A VARIATIONAL METHOD FOR HYPERBOLICALLY CONVEX FUNCTIONS

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Abstract. In this paper we describe a variational method, based on Julia's formula for the Hadamard variation, for hyperbolically convex polygons. We use this variational method to prove a general theorem for solving extremal problems for hyperbolically convex functions. Special cases of this theorem provide independent proofs for controlling growth and distortion for hyperbolically convex functions.

1. INTRODUCTION

A classical problem in Geometric Function Theory is to maximize the value of a given functional over a given class of analytic functions. Recent papers have extended this problem and its study to functionals on hyperbolically convex functions. In particular, these functions were studied by Beardon in [3], Ma and Minda in [4, 5] and Solynin in [12, 13]. More recently, they have been studied by Mejía and Pommerenke in [6, 7, 8, 9, 10] and Mejía, Pommerenke, and Vasiliyev in [11]. There have been a number of open problems and conjectures in these papers. A critical obstacle to these studies has been the lack of a suitable variational method for this class.

In this paper, we develop a variational technique, based on Julia's formula for the Hadamard variation, that can be used to overcome this obstacle and to resolve a number of these problems and conjectures. We will then use this variational method to prove a general theorem, which includes as special cases a number of the results refered to in the referenced papers in the introductory paragraph.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk in $\mathbb{C}$ and let $\mathbb{T} = \partial\mathbb{D}$. The hyperbolic plane can be viewed as $\mathbb{D}$ with the imposed hyperbolic metric $\lambda(z)|dz| = \frac{2|dz|}{1-|z|^2}$. Under this metric, hyperbolic geodesics in $\mathbb{D}$ are connected subarcs of Euclidean circles which intersect $\mathbb{T}$ orthogonally. A set $S \subset \mathbb{D}$ is *hyperbolically convex* if for any two points z_1 and z_2 in S, the hyperbolic geodesic connecting z_1 to z_2 lies entirely inside of S.

We will say that a function $f : \mathbb{D} \to \mathbb{D}$ is *hyperbolically convex* if f is analytic and univalent on $\mathbb{D}$ and if $f(\mathbb{D})$ is hyperbolically convex. The set of all hyperbolically convex functions f which satisfy $f(0) = 0$ will be denoted by H. Interesting examples are the normal fundamental domains of Fuchsian groups in $\mathbb{D}$.

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A *hyperbolic polygon* is a simply connected subset of $\mathbb{D}$, which contains the origin and which is bounded by a Jordan curve consisting of a finite collection of hyperbolic segments and arcs of the unit circle. The hyperbolic segments internal to $\mathbb{D}$ will be referred to as *proper sides*. We will let H_{poly} denote the subset of H of all functions mapping $\mathbb{D}$ onto hyperbolic polygons. Further, we will let H_n denote the subclass of H_{poly} of all functions mapping $\mathbb{D}$ onto polygons with at most n proper sides. It is easily seen that H_{poly} is dense in H. Furthermore, $H \cup \{0\}$ and $H_n \cup \{0\}$, for each n, are compact.

Our main theorem is, with $\Re\{z\}$ = real part z,

Theorem 1.1. *Let Φ be entire. For $z \in \mathbb{D} \setminus \{0\}$ and $a_k \in \mathbb{R}, k = 0 \ldots n$, let*

$$F(f, z) = \sum_{k=0}^{n} a_k \log \frac{f^{(k)}(z)}{f'(0)} \tag{1}$$

and let

$$Q(\zeta) = \sum_{k=0}^{n} a_k \left(\frac{G^{(k)}(\zeta, z)}{f^{(k)}(z)} - 1 \right), \tag{2}$$

where

$$G^{(k)}(\zeta, z) = \frac{\partial^{(k)} \left(z f'(z) \frac{\zeta+z}{\zeta-z} \right)}{\partial z^{(k)}}. \tag{3}$$

Let $f \in H$ be extremal for

$$L(f) = \Re \{\Phi \circ F(f, z)\} \tag{4}$$

over H such that

1. $\Phi' \circ F(f, z) \in \mathbb{R} \setminus \{0\}$,
2. *Q maps $\mathbb{T}$ to a curve Λ such that Λ traversely crosses the imaginary axis at most twice.*

Then, the extremal value for L over H can be obtained from a hyperbolically convex f which maps $\mathbb{D}$ onto a hyperbolic polygon with at most one proper side.

The proofs of the following two corollaries will be discussed in Section 3.

