A proof of Campbell's subordination conjecture

Roger W. Barnard   Kent Pearce

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Abstract

In the early 70's, D.M. Campbell published three papers on majorization-subordination results for locally univalent functions. In particular, he showed that if F is linearly invariant of order α and if f is subordinate to F on $\{z : |z| < 1\}$, then f' is majorized by F' on $\{z : |z| < m(\alpha)\}$ where $m(\alpha) = \alpha + 1 - \sqrt{\alpha^2 + 2\alpha}$, provided $\alpha \geq 1.65$. He conjectured, in fact, that this result also held for $1.65 > \alpha \geq 1$. We review Campbell's proof and why the restriction $\alpha \geq 1.65$ arose in the proof. We then affirmatively verify Campbell's conjecture in Theorem 1.

Key words: majorization, subordination, locally univalent, Campbell
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1 Introduction

Let $\mathcal{A}(\mathbb{D})$ denote the set of functions which are analytic on the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. A function $\phi \in \mathcal{A}(\mathbb{D})$ is a *Schwarz function* if $|\phi(z)| \leq |z|$ for $z \in \mathbb{D}$. Let $f, F \in \mathcal{A}(\mathbb{D})$. For $0 < r \leq 1$, the function f is *majorized* by F on $|z| < r$ if $|f(z)| \leq |F(z)|$ on $|z| < r$. The function f is *subordinate* to F on $\mathbb{D}$, if there exists a Schwarz function ϕ such that $f = F \circ \phi$ – we write $f \prec F$.

Let $\mathcal{U}_\alpha$ denote the subset of $\mathcal{A}(\mathbb{D})$ of all locally univalent functions of order α with the normalization $f(z) = z + \cdots$. The family $\mathcal{U}_\alpha$ is known as the universal linear invariant family of order α [7]. In the early 70's Campbell published three papers on majorization-subordination for locally univalent functions [2, 3, 4] in which he established relationships between majorization and subordination for functions in $\mathcal{U}_\alpha$. He showed that many of the classical results on majorization and subordination for univalent functions (see [6, 8]) hold in the more general setting of locally univalent functions. He remarked "Our investigation shows that the important datum for majorization-subordination theory, is not *univalence*, but the order of a linearly invariant family."

In particular, he proved:

Theorem A. *If F is locally univalent of order α for $\alpha \geq 1$ and if f is majorized by F on $\mathbb{D}$, then f' is majorized by F' on $\{z : |z| < n(\alpha)\}$ where* $n(\alpha) = \dfrac{(\alpha + 1)^{1/\alpha} - 1}{(\alpha + 1)^{1/\alpha} + 1}.$

He further showed:

Theorem B. *If F is locally univalent of order α for $\alpha \geq 1.65$ and if f is subordinate to F on $\mathbb{D}$, then f' is majorized by F' on $\{z : |z| < m(\alpha)\}$ where $m(\alpha) = \alpha + 1 - \sqrt{\alpha^2 + 2\alpha}$.*

The bounds given in Theorem A and Theorem B are sharp for each α.

The restriction $\alpha \geq 1.65$ in Theorem B arose out of the proof that Campbell constructed. He conjectured that, in fact, Theorem B also held for $1.65 > \alpha \geq 1$.

We will employ the following notation throughout the paper: Let $m = m(\alpha) = \alpha + 1 - \sqrt{\alpha^2 + 2\alpha}$ for $\alpha \geq 1$. Let f, F denote functions in $\mathcal{A}(\mathbb{D})$ such that f is subordinate to F on $\mathbb{D}$ and $F \in \mathcal{U}_\alpha$ for some $\alpha \geq 1$. Let ϕ denote the Schwarz function in $\mathcal{A}(\mathbb{D})$ such that $f = F \circ \phi$. Let $a = f'(0)$. We will assume that f has been rotated so that $0 \leq a \leq 1$. Then, we can write $\phi(z) = z\dfrac{a + \omega(z)}{1 + a\omega(z)}$ where $\omega \in \mathcal{A}(\mathbb{D})$ such that ω is a Schwarz function. For $z = xe^{it} \in \mathbb{D}$ with $0 \leq x \leq m$, let $c = \omega(z) = re^{i\theta}$. We have $0 \leq r \leq x \leq m$. Note, at z we have $f(z) = F(\phi(z))$ and $\phi(z) = z\dfrac{a + c}{1 + ac}$.

In [4], Campbell established two lemmas (Lemma 1 and Lemma 2) for estimating $|\phi'(z)|$ in terms of a and c, depending on whether a was "small" or "large". Then, referencing Pommerenke's classical paper [7] on linearly-invariant families, he established the following fundamental inequality

$$\left|\frac{f'(z)}{F'(z)}\right| \leq \frac{1 - x^2}{1 - |\phi(z)|^2}\left(\frac{|1 - \overline{\phi(z)}z| + |\phi(z) - z|}{|1 - \overline{\phi(z)}z| - |\phi(z) - z|}\right)^\alpha |\phi'(z)| \tag{1}$$

Using Lemma 2 [4], Campbell showed that

$$\left|\frac{f'(z)}{F'(z)}\right| \leq \frac{ba + 1}{b + a}\left(\frac{b - a}{b - 1}\right)^\alpha = k(a, \alpha, b) \tag{2}$$

where $b = \dfrac{1 + x^2}{2x} \geq \alpha + 1$, since $x \leq m$. In particular, Campbell showed that $k(a, \alpha, b)$ is decreasing in b. Hence, $k(a, \alpha, b) \leq k(a, \alpha, \alpha + 1)$. Following Campbell, we note that

$$\frac{\partial k(a, \alpha, \alpha + 1)}{\partial a} = \frac{(\alpha + 1 - a)^{\alpha - 1}}{(\alpha)^\alpha}\frac{p(a, \alpha)}{(\alpha + 1 + a)^2}$$

where

$$p(a, \alpha) = -\alpha(\alpha + 1)a^2 - (\alpha^3 + 3\alpha^2 + 4\alpha)a + \alpha(\alpha + 1)^2. \tag{3}$$

Let $R = \{(a, \alpha) : p(a, \alpha) \geq 0\}$, i.e., R is the set where $k(a, \alpha, \alpha + 1)$ is increasing in a. Campbell showed that R contains the set $\{(a, \alpha) : 0 \leq a \leq 0.4, \alpha \geq 1\}$.

