Lecture 16

Sec 4.5

Basis for a vector space

Suppose a collection of vectors $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ in a vector space V is given.

We say that S is a basis for V if (1) $\text{Span}(S) = V$, and

(2) S is linearly independent.

Example 1 Let $S = \{(1,0),\ (0,1)\}$ with $V = \mathbb{R}^2$.

Solution Let us check whether: (1) $\text{Span}(S) = \mathbb{R}^2$?

A random vector in $\mathbb{R}^2$ is $\mathbf{v} = (v_1, v_2)$.

We are looking for scalars c_1, c_2 so that $c_1\,(1,0) + c_2\,(0,1) = (v_1, v_2)$.

Clearly, we select $c_1 = v_1$ and $c_2 = v_2$.

(2) Is S linearly independent? Consider $c_1\,(1,0) + c_2\,(0,1) = (0,0)$.

Clearly, $c_1 = 0$ and $c_2 = 0$.

Hence, S is a basis for $\mathbb{R}^2$.

Recall (from Section 3.3)

If A is an $n \times n$ matrix, then the following conditions are equivalent.

1. A is invertible
2. $A\mathbf{c} = \mathbf{v}$ has a unique solution for every $n \times 1$ vector $\mathbf{v}$.
3. $A\mathbf{c} = \mathbf{0}$ has only the trivial solution $\mathbf{c} = \mathbf{0}$.
4. A is row equivalent to I_n.
5. A can be written as a product of elementary matrices.
6. $\det A \neq 0$.

Standard basis for $\mathbb{R}^n$

The set $S = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ where

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, \cdots, 0) \\ \mathbf{e}_2 &= (0, 1, \cdots, 0) \\ &\vdots \\ \mathbf{e}_n &= (0, 0, \cdots, 1) \end{aligned}$$

is called the standard basis for $\mathbb{R}^n$.

Non-standard basis for $\mathbb{R}^n$

Consider the set $S = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$. Form the $n \times n$ matrix $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$.

(1) S spans $\mathbb{R}^n$ if $A\,\mathbf{c} = \mathbf{b}$ has a solution for any vector $\mathbf{b}$.

(2) S is linearly independent if $A\,\mathbf{c} = \mathbf{0}$ has the only solution $\mathbf{c} = \mathbf{0}$.

We know that both are equivalent to: A is nonsingular.

Thus the column vectors of a generic nonsingular $n \times n$ matrix is a non-standard basis for $\mathbb{R}^n$

Example 1 Let $S = \{(1,2,3),\ (0,1,2),\ (-2,0,1)\}$ with $V = \mathbb{R}^3$.

Solution Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

Let us check whether A is nonsingular.

$$|A| = 1\,(1) - 2\,(1)$$

$$= -1 \neq 0$$

Hence, S is a basis for $\mathbb{R}^3$. $\square$

Example 2 Let $S = \{(1,2,3),\ (0,1,2),\ (-1,0,1)\}$ with $V = \mathbb{R}^3$.

Solution Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

$$|A| = 1\,(1) - 1\,(1) \ = 0$$

Hence, S is not a basis for $\mathbb{R}^3$. $\square$

Example 3 Let $S = \{(1,2,3), (0,1,2)\}$ with $V = \mathbb{R}^3$.

Solution If we consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$

we see that we cannot check for nonsingularity as the matrix is not square.

It is easy to see that the set S is linearly independent.

(Solve $A\,\mathbf{c} = \mathbf{0}$ and show that the solution is $\mathbf{c} = \mathbf{0}$.)

So, we suspect that $Span(S) \neq \mathbb{R}^3$.

We solve $A\,\mathbf{c} = \mathbf{b}$ where $\mathbf{b}$ is a random vector in $\mathbb{R}^3$.

We consider the augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 2 & 1 & b_2 \\ 3 & 2 & b_3 \end{array}\right]$

The row echelon form is: $\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & * \\ 0 & 0 & * \end{array}\right]$

The system is inconsistent. $\text{Span}(S) \neq \mathbb{R}^3$.