Corollary 1.1. *Let $z \in \mathbb{D} \setminus 0$ and let $f \in H$ be extremal for $L(f) = |\dfrac{f(z)}{f'(0)}|$ over H. Then, the extremal value for L over H can be obtained from a hyperbolically convex f which maps $\mathbb{D}$ onto a hyperbolic polygon with exactly one proper side.*

Corollary 1.2. *Let $z \in \mathbb{D} \setminus 0$ and let $f \in H$ be extremal for $L(f) = |\dfrac{f'(z)}{f'(0)}|$ over H. Then, the extremal value for L over H can be obtained from a hyperbolically convex f which maps $\mathbb{D}$ onto a hyperbolic polygon with exactly one proper side.*

Remark The function f in Theorem 1.1 and Corollaries 1.1 and 1.2 is given by, up to rotation,

$$f(z) = k_\alpha(z) \equiv \frac{2\alpha z}{(1-z) + \sqrt{(1-z)^2 + 4\alpha^2 z}}$$

We note that Corollary 1.1 gives the standard growth, covering and early coefficient theorems obtained by Ma and Minda [4, 5]. Corollary 1.2 gives a new independent proof of the frequently stated open problem [5, 6, 8, 10] recently solved by Mejía, Pommerenke, and Vasileyev [11], which required deep methods from extremal length and reduced module theory.

Remark The scope of Theorem 1.1 can be extended in the following fashion: if the second of the itemized hypotheses (the hypothesis which describes the geometry of the image of $\mathbb{T}$ under the kernel Q) is generalized to

(2) Q maps $\mathbb{T}$ to a curve Λ such that Λ traversely crosses the imaginary axis at most $2N$ times.

then the conclusion of theorem generalizes to

Then, the extremal value for L over H can be obtained from a hyperbolically convex f which maps $\mathbb{D}$ onto a hyperbolic polygon with at most N proper sides.

2. Variations for H_{poly}

Julia's variational formula is developed as follows: Let $f \in H_{poly}$ map $\mathbb{D} \to \Omega \subseteq \mathbb{D}$ such that $\partial\Omega$ is piecewise analytic with right and left tangents at all points. For $w \in \partial\Omega$, let $n(w)$ be the outward unit normal where it exists and the zero vector where it does not. We define a function, piecewise differentiable, $\phi{:}\partial\Omega \to \mathbb{R}$ with $\phi(w_j) = 0$ where $\{w_j\}$ is the collection of points at which $\partial\Omega$ is not analytic. We can define a new curve $\widehat{\partial\Omega_\epsilon}$ pointwise by letting $\widehat{w_\epsilon} = w + \epsilon\phi(w)n(w)$. By choosing ϵ sufficiently small, $\widehat{\partial\Omega_\epsilon}$ is a Jordan curve. We now define $\widehat{\Omega_\epsilon}$ to be the region bounded by $\widehat{\partial\Omega_\epsilon}$. We define $\widehat{f_\epsilon}$ to be the Riemann map sending $\mathbb{D}$ onto $\widehat{\Omega_\epsilon}$ such that $\hat{f}_\epsilon(0) = 0$.

Julia's result (which was really a generalization of Hadamard's work with Green's functions) was that we can write $\widehat{f_\epsilon}$ as a variation of our original f. In particular,

$$\widehat{f_\epsilon}(z) = f(z) + \frac{\epsilon z f'(z)}{2\pi i} \int \frac{\zeta + z}{\zeta - z} \frac{\phi(w)n(w)}{[\zeta f'(\zeta)]^2} dw + o(\epsilon), \tag{5}$$

where $w = f(\zeta)$ for $\zeta = e^{i\theta}$, $0 \le \theta < 2\pi$,and $o(\epsilon)$ is analytic for $z \in \mathbb{D}$.

Equation (5) can be rewritten as

$$\widehat{f_\epsilon}(z) = f(z) + \epsilon \int z f'(z) \frac{\zeta + z}{\zeta - z} d\Psi + o(\epsilon). \tag{6}$$

where $d\Psi$ is a positive measure on $\mathbb{T}$.

The problem encountered in using the method of Julia variations with hyperbolically convex functions is the difficulty in finding Julia variations on the sides of the approximating polygons that leave the varied functions in the original class. We will describe two basic types of variations which preserve the class H_{poly}. One of these will preserve the number of sides in the varied polygon. The other will increase it by one.

For each type of variation, there are three cases with which we will need to consider, depending on the angles at the ends of the sides being varied. The first case is when a single side meets the boundary of the unit circle at an angle of $\pi/2$. The next case is the one in which two sides meet on the interior of the disk at an angle lying beytween 0 and π. Finally, we deal with the case in which the two sides meet on the boundary of the disk at a zero angle. This case must be subdivided into two variations, one in which the side is pushed out thus turning the cusp into two right angles and one in which the meeting of the two sides is moved into the disk and the angle is increased to a positive angle.

We will first introduce the class preserving variations for H_n, i.e., variations which for f in H_n will produce varied functions f_ϵ which again are in H_n. We will use these in the proof of Theorem 1.1 to reduce the number of sides in the extremal domain to at most two. After the class preserving variations, we define the variations which increase the number of sides, i.e., variations which for f in H_n will produce varied functions f_ϵ which are in H_{n+1}. These we will use, in the manner described by Barnard, Cole, Pearce, and Williams [2], to reduce the possible extremal domains from polygons with at most two sides to those having at most one side.