Let $C_1 = \{(a, \alpha) \in R : k(a, \alpha, \alpha + 1) = 1\}$ and $A_1 = \{(a, \alpha) \in R : k(a, \alpha, \alpha + 1) \leq 1\}$. For $(a, \alpha) \in A_1$, we have $|f'(z)/F'(z)| \leq 1$. In this notation, Campbell explicitly verified that the rectangles

$$\begin{aligned} R_1 &= \{(a, \alpha) : 0 \leq a \leq 3/20, 1.65 \leq \alpha \leq 2\} \\ R_2 &= \{(a, \alpha) : 0 \leq a \leq 1/6, 2 \leq \alpha \leq 3\} \\ R_3 &= \{(a, \alpha) : 0 \leq a \leq 1/10, 3 \leq \alpha < \infty\} \end{aligned}$$
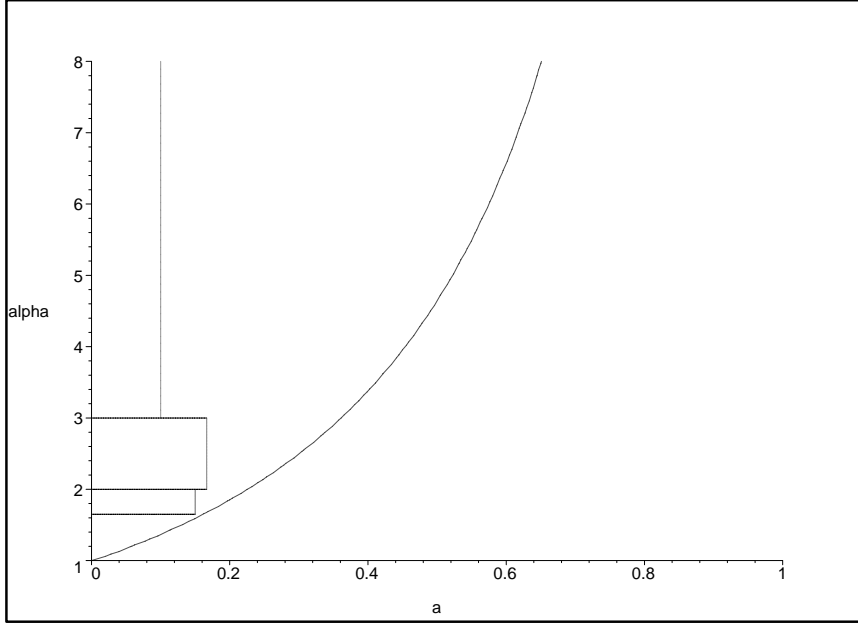
Figure 1: Rectangles R_1, R_2, R_3 contained in Region A_1

each belong to A_1. More explicitly, Campbell showed that the "right-hand" edges of R_1, R_2 and R_3 belong to A_1. Figure 1 summarizes the above containment conclusions which Campbell proved.

Using Lemma 1 [4], Campbell showed

$$\left| \frac{f'(z)}{F'(z)} \right| \leq \mathcal{G}(c, x, a) \cdot \mathcal{H}(c, x, a) = \mathcal{L}(c, x, a) \tag{4}$$

where

$$
\begin{aligned}
G(c, x, a) &= \frac{x(1-a)|1-c|}{|1 + ac - x^2(a+c)|} \\
\mathcal{G}(c, x, a) &= \left(\frac{1 + G(c, x, a)}{1 - G(c, x, a)} \right)^{\alpha - 1} \\
\mathcal{H}(c, x, a) &= (1 - x^2) \frac{|a + 2c + ac^2|(1 - x^2) + (x^2 - r^2)(1 - a^2)}{[|1 + ac - x^2(a+c)| - x(1-a)|1-c|]^2} \\
\mathcal{L}(c, x, a) &= \mathcal{G}(c, x, a) \cdot \mathcal{H}(c, x, a)
\end{aligned}
$$

Campbell showed in a separate lemma (Lemma 3 [4]) that $\mathcal{L}(c, x, a)$ as a function of θ maximizes at $\theta = 0$, i.e., $\mathcal{L}(r, x, a) \geq \mathcal{L}(c, x, a)$ for $c = re^{i\theta}$, for $0 \leq r \leq x \leq m$, on the set defined by $Q(a, \alpha) \geq 0$ where

$$Q(a, \alpha) = -(1 + \alpha + 2\alpha^2 + \alpha^3)a^2 - (2 - 2\alpha - 5\alpha^2 - 2\alpha^3)a - (\alpha + 1). \tag{5}$$
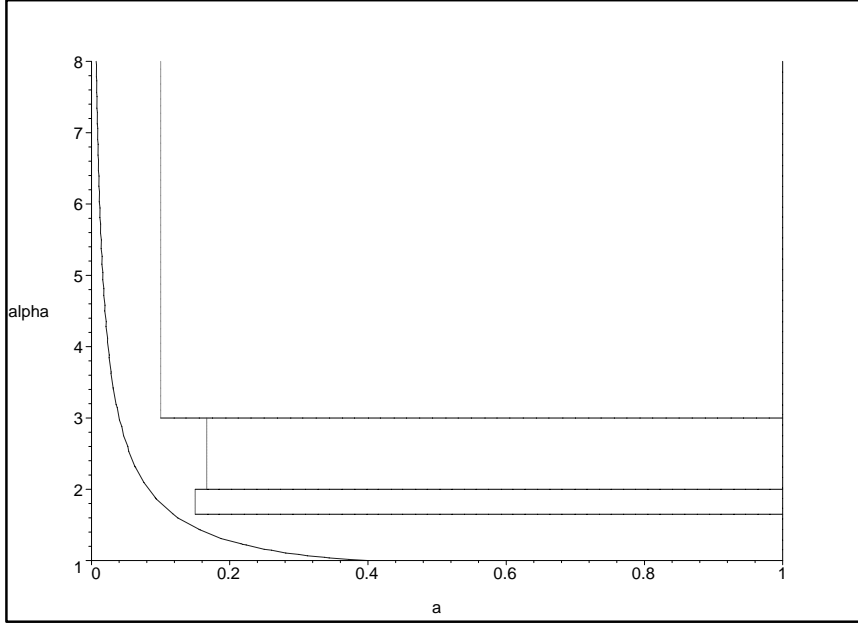
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Figure 2: Rectangles $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3$ contained in Region A_2

Finally, Campbell showed that $\mathcal{L}(r, x, a)$ is an increasing function of a provided that $x \leq m$. A calculation shows that $\mathcal{L}(r, x, 1) \equiv 1$.

Let $C_2 = \{(a, \alpha) : Q(c, x, a) = 0\}$ and $A_2 = \{(a, \alpha) : Q(c, x, a) \geq 0\}$. For $(a, \alpha) \in A_2$, we have $|f'(z)/F'(z)| \leq 1$. In this notation, Campbell explicitly verified that the rectangles

$$
\begin{aligned}
\tilde{R}_1 &= \{(a, \alpha) : 3/20 \leq a \leq 1, 1.65 \leq \alpha \leq 2\} \\
\tilde{R}_2 &= \{(a, \alpha) : 1/6 \leq a \leq 1, 2 \leq \alpha \leq 3\} \\
\tilde{R}_3 &= \{(a, \alpha) : 1/10 \leq a \leq 1, 3 \leq \alpha < \infty\}
\end{aligned}
$$

each belong to A_2. More explicitly, Campbell showed that the "left-hand" edges of $\tilde{R}_1, \tilde{R}_2$ and $\tilde{R}_3$ belong to A_2.

