HYPERGEOMETRIC TYPE IDENTITIES IN THE p-ADIC SETTING AND MODULAR FORMS

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ABSTRACT. We prove hypergeometric type identities for a function defined in terms of quotients of the p-adic gamma function. We use these identities to prove a supercongruence conjecture of Rodriguez-Villegas between a truncated $_4F_3$ hypergeometric series and the Fourier coefficients of a certain weight four modular form.

1. INTRODUCTION

Hypergeometric functions over finite fields were first introduced by Greene [11]. Functions of this type were also defined by Katz [13], about the same time, and more recently by the second author [23]. These functions are analogues of classical hypergeometric functions and were first developed to simplify character sum evaluations. Since then they have been applied to a number of areas of mathematics but the two areas of most interest to the authors are their transformation properties [11, 23], which often mirror classical hypergeometric transformations, and their connections to modular forms [1, 6, 7, 8, 9, 10, 17, 19, 20, 22, 25].

The arguments in finite field hypergeometric functions are multiplicative characters, and, consequently, many of the results involving these functions are restricted to primes in certain congruence classes (see for example [6, 7, 8, 17, 18, 19, 20, 28]) to facilitate the existence of characters of specific orders.

To overcome these restrictions, the second author defined the following function [21, 24] which extends finite field hypergeometric functions to the p-adic setting, and can often allow results involving finite field hypergeometric functions to be extended to a wider class of primes [21, 22, 24]. Let $\mathbb{F}_p$ denote the finite field with p elements, where p is prime. Let $\Gamma_p(\cdot)$ denote Morita's p-adic gamma function and let ω denote the Teichmüller character of $\mathbb{F}_p$ with $\overline{\omega}$ denoting its character inverse. For $x \in \mathbb{Q}$ we let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ denote the fractional part of x, i.e. $x - \lfloor x \rfloor$.

Definition 1.1. [24, Definition 1.1] Let p be an odd prime and let $s \in \mathbb{F}_p$. For $n \in \mathbb{Z}^+$ and $1 \leq i \leq n$, let $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Then we define

$$
{}_nG_n\left[\begin{array}{cccc} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \end{array} \Big| \, s \right]_p := \frac{-1}{p-1} \sum_{j=0}^{p-2} (-1)^{jn} \, \overline{\omega}^j(s)
$$

$$
\times \prod_{i=1}^{n} \frac{\Gamma_p\big(\langle a_i - \frac{j}{p-1} \rangle\big)}{\Gamma_p\big(\langle a_i \rangle\big)} \frac{\Gamma_p\big(\langle -b_i + \frac{j}{p-1} \rangle\big)}{\Gamma_p\big(\langle -b_i \rangle\big)} (-p)^{-\lfloor \langle a_i \rangle - \frac{j}{p-1} \rfloor - \lfloor \langle -b_i \rangle + \frac{j}{p-1} \rfloor}.
$$

Throughout the paper we will refer to this function as $_nG_n[\cdots]$ and we will omit the commas between the parameters for convenience. We note that the value of $_nG_n[\cdots]$ depends only on the fractional part of the a and b parameters, and is invariant if we change the order of the parameters.

Results involving finite field hypergeometric functions can readily be converted to expressions involving $_nG_n[\cdots]$ using Lemma 4.2, which we will see in Section 4. However these new expressions in $_nG_n[\cdots]$ will still only be valid for primes p where the original characters existed over $\mathbb{F}_p$, i.e., for primes in certain congruence classes. It is a non-trivial exercise to then extend these results to almost all primes.

The purpose of this paper is threefold. Firstly, we establish certain transformations and identities for $_nG_n[\cdots]$ in full generality, i.e., for almost all primes and beyond expressions which can be implied directly by any existing transformations for finite field hypergeometric functions. Secondly, we use these transformations to prove a linear relation between a certain $_4G_4[\cdots]_p$ and the p-th Fourier coefficients of a weight 4 newform on $\Gamma_0(16)$. As a result, we prove a supercongruence conjecture of Rodriguez-Villegas between a truncated $_4F_3$ classical hypergeometric series and the p-th Fourier coefficients of the same weight four modular form, modulo p^3.

The rest of this paper is organized as follows. In Section 2 we expand on the discussion above and state our results. In Section 3 we recall some basic properties of multiplicative characters, Gauss sums and the p-adic gamma function. We discuss the role of finite field hypergeometric functions in proving supercongruences, and their relationship to $_nG_n[\cdots]$, in Section 4. The proofs of our main results are contained in Section 5. Finally, we make some closing remarks in Section 6.

2. Statement of Results

We first define the truncated generalized hypergeometric series. For a complex number a and a non-negative integer n let $(a)_n$ denote the rising factorial defined by

$$(a)_0 := 1 \quad \text{and} \quad (a)_n := a(a+1)(a+2)\cdots(a+n-1) \text{ for } n > 0.$$

Then, for complex numbers a_i, b_j and z, with none of the b_j being negative integers or zero, we define the truncated generalized hypergeometric series

$$_rF_s \left[\begin{array}{ccccc} a_1, & a_2, & a_3, & \ldots, & a_r \\ & b_1, & b_2, & \ldots, & b_s \end{array} \, \middle| \, z \right]_m := \sum_{n=0}^{m} \frac{(a_1)_n (a_2)_n (a_3)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n} \, \frac{z^n}{n!}.$$

When we drop the subscript m we will use the same notation to mean the (un-truncated) generalized hypergeometric series, which is an infinite series. When the arguments are obvious we will omit the commas between them for convenience.