The analysis of the first two cases is similar, so we will discuss those concurrently. We will illustrate by varying a side meeting the boundary of the disk on one side with an angle of $\pi/2$ and meeting internally another side with an angle of θ with $0 < \theta < \pi$ on the other (although the analysis works identically with any permutation of the two sorts of corners). See Figure 1 below.

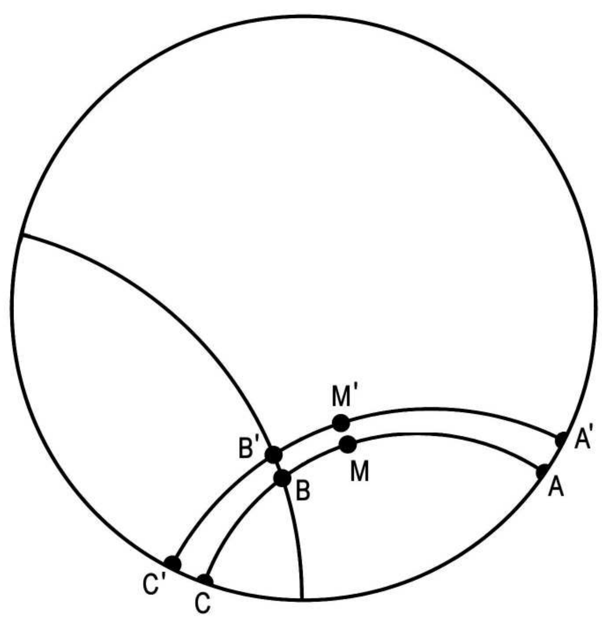

FIGURE 1. Variation at an Internal Angle

We consider side $\widehat{AB}$ of our hyperbolically convex polygon Ω. We label the point on the geodesic continuation of $\widehat{AB}$ to $\mathbb{T}$ as the point C, allowing for the possiblility that $B = C$. To perform our variation we will take the midpoint of $\widehat{AC}$ and call it M. Our variation will consist of moving M radially by a fixed small distance $\epsilon\phi(M)$ for constant $\phi(M)$. This $\phi(M)$ is chosen sufficiently small to assure that the varied polygon retains the same number of sides as the original. This will give us the new point $M' = M + \epsilon\phi(M)n(M)$. Having defined the variation at M, we now define the variation $\phi(w)$ for all other w on $\widehat{AB}$.

For a given ϵ we will define a new curve $\widehat{A'B'}$ which is the arc of the unique hyperbolic geodesic through M' having M' as the midpoint of the extension $\widehat{A'C'}$ and connecting A' to B' the resulting endpoints on $\partial\Omega_\epsilon$ in the interior variation or the necessary extension of the original connecting sides of $\partial\Omega$ in the exterior case. We then define the variation $\phi(w, \epsilon)$ to be the distance to the point on $\widehat{A'B'}$ which is on the line extended along the normal $n(w)$.

Lemma 2.1. *For ϵ small, expanding $\phi(w, \epsilon)$ as a power series about $\epsilon = 0$ gives*

$$\phi(w, \epsilon) = \frac{\partial\phi(w, 0)}{\partial\epsilon}\epsilon + o(\epsilon)$$

with $\frac{\partial\phi(w,0)}{\partial\epsilon} \neq 0$.

Proof: Without loss of generality we will consider only the case when $\epsilon < 0$. To simplify constructions and descriptions we will also assume that M and M' are both real and negative. We will start by considering the circle Λ_0 in the plane concentric with our original geodesic through the point M'. We will define $\tilde{\phi}(w)$ to be the radial distance from w to Λ_0. Note that clearly we have

$$\tilde{\phi}(w) = \epsilon\phi(M).$$

Our strategy is to show that for each $w \in \widehat{AB}$, we have that $\phi(w) > \tilde{\phi}(w)$. Since $\phi(w)$ is sufficiently smooth, we may expand it in as a first order Taylor polynomial. Suppose that $\frac{\partial\phi(w,0)}{\partial\epsilon} = 0$. Then, on expanding as a function of ϵ we get that $\phi(w, \epsilon) = o(\epsilon)$.

So if we divide $\phi(w, \epsilon)$ by $\tilde{\phi}(w)$ and take the limit as ϵ goes to zero from below, we get zero, by the definition of $o(\epsilon)$. However, as we will show $\phi(w) > \tilde{\phi}(w)$, the quotient must be greater than one for every sufficiently small ϵ. Thus, the limit, if it exists at all, must be greater than or equal to one. Hence, we will have a contradiction. The assumption that $\frac{\partial\phi(w,0)}{\partial\epsilon} = 0$ must fail and we have our result.

Showing that $\tilde{\phi}(w) < \phi(w)$ comes from a simple geometric construction. We show that except at M', the curve $\widehat{A'B'}$ will lie outside of Λ_0. Since $\widehat{AB}$ lies inside of Λ_0, the distance from w to Λ_0 is less than the distance to $\widehat{A'B'}$ and we are done.