Figure 2 summarizes the above containment conclusions which Campbell proved. Note, the "right-hand" edges of the rectangles R_1, R_2, R_3 were precisely the "left-hand" edges of the rectangles $\tilde{R}_1, \tilde{R}_2$ and $\tilde{R}_3$. Figure 3 summarizes both of the above containment geometries.

Summarizing the above, Campbell proved that subordination $f \prec F$ for $F \in \mathcal{U}_\alpha$ for $1.65 \leq \alpha < \infty$ implies majorization $|f'(z)| \leq |F'(z)|$ for $|z| \leq m$.

Campbell conjectured that subordination $f \prec F$ for $F \in \mathcal{U}_\alpha$ for $1 \leq \alpha < 1.65$ implies majorization $|f'(z)| \leq |F'(z)|$ for $|z| \leq m$. In [1], this conjecture was verified for the case $\alpha = 1$. For this case, the family $\mathcal{U}_1$ consists of normalized convex functions. The proof for the case $\alpha = 1$ given in [1] depended heavily on the convexity of the superordinate
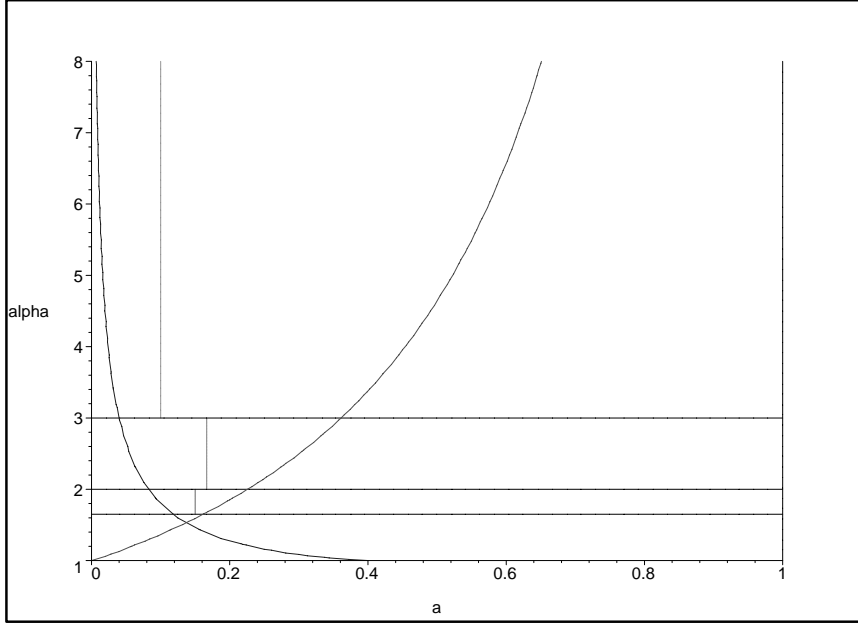
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Figure 3: Rectangles for $\alpha \geq 1.65$

function and was not extendable for $\alpha > 1$. Thus, what remained was to verify Campbell's conjecture for the range $1 < \alpha < 1.65$.

Campbell's arguments, as summarized above, imply, in fact, that for $(a, \alpha) \in A_1 \cup A_2$ that majorization holds even for $\alpha < 1.65$. Thus, to verify Campbell's conjecture what remained to be shown was that if (a, α) belongs to the parameter region $\{(a, \alpha) : 0 \leq a \leq 1, \alpha \geq 0\}$ and if $(a, \alpha) \notin A_1 \cup A_2$, then subordination $f \prec F$ for $F \in \mathcal{U}_\alpha$ implies majorization $|f'(z)| \leq |F'(z)|$ for $|z| \leq m$.

2 Preliminary Step

The boundary curves defining the region $A_1 \cup A_2$ are given implicitly via the bivariate polynomials p and Q, as described above, and are difficult to handle analytically. The preliminary step to verifying Campbell's conjecture consists of identifying two explicit line segments L_1 and L_2 and analytically verifying that L_1 lies in A_1 and that L_2 lies in A_2. See Figure 4.

Lemma 2.1 *Case A_1. Let $A_1 = \{(a, \alpha) : p(a, \alpha) \leq 1\}$ where p is defined by (3). Let $l_1(\alpha) = \dfrac{\alpha - 1}{5}, 1 \leq \alpha \leq 1.65$ and $L_1 = \{(a, \alpha) : a = l_1(\alpha), 1 \leq \alpha \leq 1.65\}$. Then, $L_1 \subset A_1$.*

Case A_2. Let $A_2 = \{(a, \alpha) : Q(a, \alpha) \geq 0\}$ where Q is defined by (5). Let $l_2(\alpha) = \dfrac{4 - 2\alpha}{5}$, $1 \leq \alpha \leq 1.65$ and $L_2 = \{(a, \alpha) : a = l_2(\alpha), 1 \leq \alpha \leq 1.65\}$. Then, $L_2 \subset A_2$.

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Proof *Case A_1.* For $1 \leq \alpha \leq 1.65$, define $k_1(\alpha) = p(l_1(\alpha), \alpha)$. Then, we have

$$k_1(\alpha) = \frac{((\alpha + 1)(\frac{\alpha - 1}{5}) + 1)\left(\frac{4\alpha + 6}{5\alpha}\right)^\alpha}{\frac{6\alpha + 4}{5}}$$

As a first step, to assure that our arguments will require only finite arithmetic with no numerical approximations, thus justifying our use of Sturm sequence arguments, we will make estmates with rational functions whose coefficients are integers. We will show for $1 \leq \alpha \leq 1.65$, that

$$\left(\frac{4\alpha + 6}{5\alpha}\right)^\alpha \leq p_2(\alpha) \tag{6}$$

where $p_2(\alpha) = 2 + \frac{1}{5}(\alpha - 1) - \frac{1}{5}(\alpha - 1)^2$. Then, we will have for $1 \leq \alpha \leq 1.65$

$$k_1(\alpha) \leq k_2(\alpha) = \frac{((\alpha + 1)(\frac{\alpha - 1}{5}) + 1)p_2(\alpha)}{\frac{6\alpha + 4}{5}} = \frac{n_2(\alpha)}{d_2(\alpha)}.$$

However, $k_2(\alpha)$ is rational in α. A straightforward rearrangement shows that $d_2(\alpha) - n_2(\alpha) = (\alpha - 1)(\alpha - 2)(\alpha^2 - 6) \geq 0$.

