz^n-\alpha\right|<\beta$$

which gives that $G'(z)=\frac{f(z)}{z}*H(z)$ with

$$|G'(z)-1+2(1-\beta')(1-\alpha)|\leq 2\beta(1-\beta')$$

and the desired conclusion follows. ☐

THEOREM 4.8. *Let $f\in\mathcal{U}(\lambda)$, $a=|f''(0)|/2\leq 1$ with $0\leq\lambda+|a|\leq 1$. Suppose that $-1<c\leq 0$ and G denotes the Bernardi transform of $f\in\mathcal{A}$ defined by* (4.2). *Then we have*

(1) $|G'(z)-1+2(1-\beta')(1-\alpha)|\leq 2\beta(1-\beta')$ *if* $0\leq\lambda+|a|<1$

(2) $\mathrm{Re}\, G'(z)>\beta'$ *if* $\lambda+|a|=1$,

where $\alpha=1/(1-(|a|+\lambda)^2)$, $\beta=\alpha(|a|+\lambda)$ and β' is given by (4.5).

Proof. Suppose that $f\in\mathcal{U}(\lambda)$. Then, we can write

$$-z\left(\frac{z}{f(z)}\right)'+\frac{z}{f(z)}=\left(\frac{z}{f(z)}\right)^2f'(z)=1+\lambda w(z) \tag{4.9}$$

where w is a Schwarz function with an additional condition $w'(0)=0$. We observe from Schwarz lemma that $|w(z)|\leq|z|^2$. It can be easily seen that

$$\frac{z}{f(z)}=1-az-\lambda\int_0^1\frac{w(tz)}{t^2}dt$$

and therefore,

$$\left|\frac{z}{f(z)}-1\right|\leq|a|+\lambda$$

(strict inequality if $0<|a|+\lambda$). By a calculation it follows that

$$\left|\frac{f(z)}{z}-\alpha\right|\leq\beta \quad\text{if}\quad 0\leq\lambda+|a|<1$$

and

$$\mathrm{Re}\left(\frac{f(z)}{z}\right)>\frac{1}{2}\quad\text{if}\quad\lambda+|a|=1.$$

The desired conclusion follows from Theorem 4.7. ☐

In particular, we have

COROLLARY 4.10. *Let $f\in\mathcal{U}(\lambda)$ with $f''(0)=0$ with $0\leq\lambda\leq 1$. Suppose that $-1<c\leq 0$ and G denotes the Bernardi transform of $f\in\mathcal{A}$ defined by* (4.2). *Then we have the following:*

(1) $\left|G'(z) - \dfrac{1-\lambda^2 c(1-F(1,1+c;2+c;-1))}{1-\lambda^2}\right| \leq \dfrac{\lambda(1-c+cF(1,1+c;2+c;-1))}{1-\lambda^2}$ *if* $0 \leq \lambda < 1$

(2) $\operatorname{Re} G'(z) > \dfrac{1+c}{2} - cF(1,1+c;2+c;-1)$ *if* $\lambda = 1$.

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