The construction (see Figure 2) will illustrate this. Note first that $\widehat{A'B'}$ and Λ_0 both go through the point M'. Also observe that Λ_0 and $\widehat{AB}$ have the same center, m. The circle, Λ_1, containing $\widehat{A'B'}$ is normal to the unit circle at C'. The tangent line l' to $\mathbb{T}$ at C' is therefore a radius of Λ_1. Note that since $M' > M$, we have that the center, m' of $\widehat{A'B'}$ lies to the left of the center of Λ_0 and hence has a greater radius. As the two circles are tangent at M', we have that the circle, Λ_0, with the smaller radius lies entirely inside the disk bounded by $\widehat{A'B'}$. This gives us that

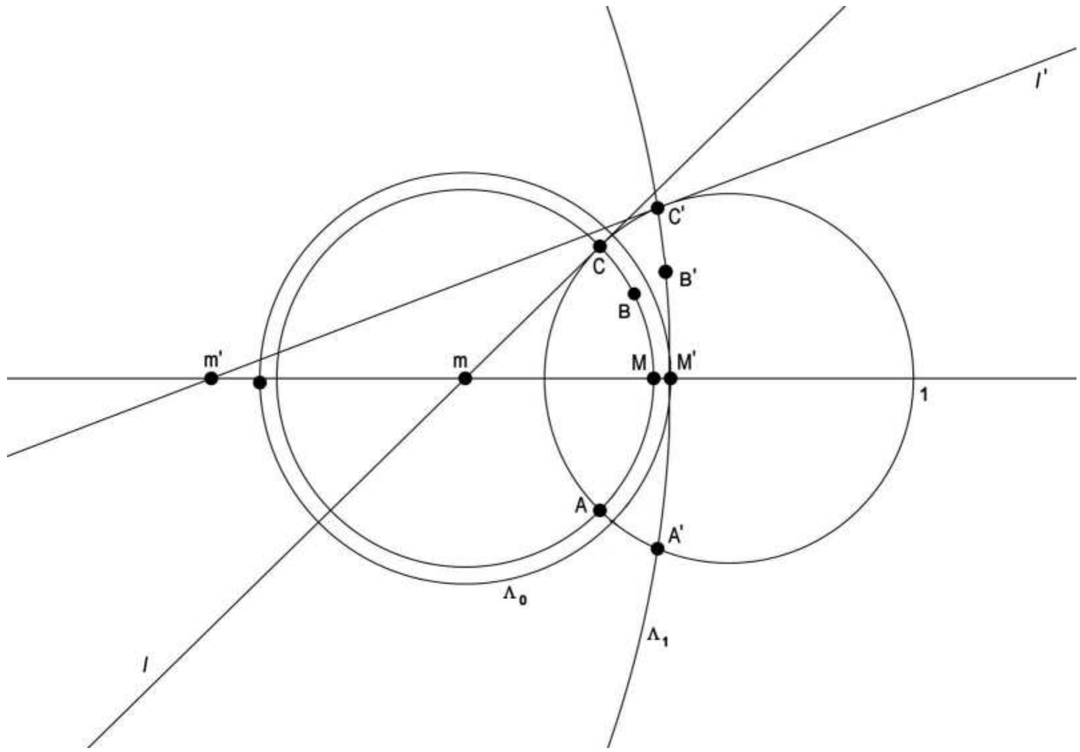

FIGURE 2. Construction of $\widehat{AB}$, Λ_0, and $\widehat{A'B'}$

$\widehat{A'B'}$ lies inside of the hyperbolic polygon whose internal boundary is the arc of Λ_0 internal to $\mathbb{D}$ and we are done. ■

With this, we can now write our variation at w as

$$w' = w + \frac{\partial \phi(w,0)}{\partial \epsilon}\eta(w)\epsilon + o(\epsilon).$$

We can then absorb the $o(\epsilon)$ term into the error term in the Julia Variation formula. Although the variations necessary to produce domains that are in the original class are not always strictly normal, it was shown by Barnard and Lewis [1], that the error introduced for small ϵ is of order $o(\epsilon)$ and thus may also be absorbed into the $o(\epsilon)$ term in the variational formula.

As the previous analysis dealt with both the first two cases, we are left only with the case in which the two sides meet at a cusp. This in turn will be dealt with in two steps. In the first case, we take $\epsilon > 0$ and move the arc of the circle outwards. The second case, of course, is that we take $\epsilon < 0$ and move the arc of the circle towards the middle of $\mathbb{D}$ (see Figure 3).