To verify (6), we will show that

$$f_2(\alpha) = \log p_2(\alpha) - \alpha \log \frac{4\alpha + 6}{5\alpha} \geq 0$$

However,

$$f_2'(\alpha) = \frac{7\alpha^2 - 9\alpha - 33}{(\alpha^2 - 3\alpha - 8)(2\alpha + 3)} - \log \frac{4\alpha + 6}{5\alpha}$$

We will show for $1 \leq \alpha \leq 1.65$ that

$$\log \frac{4\alpha + 6}{5\alpha} \leq p_3(\alpha) \tag{7}$$

where $p_3(\alpha) = \frac{695}{1000} - \frac{52}{100}(\alpha - 1) + \frac{1}{5}(\alpha - 1)^2$.

Hence,

$$f_2'(\alpha) \geq g(\alpha) = \frac{7\alpha^2 - 9\alpha - 33}{(\alpha^2 - 3\alpha - 8)(2\alpha + 3)} - p_3(\alpha)$$

However, a Sturm sequence argument, using only finite arithmetic (see [5]), shows for $1 \leq \alpha \leq 1.65$ that $g(\alpha) \geq 0$.

Finally, to verify (7), we note that if we let $p_4(\alpha) = p_3(\alpha) - \log \frac{4\alpha + 6}{5\alpha}$, then a Sturm sequence argument shows that, $p_4'(\alpha) \geq 0$ and a calculation shows that $p_4(1) > 0$.

Case A_2. For $1 \leq \alpha \leq 1.65$, define $j_1(\alpha) = Q(l_2(\alpha), \alpha)$. Then,

$$j_1(\alpha) = -\frac{4}{25}\alpha^5 - \frac{12}{25}\alpha^4 + \frac{2}{25}\alpha^3 + \frac{12}{5}\alpha^2 + \frac{7}{5}\alpha - \frac{81}{25}.$$

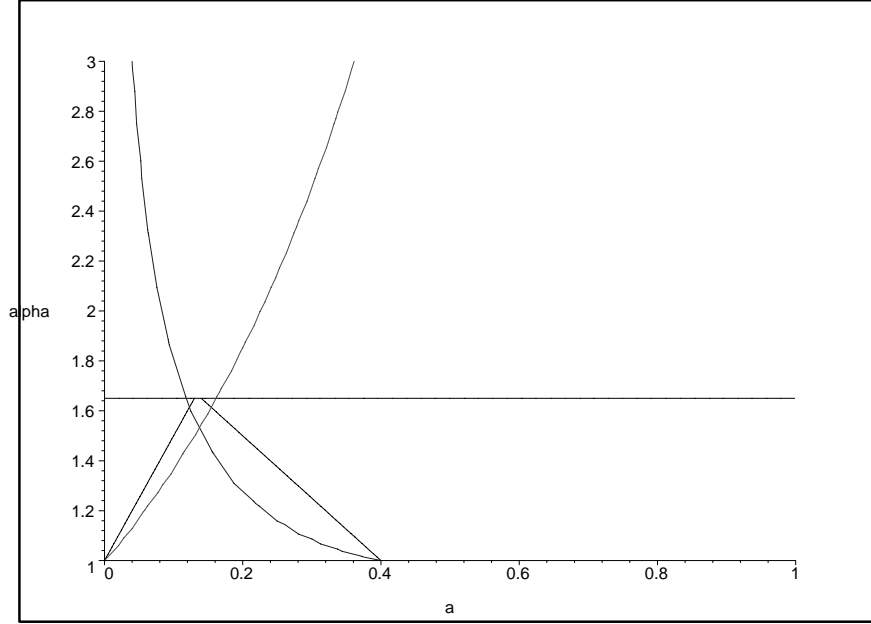A Sturm sequence argument shows that $j_1(\alpha) \geq 0$. $\square$

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Figure 4: Critical Trapezoid

Campbell's arguments imply for (a, α) to the "left" of L_1 or for (a, α) to the "right" of L_2, that subordination $f \prec F$ for $F \in \mathcal{U}_\alpha$ implies $|f'(z)| \leq |F'(z)|$ for $|z| \leq m$. Let T be the critical trapezoid (see Figure 4) given by

$$T = \{(a, \alpha) : l_1(\alpha) \leq a \leq l_2(\alpha), 1 \leq \alpha \leq 1.65\}. \tag{8}$$

We will now prove

Theorem 1. *Let (a, α) belong to the critical trapezoid T given by (8). Let F be locally univalent of order α. Then, if f is subordinate to F on $\mathbb{D}$, then f' is majorized by F' on $\{z : |z| < m(\alpha)\}$ where $m(\alpha) = \alpha + 1 - \sqrt{\alpha^2 + 2\alpha}$. More specifically, for $|z| < m(\alpha)$*

$$\left| \frac{f'(z)}{F'(z)} \right| \leq \max_{(a,\alpha)\in T} \mathcal{G}(c, x, a) \cdot \max_{(a,\alpha)\in T} \mathcal{H}(c, x, a) \leq 1 \tag{9}$$

Theorem 1, combined with the above observations about Campbell's original arguments, implies Campbell's conjecture.

The proof of Theorem 1 consists of four steps

- Step 1. Constructing a rational function upper bound estimate for $G(c, x, a)$.

- Step 2. Using the estimate for $G(c, x, a)$ from Step 1, constructing a rational function upper bound estimate for $\mathcal{G}(c, x, a)$.

- Step 3. Constructing a rational function upper bound estimate for $\mathcal{H}(c, x, a)$.

- Step 4. Using the estimates for $\mathcal{G}(c, x, a)$ and $\mathcal{H}(c, x, a)$ from Step 2 and Step 3, verifing that the right-hand inequality in (9) holds.

3 Step 1

For $(a, \alpha) \in T$ and $c = re^{i\theta}, 0 \leq r \leq x \leq m$, we will construct a rational function upper bound estimate for $G(c, x, a)$; specifically, we will show that $G(c, x, a) \leq \dfrac{6 - \alpha}{6 + 9\alpha}$.