The set S is not a basis for $\mathbb{R}^3$.

Example 3 Let $S = \{(1,2,3),\ (0,1,2), (1,0,0), (-1,-1,-1)\}$ with $V = \mathbb{R}^3$.

Solution If we consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 0 & -1 \end{bmatrix}$

we see that we cannot check for nonsingularity as the matrix is not square.

Let us check whether the set S is linearly independent.

We solve $A\,\mathbf{c} = \mathbf{0}$.

The augmented matrix is: $\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 2 & 1 & 0 & -1 & 0 \\ 3 & 2 & 0 & -1 & 0 \end{array}\right]$

The reduced row echelon form is: $\left[\begin{array}{cccr|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$

As there is a free variable, the set S is linearly dependent.

The set S is not a basis for $\mathbb{R}^3$.

Standard basis for P_n

$$P_n = \{a_n\, x^n + a_{n-1}\, x^{n-1} + \cdots + a_1\, x + a_0 \mid a_0,\, a_1, \cdots, a_n \in \mathbb{R}\}.$$

We set up a one-one correspondence between P_n and $\mathbb{R}^{n+1}$.

$$\begin{array}{ccc} x^n & \longleftrightarrow & \mathbf{e}_1 \\ x^{n-1} & \longleftrightarrow & \mathbf{e}_2 \\ & \vdots & \\ x^1 & \longleftrightarrow & \mathbf{e}_n \\ 1 & \longleftrightarrow & \mathbf{e}_{n+1} \end{array}$$

So, $a_n\, x^n + a_{n-1}\, x^{n-1} + \cdots + a_1\, x + a_0 \quad \longleftrightarrow \quad (a_n, a_{n-1}, \cdots, a_0)$

The above one-to-one correspondence shows how to obtain standard basis vectors in P_n from standard basis vectors for $\mathbb{R}^{n+1}$.

Standard basis for $M_{m,n}$ – the set of $m \times n$ matrices

We set up a one-one correspondence between $M_{m,n}$ and $\mathbb{R}^{mn}$.

Let $E(i,j)$ denote the matrix with 1 in the (i,j) location, and zeros elsewhere.

$$E(i,j) \longleftrightarrow \mathbf{e}_{i+(j-1)m}$$

$$\begin{aligned}
\text{So, } E(1,1) &\longleftrightarrow \mathbf{e}_1 \\
E(2,1) &\longleftrightarrow \mathbf{e}_2 \\
&\vdots \\
E(m,1) &\longleftrightarrow \mathbf{e}_m \\
E(1,2) &\longleftrightarrow \mathbf{e}_{m+1} \\
&\vdots \\
E(m,n) &\longleftrightarrow \mathbf{e}_{mn}
\end{aligned}$$

The above one-to-one correspondence shows how to obtain standard basis vectors in $M_{m,n}$ from standard basis vectors for $\mathbb{R}^{mn}$.

Properties of a basis

- Unique representation of a vector in the vector space using basis vectors.

Example $\mathbf{v} = (1, 2, -1)$ in $\mathbb{R}^3$ is written as $\mathbf{v} = 1\,\mathbf{e}_1 + 2\,\mathbf{e}_2 - 1\,\mathbf{e}_3$.

This is the only possible representation using the standard basis.

Example With the basis $S = \{(1,0,0), (1,1,0), (1,1,1)\}$

we consider $c_1\,(1,0,0) + c_2\,(1,1,0) + c_3\,(1,1,1) = (1,2,-1)$.

The system is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

The only solution is $c_3 = -1$, $c_2 = 3$, and $c_1 = -1$.

So, $\mathbf{v} = -1\,(1,0,0) + 3\,(1,1,0) - 1\,(1,1,1)$.

Easy proof Let $S = \{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ be a basis for a vector space V. Let $\mathbf{v} \in V$.

Suppose $\mathbf{v} = c_1\,\mathbf{u}_1 + \cdots + c_k\,\mathbf{u}_k$, and $\mathbf{v} = d_1\,\mathbf{u}_1 + \cdots + d_k\,\mathbf{u}_k$.