In the first case, $\epsilon > 0$, we are actually removing the cusp and turning it into two separate right angles at A and A_1. Note that this does not increase the number of new sides, as the new "side" lies on $\mathbb{T}$ and thus does not count as a proper side of the polygon. Since this variation can be done normally without moving the vertex at the cusp, the previous arguments hold. In the second case, we will pull the side

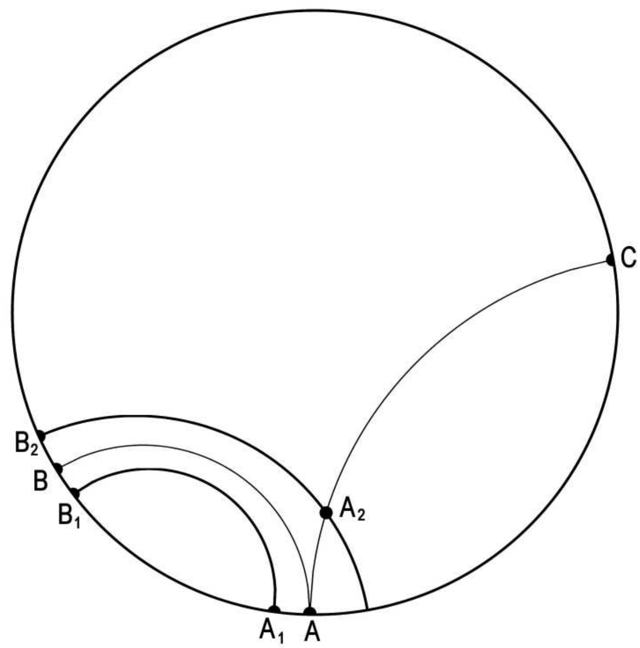

FIGURE 3. Variations at a Cusp

slightly into the disk. The arguments of Barnard and Lewis for controlling the error rates are valid in this case also (where we have a bounded cusp with zero opening). Thus, we have a valid application of the Julia Variation Formula for all of our class preserving variations of various angles in our polygons.

We end this section with a final variation we can apply with all three types of intersection. We will add a new small side to our polygon "cutting off" a vertex z_0. This variation, unlike those previously described, will not preserve the class H_n but will leave the varied function in H_{n+1}. We choose a point z_1 on the side of the polygon we are varying, some fixed small (Euclidean) distance δ from z_0. (See Figure 4). Then, choose a point z_2 on either the next side (if the vertex occured at a cusp or in the interior of $\mathbb{D}$) or along the arc of $\mathbb{T}$ (if the vertex was a right angle on the boundary). Choose z_2 some small distance ϵ from z_0 along the new side. Finally join z_1 with z_2 with a hyperbolic geodesic. The variation will "pivot"the new side $\widehat{z_1 z_2}$ about z_1 into the polygonal domain. The analysis of the error for these variations follows very much the same path as for the previous cases.

3. PROOFS

Proof of Theorem 1.1 Suppose that $f \in H_n$ is extremal for (4) over H_n for some $n \geq 3$ and that $f(\mathbb{D})$ has at least 3 proper sides. Choose one of the proper sides of $f(\mathbb{D})$, say Γ, and let $\gamma = f^{-1}(\Gamma)$. We apply one of the class preserving variations described in the previous section to Γ. From equations (3) and (6), we have for each $k, 0 \leq k \leq n$,

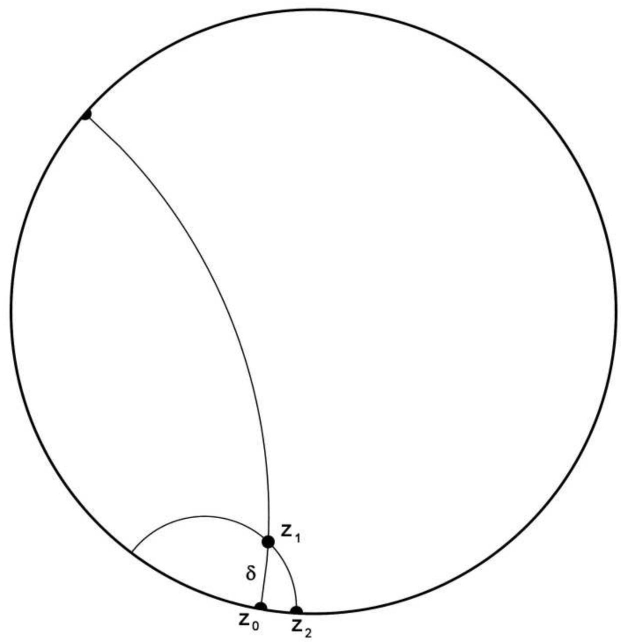

FIGURE 4. Side Adding Variation (Right Angle Illustrated)

$$f_\epsilon^{(k)}(z) = f^{(k)}(z) + \epsilon \int_\gamma G^{(k)}(\zeta, z)d\Psi + o(\epsilon)$$

which can be rewritten as

$$f_\epsilon^{(k)}(z) = f^{(k)}(z)\left\{1 + \epsilon \int_\gamma \frac{G^{(k)}(\zeta, z)}{f^{(k)}(z)}d\Psi\right\} + o(\epsilon)$$

Using this last equation, we can write

$$\log \frac{f_\epsilon^{(k)}(z)}{f'_\epsilon(0)} = \log \frac{f^{(k)}(z)}{f'(0)} + \log \frac{1 + \epsilon \int_\gamma \frac{G^{(k)}(\zeta,z)}{f^{(k)}(z)}d\Psi + o(\epsilon)}{1 + \epsilon \int_\gamma d\Psi + o(\epsilon)}. \tag{7}$$