Recall that $G(c, x, a) = \dfrac{x(1 - a)|1 - c|}{|1 + ac - x^2(a + c)|}$. First, we will show that $G(c, x, a) \leq G(-r, x, a)$, i.e., that $G(c, x, a)$, as a function of the first argument, maximizes at $c = -r$. It suffices to show that $G_1 = \dfrac{|1 - c|^2}{|1 + ac - x^2(a + c)|^2}$ maximizes at $c = -r$. Rewriting, we have

$$G_1 = \frac{1 - 2r\cos\theta + r^2}{(1 - ax^2)^2 + 2(1 - ax^2)(a - x^2)r\cos\theta + (a - x^2)^2 r^2} = \frac{\mu_1 + \nu_1\cos\theta}{\sigma_1 + \tau_1\cos\theta}.$$

Hence,

$$\frac{\partial G_1}{\partial\theta} = \frac{\mu_1\tau_1 - \nu_1\sigma_1}{(\sigma_1 + \tau_1\cos\theta)^2}(\sin\theta).$$

Rewriting, we have

$$\mu_1\tau_1 - \nu_1\sigma_1 = 2r(1 + a)(1 - x^2)[1 - ax^2 + (a - x^2)r^2]$$

If $a \geq x^2$, then clearly $\mu_1\tau_1 - \nu_1\sigma_1 \geq 0$. On the other hand, if $a < x^2$, then

$$
\begin{aligned}
\mu_1\tau_1 - \nu_1\sigma_1 &= 2r(1 + a)(1 - x^2)[1 - ax^2 + (a - x^2)r^2] \\
&\geq 2r(1 + a)(1 - x^2)[1 - ax^2 + (a - x^2)1] \\
&= 2r(1 + a)^2(1 - x^2)^2 \\
&\geq 0
\end{aligned}
$$

Similarly, by taking derivatives it follows that $G(-r, x, a)$, $G(-x, x, a)$ and $G(-m, m, a)$ are increasing in r, increasing in x and decreasing in a, resp.

Thus, on T we have $G(c, x, a) \leq G(-m, m, l_1(\alpha))$. It is a straightforward calculation, using the relationship that $1 + m^2 = 2(\alpha + 1)m$, that $G(-m, m, l_1(\alpha)) = \dfrac{6 - \alpha}{6 + 9\alpha}$.

4 Step 2

For $(a, \alpha) \in T$ and $c = re^{i\theta}, 0 \leq r \leq x \leq m$, we will construct a rational function upper bound estimate for $\mathcal{G}(c, x, a)$.

Recall that $\mathcal{G}(c, x, a) = \left(\dfrac{1 + G(c, x, a)}{1 - G(c, x, a)} \right)^{\alpha - 1}$. It is easily verified, for $\alpha \geq 1$, that $g(y) = \left(\dfrac{1 + y}{1 - y} \right)^{\alpha - 1}$ is an non-decreasing function of y. In Step 1, we showed for $(a, \alpha) \in T$ that $G(c, x, a) \leq \dfrac{6 - \alpha}{6 + 9\alpha}$.

Again, to assure that our arguments will require only finite arithmetic with no numerical approximations, we will make estmates with rational functions whose coefficients are integers. Let $l(y) = 1 + \dfrac{21}{10}(\alpha - 1)(1 + \dfrac{\alpha - 1}{4})y$. We will show that $g(\dfrac{6 - \alpha}{6 + 9\alpha}) \leq l(\dfrac{6 - \alpha}{6 + 9\alpha})$. Hence, on T we will have that $\mathcal{G}(c, x, a) \leq l(\dfrac{6 - \alpha}{6 + 9\alpha})$.

Let $h(\alpha) = l(\dfrac{6 - \alpha}{6 + 9\alpha}) - g(\dfrac{6 - \alpha}{6 + 9\alpha})$. We can rewrite h as $h(\alpha) = \dfrac{n(\alpha)}{40(2 + 3\alpha)}$ where

$$n(\alpha) = -46 + 225\alpha + 28\alpha^2 - 7\alpha^3 - 40(2 + 3\alpha) \left(\frac{2(3 + 2\alpha)}{5\alpha} \right)^{\alpha - 1}.$$

We will show explicitly for $1 \leq \alpha \leq 1.65$ that n is an non-negative increasing function of α. Let $\beta = \left(\dfrac{2(3 + 2\alpha)}{5\alpha} \right)^{\alpha - 1}$ and $\gamma = \log \dfrac{2(3 + 2\alpha)}{5\alpha}$. Then, we can write

$$n'(\alpha) = \frac{n_1(\alpha)}{\alpha(3 + 2\alpha)} = \frac{A_0 + \beta(B_0 + \gamma C_0)}{\alpha(3 + 2\alpha)}$$

where

$$
\begin{aligned}
A_0 &= 675\alpha + 618\alpha^2 + 49\alpha^3 - 42\alpha^4 \geq 0 \\
B_0 &= -240 - 480\alpha + 120\alpha^2 \leq 0 \\
C_0 &= -240\alpha - 520\alpha^2 - 240\alpha^3 \leq 0.
\end{aligned}
$$

It is sufficient to show that n_1 is non-negative.

In a separate lemma, in the appendix, we will show for $1 \leq \alpha \leq 1.65$ the following upper bounds for β and γ hold

$$\beta \leq \beta_1 = 1 + \frac{694}{1000}(\alpha - 1) - \frac{361}{1000}(\alpha - 1)^2 + \frac{69}{1000}(\alpha - 1)^3 \tag{10}$$

$$\gamma \leq \gamma_1 = \frac{6932}{10000} - \frac{5992}{10000}(\alpha - 1) + \frac{42}{100}(\alpha - 1)^2 - \frac{2795}{10000}(\alpha - 1)^3 \tag{11}$$
$$+ \frac{111}{1000}(\alpha - 1)^4$$

Using the upper bounds for β and γ in (10) and (11), we have

$$n_1(\alpha) \geq n_2(\alpha) = A_0 + \beta_1(B_0 + \gamma_1 C_0)$$

However, a Sturm sequence argument shows for $1 \leq \alpha \leq 1.65$ that $n_2(\alpha) \geq 0$.

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5 Step 3

For $(a, \alpha) \in T$ and $c = re^{i\theta}, 0 \le r \le x \le m$, we will construct a rational function upper bound estimate for $\mathcal{H}(c, x, a)$.

The construction of the upper bound for $\mathcal{H}(c, x, a)$ in Step 3 consists of five separate substeps.

- Substep 3-1. For fixed (a, α) and fixed x, $\mathcal{H}(c, x, a)$ is a function of $c = re^{i\theta}$ where $0 \le r \le x$ and $0 \le \theta \le \pi$. We show that any maximal value of $\mathcal{H}(c, x, a)$ must occur on the boundary of the region of variability for c.

- Substep 3-2. For fixed (a, α) and x, we show that on the boundary of the region of variability for c, the maximal value of $\mathcal{H}(c, x, a)$ must occur on the portion of the boundary of variability for c where $r = x$.

- Substep 3-3. For fixed (a, α) and for $c = xe^{i\theta}$, $\mathcal{H}(c, x, a)$ is a function of (x, θ) where $0 \le x \le m$ and $0 \le \theta \le \pi$. We show that any maximal value of $\mathcal{H}(c, x, a)$ must occur on the boundary of the region of variability for (x, θ).