Hence $(c_1 - d_1)\,\mathbf{u}_1 + \cdots + (c_k - d_k)\,\mathbf{u}_k = \mathbf{0}$.

As S is a basis, $c_1 = d_1$ etc.

- If a basis has n vectors, then any collection of more than n vectors is not a basis.

Example In $\mathbb{R}^3$, the standard basis has 3 vectors.

Let $S = \{(1,2,3),\ (0,1,2), (1,0,0), (-1,-1,-1)\}$.

From 2 slides ago, the set S is linearly dependent. The set S is not a basis for $\mathbb{R}^3$.

Easy proof Let $S = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis for a vector space V.

Consider the collection $T = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ where $m > n$.

As S is a basis,
$$\begin{aligned}\mathbf{v}_1 &= c_{11}\,\mathbf{u}_1 + \cdots + c_{n1}\,\mathbf{u}_n.\\ \mathbf{v}_2 &= c_{12}\,\mathbf{u}_1 + \cdots + c_{n2}\,\mathbf{u}_n.\\ &\vdots\\ \mathbf{v}_m &= c_{1m}\,\mathbf{u}_1 + \cdots + c_{nm}\,\mathbf{u}_n.\end{aligned}$$

Consider the equation $k_1\,\mathbf{v}_1 + \cdots + k_m\,\mathbf{v}_m = \mathbf{0}$

This yields the system: $$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nm} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

As $m > n$, there are free variables, and $\mathbf{k} = \mathbf{0}$ is not the only solution.

The set S is linearly dependent and not a basis for V.

- If any one basis has n vectors, then any other basis also has n vectors.

Suppose S and T are two basis sets for a vector space V.

Suppose S has n vectors and T has m vectors.

Case I: $m > n$

T cannot be a basis as it has more than n vectors, and S is a basis.

Case II: $m < n$

S cannot be a basis as it has more than m vectors, and T is a basis.

Case III: $m = n$

Both basis have exactly the same number of vectors.

Dimension of a vector space

It is the number of vectors in any basis set.

Special case: If V is only the zero vector, then the dimension is 0.

Let V be a vector space of dimension n. We assume it can be represented using $\mathbb{R}^n$.

Suppose $S = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a linearly independent set of vectors.

If $\text{Span}(S) = V$, then S is a basis for V.

Let $\mathbf{v} \in V$ and consider the equation $c_1\,\mathbf{u}_1 + \cdots + c_n\,\mathbf{u}_n = \mathbf{v}$.

Let $A = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_n]$ be an $n \times n$ matrix.

The above system is equivalent to $A\,\mathbf{c} = \mathbf{v}$.

From 3.3, we know that a unique solution exists if A is non-singular.

From 3.3, the condition of nonsingularity is equivalent to $A\,\mathbf{c} = \mathbf{0}$ has only the zero solution.

$A\,\mathbf{c} = \mathbf{0}$ has only the zero solution if S is linearly independent.

S is linearly independent set $\implies$ S is a basis for V.

Let V be a vector space of dimension n. We assume it can be represented using $\mathbb{R}^n$.

Suppose $S = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ spans V.

For any $\mathbf{v} \in V$, the equation $c_1\,\mathbf{u}_1 + \cdots + c_n\,\mathbf{u}_n = \mathbf{v}$ has a solution.

To show S is linearly independent, consider the equation $c_1\,\mathbf{u}_1 + \cdots + c_n\,\mathbf{u}_n = \mathbf{0}$.

Let $A = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_n]$ be an $n \times n$ matrix.

The above system is equivalent to $A\,\mathbf{c} = \mathbf{0}$.

From 3.3, if there is a non-zero solution $\mathbf{c}$, then A must be singular.

From 3.3, A is singular if A is not row equivalent to identity.

If A is not row equivalent to identity, the reduced row echelon form of A has a row of zeros.

Hence $A\,\mathbf{c} = \mathbf{v}$ is inconsistent for some $\mathbf{v}$ which is a contradiction.

S is spanning set $\implies$ S is a basis for V.