Expanding the right hand side of (7) as a series in ϵ, for sufficiently small values of ϵ, gives

$$\log \frac{f_\epsilon^{(k)}(z)}{f'_\epsilon(0)} = \log \frac{f^{(k)}(z)}{f'(0)} + \epsilon \int_\gamma \left(\frac{G^{(k)}(\zeta, z)}{f^{(k)}(z)} - 1\right) d\Psi + o(\epsilon)$$

Hence, we can write using (1), (2) and (4)

$$L(f_\epsilon) = \Re\left\{\Phi \circ \left(F(f, z) + \epsilon \int_\gamma Q(\zeta)d\Psi + o(\epsilon)\right)\right\}$$

If $\frac{\partial L(f_\epsilon)}{\partial \epsilon}|_{\epsilon=0}$ is non-zero, then the value of $L(f_\epsilon)$ can be made larger than the value of $L(f)$, which will imply that f cannot be extremal for (4) in H_n. Using the above representation for $L(f_\epsilon)$ and the fact that Φ is entire, we can differentiate $L(f_\epsilon)$ as a function of ϵ and obtain

$$\frac{\partial L(f_\epsilon)}{\partial \epsilon}|_{\epsilon=0} = \Re\left\{(\Phi' \circ F(f,z)) \int_\gamma Q(\zeta) d\Psi\right\}.$$

By hypothesis we have that the first term, $\Phi' \circ F(f,z)$ is real and nonzero. So we can pass the $\Re$ operator through to the integral and through the integral as $d\Psi$ is a real measure. Thus, for the derivative to be zero, we must have $\int_\gamma \Re\{Q(\zeta)\}\, d\Psi$ to be zero. As $d\Psi$ is real valued, we have a real-valued integrand and a real-valued measure.

We are assuming that f is extremal for (4) in H_n and considering the case where f has at least three proper sides, say $\Gamma_j, j = 1,2,3$. We now observe that we can vary each side Γ_j separately. Let γ_j be the arc $\left[e^{i\alpha_j}, e^{i\beta_j}\right]$, the preimage of Γ_j under f, $j = 1,2,3$. Applying the class preserving variation to each side Γ_j yields the requirement, under the supposition that f is extremal in H_n,

$$\int_{\gamma_j} \Re\{Q(\zeta)\} d\Psi = \int_{\left[e^{i\alpha_j}, e^{i\beta_j}\right]} \Re\{Q(\zeta)\} d\Psi(e^{i\theta}) = 0, j = 1,2,3.$$

Applying the mean value theorem for integrals we obtain

$$\int_{\left[e^{i\alpha_j}, e^{i\beta_j}\right]} \Re\{Q(\zeta)\}\, d\Psi(e^{i\theta}) = \Re\{Q(\zeta(\theta))\}\, |_{\theta=\theta_j} \int_{\left[e^{i\alpha_j}, e^{i\beta_j}\right]} d\Psi$$

where $\alpha_j < \theta_j < \beta_j$. Note that as $\alpha_j \neq \beta_j$, we have $\int_{\left[e^{i\alpha_j}, e^{i\beta_j}\right]} d\Psi > 0$. Thus, the only way our integral can be zero is for $\Re\{Q(\zeta(\theta))\}\, |_{\theta=\theta_j}$ to be zero.

Since we can perform the appropriate class preserving variation described above on each proper side $\Gamma_j, j = 1,2,3$, we must have

$$\frac{\partial L(f_\epsilon)}{\partial \epsilon}|_{\epsilon=0} = \Re\{\Phi' \circ F(f,z)\{\,\Re\{Q(e^{i\theta_j})\} \int_{\left[e^{i\alpha_j}, e^{i\beta_j}\right]} d\Psi = 0, j = 1,2,3 \tag{8}$$

where θ_j lies in the interval $(\alpha_j,\, \beta_j)$. Thus, $\frac{\partial L(f_\epsilon)}{\partial \epsilon}|_{\epsilon=0}$ can only be zero at a root of $\Re\{Q(\zeta)\} = 0$. By hypothesis the kernel Q of our integral maps $\mathbb{T}$ to a curve Λ such that Λ intersects the imaginary axis only twice. Since $\Re\{Q(e^{i\theta_j})\}$ can equal 0 for only two our our three sides, there exists a third side we can push either in or out and increase the value of L for some function f_ϵ near f, using our variational argument. Thus, f is not extremal for L, i.e., if f is extremal in H_n, $n \geq 3$, then $f \in H_2 \subset H_n$.

We now have that the extremal f can have at most two proper sides. We will now argue that f can actually have at most one, using an argument from Barnard *et al.* [2]. Consider H_n, $n \geq 3$, and let f be extremal in H_n for (4). By the above argument, $f(\mathbb{D})$ can have at most two sides. Suppose $f(\mathbb{D})$ has exactly two proper sides. If the image under the kernel Q of the preimage of either side is entirely on one side or the other of the imaginary axis, then by our previous arguments, we can increase the value achieved by $L(f_\epsilon)$ and hence f is not extremal. So we conclude that both images intersect the imaginary axis. Thus, for each proper side Γ of $f(\mathbb{D})$, we must have that the image under the kernel Q of the preimage of one endpoint of Γ lies in the left-half plane and the image under the kernel Q of the preimage of other endpoint Γ lies right-half plane.