- Substep 3-4. For fixed (a, α) and for $c = xe^{i\theta}$, we show that on the boundary of the region of variability for (x, θ), the maximal value of $\mathcal{H}(c, x, a)$ must occur on the portion of the boundary of variability for (x, θ) where $x = m$.

- Substep 3-5. We then construct a rational function upper bound estimate for $\mathcal{H}(me^{i\theta}, m, a)$.

To simplify notation, we will write $\mathcal{H}(c, x, a) = (1 - x^2) \dfrac{C(c, x, a)}{[B(c, x, a)]^2}$ where

$$
\begin{aligned}
C(c, x, a) &= |a + 2c + ac^2|(1 - x^2) + (x^2 - r^2)(1 - a^2) \\
B(c, x, a) &= |1 + ac - x^2(a + c)| - x(1 - a)|1 - c|
\end{aligned}
$$

5.1 Substep 3-1

Let $(a, \alpha) \in T$ be fixed and let $x \le m$ be fixed. For $c = re^{i\theta}$, we first show that $\mathcal{H}(c, x, a)$ does not have a local extreme value on the open set $O_r = \{(r, \theta) : 0 < r < x, 0 < \theta < \pi\}$.

Since a, x are fixed and the only variable parameter is $c = re^{i\theta}$, we will write, in Substep 3-1, $\mathcal{H}(r, \theta) \equiv \mathcal{H}(c, x, a), B(r, \theta) \equiv B(c, x, a)$, etc. Specifically, if $\mathcal{H}(r, \theta)$ had a local extreme value, say at (r, θ), then we would have

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial r}(r, \theta) &= 0 \\
\frac{\partial \mathcal{H}}{\partial \theta}(r, \theta) &= 0
\end{aligned}
\tag{12}
$$

However, (12) implies that

$$
\begin{aligned}
B(r, \theta) \frac{\partial C}{\partial r}(r, \theta) - 2C(r, \theta) \frac{\partial B}{\partial r}(r, \theta) &= 0 \\
B(r, \theta) \frac{\partial C}{\partial \theta}(r, \theta) - 2C(r, \theta) \frac{\partial B}{\partial \theta}(r, \theta) &= 0
\end{aligned}
$$

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which in turn implies that

$$\Delta_r = \begin{vmatrix} \frac{\partial C}{\partial r}(r,\theta) & \frac{\partial B}{\partial r}(r,\theta) \\[2mm] \frac{\partial C}{\partial \theta}(r,\theta) & \frac{\partial B}{\partial \theta}(r,\theta) \end{vmatrix} = 0 \tag{13}$$

We will show that the determinant in (13) is, however, non-zero. Specifically, computing we have

$$\Delta_r = 2r^2 \sin\theta (1-a^2)\frac{N_r}{D_r}$$

where

$$\begin{aligned} N_r &= (p_r + q_r X) + (P_r + Q_r X)G \\ D_r &= |1-c|X \\ X &= |a + 2c + ac^2| \end{aligned}$$

and p_r, q_r, P_r and Q_r are polynomials, with rational coefficients, in $r, \cos\theta, x$ and a, and $G = G(c, x, a)$.

It suffices to show that N_r is negative. Since N_r is linear in G and, from Step 1, we have that $0 \le G \le \dfrac{6-\alpha}{6+9\alpha}$, then $ng0 \le N_r \le ng1$ where

$$\begin{aligned} ng0 &= (p_r + q_r X) + (P_r + Q_r X)0 \\ ng1 &= (p_r + q_r X) + (P_r + Q_r X)\frac{6-\alpha}{6+9\alpha} \end{aligned}$$

Further, since $ng0$ and $ng1$ are each linear in X and $0 \le X \le a + 2x + ax^2$, then we have $ng00 \le ng0 \le ng01$ and $ng10 \le ng1 \le ng11$ where

$$\begin{aligned} ng00 &= (p_r + q_r 0) + (P_r + Q_r 0)0 = p_r \\ ng01 &= (p_r + q_r(a + 2x + ax^2)) + (P_r + Q_r(a + 2x + ax^2))0 = p_r + q_r(a + 2x + ax^2)) \\ ng10 &= (p_r + q_r 0) + (P_r + Q_r 0)\frac{6-\alpha}{6+9\alpha} = p_r + P_r\frac{6-\alpha}{6+9\alpha} \\ ng11 &= (p_r + q_r(a + 2x + ax^2)) + (P_r + Q_r(a + 2x + ax^2))\frac{6-\alpha}{6+9\alpha} \end{aligned}$$

In a lengthy, finite arithmetic/integer argument, using exact calcluations, in the Maple worksheet *campbell-maple-substep1.mws*[1], we explicitly verify that each of the factors $ng00, ng01, ng10, ng11$ are negative.

5.2 Substep 3-2

Since $\mathcal{H}(c, x, a)$ does not have a local extreme value on the open set $O_r = \{(r, \theta) : 0 < r < x, 0 < \theta < \pi\}$, then any extreme value of $\mathcal{H}(c, x, a)$ must occur on either

2a) the set $E(r)^+ = \{(r, 0) : 0 \le r \le x\}$
2b) the set $E(r)^- = \{(-r, 0) : 0 \le r \le x\}$
2c) the set $E(x, \theta) = \{(x, \theta) : 0 \le \theta \le \pi\}$

A straightforward calculus argument shows that on the boundary of the set O_r the maximum value of $\mathcal{H}(c, x, a)$ must occur on on the set $E(x, \theta)$. Formal details of that argument are given in the Maple worksheets *campbell-maple-substep2a.mws, campbell-maple-substep2b.mws.*

5.3   Substep 3-3

Consider the set $E(x, \theta)$. Let $(a, \alpha) \in T$ be fixed. For $c = xe^{i\theta}$ we will show that $\mathcal{H}(c, x, a)$ does not have a local extreme value on the open set $O_x = \{(x, \theta) : 0 < x < m, 0 < \theta < \pi\}$.

Since a is fixed and the only variable parameters are x and θ, we will write, in Substep 3-3, $\mathcal{H}(x, \theta) \equiv \mathcal{H}(xe^{i\theta}, x, a), B(r, \theta) \equiv B(xe^{i\theta}, x, a)$, etc. Specifically, if $\mathcal{H}(c, x, a)$ had a local extreme value, say at (x, θ), then we would have

$$\frac{\partial \mathcal{H}}{\partial x}(x, \theta) = 0 \tag{14}$$

$$\frac{\partial \mathcal{H}}{\partial \theta}(x, \theta) = 0$$

However, (14) implies that

$$B(x, \theta)\frac{\partial C}{\partial x}(x, \theta) - 2C(x, \theta)\frac{\partial B}{\partial x}(x, \theta) = 0$$

$$B(x, \theta)\frac{\partial C}{\partial \theta}(x, \theta) - 2C(x, \theta)\frac{\partial B}{\partial \theta}(x, \theta) = 0$$

which in turn implies that

$$\Delta_x = \begin{vmatrix} \frac{\partial C}{\partial x}(x, \theta) & \frac{\partial B}{\partial x}(x, \theta) \\[2mm] \frac{\partial C}{\partial \theta}(x, \theta) & \frac{\partial B}{\partial \theta}(x, \theta) \end{vmatrix} = 0 \tag{15}$$

We will show that the determinant in (15) is, however, non-zero. Specifically, computing we have

$$\Delta_x = 2x\sin\theta(1 - x^2)\frac{N_x}{D_x}$$

where $N_x = P_x + Q_x G, D_x = |1 - c||a + 2c + ac^2|$ and P_x, Q_x are polynomials, with rational coefficients, in $r, \cos\theta, x$ and a, and $G = G(c, x, a)$.

It suffices to show that N_x is negative. Since N_x is linear in G and, from Step 1, we have that $0 \le G \le \dfrac{6 - \alpha}{6 + 9\alpha}$, then $nG0 \le N_r \le nG1$ where

$$nG0 = P_x + Q_x 0 = P_x$$

$$nG1 = P_x + Q_x\frac{6 - \alpha}{6 + 9\alpha}$$

In a lengthy, finite arithmetic/integer argument, using exact calcluations, in the Maple worksheet *campbell-maple-substep3.mws*, we explicitly verify that each of the factors $nG0, nG1$ are negative.

5.4 Substep 3-4

Since $\mathcal{H}(c,x,a)$ does not have a local extreme value on the open set $O_x = \{(x,\theta) : 0 < x < m, 0 < \theta < \pi\}$, then any extreme value of $\mathcal{H}(c,x,a)$ must occur on either

    4a) the set $E(x)^+ = \{(x,0) : 0 \le x \le m\}$
    4b) the set $E(x)^- = \{(-x,0) : 0 \le x \le m\}$
    4c) the set $E(m,\theta) = \{(m,\theta) : 0 \le \theta \le \pi\}$

A straightforward calculus argument shows that on the boundary of the set O_x the maximum value of $\mathcal{H}(c,x,a)$ must occur on on the set $E(m,\theta)$. Formal details of that argument are given in the Maple worksheets *campbell-maple-substep4a.mws, campbell-maple-substep4b.mws.*

5.5 Substep 3-5

Finally, consider the set $E(m,\theta) = \{(m,\theta) : 0 \le \theta \le \pi\}$. Let $(a,\alpha) \in T$ and let

$$h_3(\alpha) = 1 - \frac{4}{5}(\alpha - 1) + \frac{13}{10}(\alpha - 1)^2 - \frac{13}{10}(\alpha - 1)^3. \tag{16}$$

In a lengthy, finite arithmetic/integer argument, using exact calcluations, in the Maple worksheet *campbell-maple-substep5.mws*, we explicitly verify the inequality that

$$\mathcal{H}(me^{i\theta}, m, a) \le h_3(\alpha).$$

We note here that in the appendix we state three specific lemmas, which give finite arithmetic/integer arguments for establishing the validity of certain multivariate polynomial inequalities, which are tacitly used in this last substep to verify that $h_3(\alpha)$ is an upper bound for $\mathcal{H}(me^{i\theta}, m, a)$ on the set $E(m,\theta)$.

6 Step 4

Using the estimates for $G(c,x,a)$ and $\mathcal{H}(c,x,a)$ from Step 2 and Step 3, we will now show that

$$\max_{(a,\alpha)\in T} \mathcal{G}(c,x,a) \cdot \max_{(a,\alpha)\in T} \mathcal{H}(c,x,a) \le 1 \tag{17}$$

Recall in Step 1 and Step 2, we showed that

$$\mathcal{G}(c,x,a) \le l(\frac{6-\alpha}{6+9\alpha}) = g_3(\alpha) = 1 + \frac{21}{10}(\alpha-1)(1 + \frac{\alpha-1}{4})\frac{6-\alpha}{6+9\alpha}.$$

Further, in Step 3 we showed

$$\mathcal{H}(c,x,a) \le h_3(\alpha) = 1 - \frac{4}{5}(\alpha-1) + \frac{13}{10}(\alpha-1)^2 - \frac{13}{10}(\alpha-1)^3.$$

It remains to show for $1 \le \alpha \le 1.65$ that $g_3(\alpha) \cdot h_3(\alpha) \le 1$. It is sufficient to show that $k_3(\alpha) \ge 0$ where

$$
\begin{aligned}
k_3(\alpha) &= 400(6+9\alpha)(1 - g_3(\alpha) \cdot h_3(\alpha)) \\
&= 8472 - 36174\alpha + 52755\alpha^2 - 29838\alpha^3 + 2874\alpha^4 + 2184\alpha^5 - 273\alpha^6
\end{aligned}
$$

However, a Sturm sequence argument verifies that $k_3(\alpha) \ge 0$.

7 Appendix

We give here for $1 \le \alpha \le 1.65$ the technical lemma which verifies the upper bounds given in Step 2 for β and γ:

Lemma 7.1 *Let* $1 \le \alpha \le 1.65$. *Let* $\beta = \left(\dfrac{2(3+2\alpha)}{5\alpha} \right)^{\alpha-1}$ *and* $\gamma = \log \dfrac{2(3+2\alpha)}{5\alpha}$. *Then,*

$$
\beta \le \beta_1 = 1 + \frac{694}{1000}(\alpha - 1) - \frac{361}{1000}(\alpha - 1)^2 + \frac{69}{1000}(\alpha - 1)^3
$$

$$
\gamma \le \gamma_1 = \frac{6932}{10000} - \frac{5992}{10000}(\alpha - 1) + \frac{42}{100}(\alpha - 1)^2 - \frac{2795}{10000}(\alpha - 1)^3
$$
$$
+ \frac{111}{1000}(\alpha - 1)^4
$$

Proof *Case* γ. We will show that $\gamma_2 = \gamma_1 - \gamma$ is a non-negative increasing function. We have

$$
\frac{d\gamma_2(\alpha)}{d\alpha} = \frac{N_1(\alpha)}{10000\alpha(3+2\alpha)}
$$
$$
= \frac{30000 - 81651\alpha + 61036\alpha^2 + 11865\alpha^3 - 30090\alpha^4 + 8880\alpha^5}{10000\alpha(3+2\alpha)}
$$

A Sturm sequence argument shows that N_1 is non-negative.