Suppose $\Phi' \circ F(f, z)$ is positive. We consider a vertex z_0 whose image under $Q \circ f^{-1}$ is in the left half plane. Apply the variation at the vertex z_0 described above which adds another side to $f(\mathbb{D})$, making sure to keep the entire image of the new side in the left half-plane. We now have the derivative (8) taken over our newly created side is positive. By our basic variational argument, the newly varied function has a greater value for L. But this means f cannot be extremal. A similar argument works if $\Phi'\left(\log \frac{f'(r)}{f'(0)}\right)$ is negative. Thus, the extremal function for L in H_n, $n \geq 3$, cannot have two proper sides. It follows therefore that the extremal function in H_n can have at most one proper side.

Since $H_2 \subset H_n$ for all $n \geq 3$, if f is extremal in H_n and is an element of H_2, it must be extremal in H_2 as well. Thus, the extremal element in H_2 has at most one proper side. Thus, the extremal value for L in each H_n is achieved by the region with at most one proper side and hence the extremal value for $H = \overline{\cup_{n\in\mathbb{N}} H_n}$ is achieved by a region with at most one proper side. This proves Theorem 1.1. ■

Proof of Corollary 1.1 In this case, $L(f) = \exp(\log(f(z)/f'(0)))$ and Q is a bilinear mapping. Hence, the hypotheses of Theorem 1.1 are satisfied. ■

Proof of Corollary 1.2 In this case, $L(f) = \exp(\log(f'(z)/f'(0)))$. Since we are finding an extremal value for $|f'(z)|$, then condition (1) of Theorem 1.1 is satisfied. Now consider

$$\begin{aligned} Q(\zeta) &= A\frac{\zeta + z}{\zeta - z} + \left(\frac{\zeta + z}{\zeta - z}\right)' - 1 \\ &= \frac{A(\zeta^2 - z^2) + 2\zeta z - (\zeta - z)^2}{(\zeta - z)^2}. \end{aligned}$$

where $A = \dfrac{z(zf'(z))'}{zf'(z)}$.

By a variant of Jack's lemma, we have that A is real. Next we multiply through by $1 = \left(\dfrac{\bar{\zeta}}{\bar{\zeta}}\right)^2$ to obtain

$$Q(\zeta) = \frac{A(|\zeta|^4 - (\bar{\zeta}z)^2) + 2|\zeta|^2\bar{\zeta}z - (|\zeta|^2 - \bar{\zeta}z)^2}{(|\zeta|^2 - \bar{\zeta}z)^2}.$$

Then, we set $w = \bar{\zeta}z$ to produce

$$\tilde{Q}(w) = \frac{A(1 - w^2) + 2w - (1 - w)^2}{(1 - w)^2}.$$

Continuing we substitute $w = re^{i\theta}$into $\tilde{Q}$ and obtain

$$\frac{A\left(1 - r^2\left(e^{i\theta}\right)^2\right) + 2\,re^{i\theta} - \left(1 - re^{i\theta}\right)^2}{\left(1 - re^{i\theta}\right)^2}.$$

We then expand the above rational function, multiply through by the conjugate of the denominator, make the substitution $re^{i\theta} = r\left(\cos(\theta) + i\sin(\theta)\right)$, and collect the real parts. What is left in the numerator when all this is done is:

$$R(\cos(\theta)) = -4\,r^2\,(\cos(\theta))^2 + \left(-2\,rA + 2\,r^3A + 6\,r^3 + 6\,r\right)\cos(\theta)$$
$$-1 - r^4A - r^4 + A - 6\,r^2.$$

Lemma 3.1. *The polynomial $R(x)$ has at most one real root smaller than one for $A \in \mathbb{R}$.*

Proof: We begin by solving the polynomial using the quadratic formula. This yields the following roots:

$$\frac{3\,r^2 + Ar^2 + 3 - A + \sqrt{5\,r^4 + 2\,Ar^4 - 6\,r^2 + A^2r^4 - 2\,A^2r^2 + 5 - 2\,A + A^2}}{4r}$$

and

$$\frac{3\,r^2 + Ar^2 + 3 - A - \sqrt{5\,r^4 + 2\,Ar^4 - 6\,r^2 + A^2r^4 - 2\,A^2r^2 + 5 - 2\,A + A^2}}{4r}$$

Since r, A, and θ are all real, we have that the roots of the polynomial occur in conjugate pairs. Hence, if the term under the radical is negative, both roots are complex and we are done. So we can assume the radicand is non-negative. If the radical is real, the lemma will hold if:

$$\frac{3\,r^2 + Ar^2 + 3 - A + \sqrt{5\,r^4 + 2\,Ar^4 - 6\,r^2 + A^2r^4 - 2\,A^2r^2 + 5 - 2\,A + A^2}}{4r} > 1$$