Case β. It suffices to show that $\beta_2 = \log \beta_1 - \log \beta \ge 0$. We will show for $1 \le \alpha \le 1.65$ that β_2 is a non-negative increasing function of α. We have

$$
\frac{d\beta_2(\alpha)}{d\alpha} = \frac{N_2(\alpha)}{\alpha((-124 + 1623\alpha - 568\alpha^2 + 69\alpha^3))(3+2\alpha)} \tag{18}
$$

where

$$
\begin{aligned}
N_2(\alpha) &= 372 - 372\alpha + 6411\alpha^2 - 3562\alpha^3 + 621\alpha^4 \\
&+ (372\alpha - 4621\alpha^2 - 1542\alpha^3 + 929\alpha^4 - 138\alpha^5)\gamma
\end{aligned}
$$

A Sturm sequence argument shows for $1 \le \alpha \le 1.65$ that the denominator of (18) is positive. We have then, since $\gamma \le \gamma_1$ that $N_2(\alpha) \ge N_3(\alpha)$ where

$$
\begin{aligned}
N_3(\alpha) &= 372 - 372\alpha + 6411\alpha^2 - 3562\alpha^3 + 621\alpha^4 \\
&+ (372\alpha - 4621\alpha^2 - 1542\alpha^3 + 929\alpha^4 - 138\alpha^5)\gamma_1
\end{aligned}
$$

A Sturm sequence argument shows for $1 \le \alpha \le 1.65$ that $N_3(\alpha) \ge 0$. $\square$

We state here three specific lemmas, which give finite arithmetic/integer arguments for establishing the validity of certain multivariate polynomial inequalities, which are tacitly used in the last substep of Step 3 to verify that $h_3(\alpha)$ is an upper bound for $\mathcal{H}(me^{i\theta}, m, a)$ on the set $E(m, \theta)$.

Lemma 7.2 *Let $p = p(x_1, x_2, x_3)$ be a polynomial with rational coefficients defined on the region $R = \{(x_1, x_2, x_3) : a_1 \leq x_1 \leq b_1, \quad a_2 \leq x_2 \leq b_2, \quad a_3 \leq x_3 \leq b_3\}$, where a_j, b_j are all rational numbers. Let M be an upper bound for $|\frac{\partial p}{\partial x_j}(x_1, x_2, x_3)|$ on R, $1 \leq j \leq 3$. Let $\delta > 0$ and suppose N_1, N_2, N_3 are chosen so that $\Delta_j = (b_j - a_j)/N_j \leq \delta$, $1 \leq j \leq 3$. Let L be the lattice $L = L(N_1, N_2, N_3) = \{(a_1 + i\Delta_1, a_2 + j\Delta_2, a_3 + k\Delta_3) : 0 \leq i \leq N_1, \quad 0 \leq j \leq N_2, \quad 0 \leq k \leq N_3\}$. Let*

$$m = \min_{(x_1, x_2, x_2) \in L} p(x_1, x_2, x_3).$$

If $m \geq \frac{3}{2}M\delta$, then $p(x_1, x_2, x_3) \geq 0$ on R.

Proof Let $(x_1, x_2, x_3) \in R$. Then, there exists $(x_1^0, x_2^0, x_3^0) \in L$ such that

$$\text{dist}((x_1^0, x_2^0, x_3^0 0), (x_1, x_2, x_3)) \leq \frac{\sqrt{3}}{2}\delta$$

. Without loss of generality we can suppose that $x_1 \geq x_1^0, x_2 \geq x_2^0, x_3 \geq x_3^0$. Define

$$p(t) = p_{\theta,\phi}(t) = p(x_1^0 + t\cos(\theta)\sin(\phi), x_2^0 + t\sin(\theta)\sin(\phi), x_3^0 + t\cos(\phi))$$

$0 < t \leq \frac{\sqrt{3}}{2}\delta, 0 < \theta < \pi/2, 0 < \phi < \pi/2$. Then, $|p'(t)| \leq M\sqrt{3}$ for $0 \leq t \leq \frac{\sqrt{3}}{2}\delta$. Hence, $p(x_1, x_2, x_3) = p_{\theta,\phi}(t) \geq p_{\theta,\phi}(0) - M\sqrt{3}\,t = p(x_1^0, x_2^0, x_3^0) - M\sqrt{3}\,t \geq m - \frac{3}{2}M\delta \geq 0$. □

Lemma 7.3 *Let $p = p(x)$ be a polynomial with rational coefficients defined on the interval $R_1(d) = \{x : 0 < x < d\}$ such that $p(0) = 0$. Let $M_1 = p'(0) > 0$ and let $-N_1$ be a lower bound for $p''(x)$ on $R_1(d)$ where $N_1 \geq 0$. Then, $p(x) > 0$ on $R_1(\delta_1)$ where $\delta_1 = \min\{\frac{2M_1}{N_1}, d\}$.*

Proof The hypotheses imply, using Taylor's remainder theorem, that
$p(x) = M_1 x + p''(c)\frac{x^2}{2}$ for some c such that $0 < c < x$. Hence, we have $\frac{p(x)}{x} > M_1 - \frac{N_1 x}{2}$.
□.

Lemma 7.4 *Let $p = p(x_1, x_2, x_3)$ be a polynomial with rational coefficients defined on the region $R_3(d) = \{(x_1, x_2, x_3) : 0 < x_1 < d\,, 0 < x_2 < d\,, 0 < x_3 < d\}$ such that $p(0, 0, 0) = 0$. Let*

$$M_3 = \min\{p_1(0, 0, 0), p_2(0, 0, 0), p_3(0, 0, 0)\} > 0$$

where p_j denotes the partial of p with respect to $x_j, j = 1, 2, 3$. Let $-N_3$ be a lower bound for $p_{ij}(x_1, x_2, x_3)$ on $R_3(d)$, $1 \leq i, j \leq 3$, where $N_3 \geq 0$. Then, $p(x_1, x_2, x_3) > 0$ on $R_3(\delta_3)$ where $\delta_3 = \min\{2M_3/(3N_3), d\}$.

Proof Let $B_3(d) = \{(x_1, x_2, x_3) \in R_3(d) : ||(x, y, z)|| < d\}$. Note that $R_3(d/\sqrt{3}) \subset B_3(d)$. On $B_3(d)$ we can write

$$p(x_1, x_2, x_3) = p(t\cos(\theta)\sin(\phi), t\sin(\theta)\sin(\phi), t\cos(\phi)) = p_{\theta,\phi}(t) = p(t)$$

where $0 < t < d, 0 < \theta < \pi/2, 0 < \phi < \pi/2$. Apply Lemma 7.3 to p on $R_1(d)$ with $M_1 = M_3$ and $N_1 = \sqrt{3}N_3$. Consequently, $p(t) > 0$ on $R_1(\delta_1)$ where $\delta_1 = 2M_3/(\sqrt{3}N_3)$. Hence, $p(x_1, x_2, x_3) > 0$ on $R_3(\delta_3/\sqrt{3})$. □

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