We multiply through by $4r$. Next we subtract $4r$ from both sides to obtain

$$3\,r^2 + Ar^2 - 4r + 3 - A + \sqrt{5\,r^4 + 2\,Ar^4 - 6\,r^2 + A^2r^4 - 2\,A^2r^2 + 5 - 2\,A + A^2} > 0 \tag{9}$$

Since the radical is non-negative, if $3r^2 + Ar^2 - 4r + 3 - A > 0$, we will be done. Observe that if we take the left hand side as a linear function in A we have $\left(r^2 - 1\right)A + 3r^2 - 4r + 3$ which is decreasing in A. Thus, for a given $r \in (0, 1)$ the expression will have its minimal value on $(-\infty, 1]$ at $A = 1$. Substituting gives: $4r^2 - 4r + 2$, which is positive for all $0 < r < 1$. Hence for $A \le 1$ we have $3r^2 + Ar^2 - 4r + 3 - A > 0$.

Now we consider the case where $A > 1$. If $3r^2 + Ar^2 - 4r + 3 - A > 0$, then we are done anyway. So assume the contrary. Subtracting $3r^2 + Ar^2 - 4r + 3 - A$ from both sides of (9) gives

$$\sqrt{5\,r^4 + 2\,Ar^4 - 6\,r^2 + A^2r^4 - 2\,A^2r^2 + 5 - 2\,A + A^2} > -\left(3r^2 + Ar^2 - 4r + 3 - A\right) \tag{10}$$

As the term inside the parenthesis is by hypothesis negative, the right hand side of (10) is positive. This gives that the inequality will be preserved if we square both sides. This gives

$$\begin{aligned}5\,r^4+2\,Ar^4-6\,r^2+A^2r^4-2\,A^2r^2+5-2\,A+A^2\\ >\left(A^2+6\,A+9\right)r^4+(-8\,A-24)\,r^3\\ +\left(-2\,A^2+34\right)r^2+(-24+8\,A)\,r+9+A^2-6\,A.\end{aligned}$$

Subtracting the right hand side to the left hand side and collecting the result as a function of A gives:

$$\left(-4\,r^4+4+8\,r^3-8\,r\right)A-4\,r^4-40\,r^2-4+24\,r^3+24\,r>0. \tag{11}$$

We note first that the right hand side is linear in A. Furthermore, for $r \in (0,\,1)$ the coefficient of A is positive. (There is a triple root at $r=1$, a single root at $r=-1$, and the polynomial evaluates to 4 at $r=0$.) This gives us that for any value of r between 0 and 1, the left hand side of (11) is an increasing function of A. If we substitute $A=1$ into the left hand side, we obtain

$$-8\,r^4+32\,r^3+16\,r-40\,r^2$$

which is easily seen to be positive on $r \in (0,\,1)$. This gives us, for the case $A>1$ that for all r in the open unit interval (10) is satisfied and thus we have a root bigger than one. ■

The above lemma gives us that for any real A, we will always have at most one real root less than one. As $-1 \le \cos(\theta) \le 1$ for real θ, we have that at most one value of $\cos(\theta)$ will give a root for $\tilde{Q}$ and hence at most 2 values of θ will be roots for $\tilde{Q}$. Geometrically, this means the "dimple" in the graph of $Q(\zeta)$ will always lie to the left of the imaginary axis (see Figure 5). Hence, Q maps $\mathbb{T}$ to a curve Λ which intersects the imaginary axis at most twice. ■

Remark The techniques introduced in this paper have been used in [2] to determine the sharp bound for the Schwarzian derivative for functions in H.

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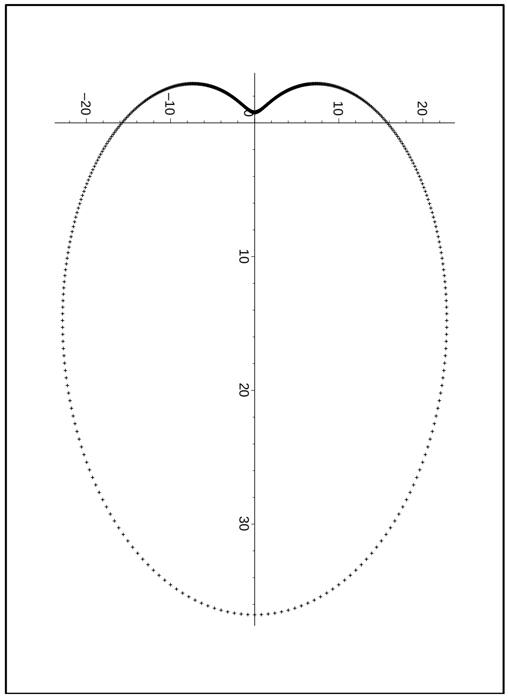

Figure 5. Image of Q for $|z| = .70$

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