In [26] Rodriguez-Villegas examined the relationship between the number of points over $\mathbb{F}_p$ on certain Calabi-Yau manifolds and truncated generalized hypergeometric series which correspond to a particular period of the manifold. In doing so, he numerically identified 14 possible supercongruences between truncated $_4F_3$ hypergeometric series and Fourier coefficients of certain weight four modular forms, modulo p^3. To date, two of these conjectures have been proven [14, 22].

The first one of these supercongruence conjectures proven, by Kilbourn [14], can be described as follows. Let $q := e^{2\pi i z}$ and let

$$f_1(z) := q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \sum_{n=1}^{\infty} a(n) q^n$$

be the unique newform in $S_4(\Gamma_0(8))$, the space of weight four cusp forms on the congruence subgroup $\Gamma_0(8)$. Then, for all odd primes p,

$$_4F_3 \left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 \end{array} \Bigg| 1 \right]_{p-1} \equiv a(p) \pmod{p^3}. \tag{2.1}$$

Kilbourn's proof of (2.1) can be interpreted in terms of finite field hypergeometric functions. However, by definition, results involving finite field hypergeometric functions will often be restricted to primes in certain congruence classes (see for example [6, 7, 8, 17, 18, 19, 20, 28]). Consequently, based on the parameters involved, it appears that these functions cannot be used to fully establish the other 13 supercongruence conjectures of Rodriguez-Villegas. (Please see Section 4 for a detailed discussion of finite field hypergeometric functions and their restrictions.) To overcome these restrictions, the second author defined $_nG_n[\cdots]$ (see Definition 1.1), which extends finite field hypergeometric functions to the p-adic setting, and can often allow results involving finite field hypergeometric functions to be extended to a wider class of primes [21, 22, 24].

Families of congruences between $_nG_n[\cdots]$ and truncated generalized hypergeometric series have been established in [21] and these congruences cover all 14 series listed in Rodriguez-Villegas' conjectures. Therefore, to establish the remaining conjectures, it suffices to link certain $_nG_n[\cdots]$ to the Fourier coefficients of the relevant modular forms. Using this method, a second of Rodriguez-Villegas' supercongruence conjectures was proved by the second author in [22]. (Please see Section 4 and (4.4) for further details.) One such family of congruences is as follows.

Theorem 2.1. [21, Theorem 2.6] *Let $d_1, d_2 \geq 2$ be integers and let p be an odd prime such that $p \equiv \pm 1 \pmod{d_1}$ and $p \equiv \pm 1 \pmod{d_2}$. If $s(p) := \Gamma_p\left(\frac{1}{d_1}\right)\Gamma_p\left(1 - \frac{1}{d_1}\right)\Gamma_p\left(\frac{1}{d_2}\right)\Gamma_p\left(1 - \frac{1}{d_2}\right) = (-1)^{\left\lfloor \frac{p-1}{d_1} \right\rfloor + \left\lfloor \frac{p-1}{d_2} \right\rfloor}$, then*

$$_4G_4 \left[\begin{array}{cccc} \frac{1}{d_1} & 1 - \frac{1}{d_1} & \frac{1}{d_2} & 1 - \frac{1}{d_2} \\ 1 & 1 & 1 & 1 \end{array} \Bigg| 1 \right]_p$$

$$\equiv {}_4F_3 \left[\begin{array}{cccc} \frac{1}{d_1} & 1 - \frac{1}{d_1} & \frac{1}{d_2} & 1 - \frac{1}{d_2} \\ & 1 & 1 & 1 \end{array} \Bigg| 1 \right]_{p-1} + s(p) p \pmod{p^3}.$$

The purpose of this paper is to establish certain identities for $_nG_n[\cdots]$ which then can be used to establish another one of Rodriguez-Villegas' supercongruence conjectures, as follows.

Theorem 2.2. *Let $f_2 \in S_4(\Gamma_0(16))$ be the unique weight 4 newform of level 16, with Fourier expansion $f_2(z) := \sum_{n=1}^{\infty} c(n)q^n$, where $q := e^{2\pi i z}$. Then, for all odd primes p,*

$$
{}_4F_3\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ & 1 & 1 & 1 \end{array}\,\middle|\,1\right]_{p-1} \equiv c(p) \pmod{p^3}.
$$

This congruence was established by Lennon [17] in the case $p \equiv 1 \pmod 4$, using finite field hypergeometric functions. If we appeal to Theorem 2.1 with $d_1 = 2$, $d_2 = 4$, then, to establish Theorem 2.2, it suffices to prove

Theorem 2.3. *Let p be an odd prime and define $s(p) = \Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)\Gamma_p\left(\frac{1}{2}\right)^2$. Then*

$$
{}_4G_4\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1 & 1 \end{array}\,\middle|\,1\right]_p - s(p) \cdot p = c(p),
$$

where $c(p)$ is as defined in Theorem 2.2.

Like Lennon, we will rely on the fact that f_2 is quadratic twist of f_1 with $c(p) = \phi_p(-1)a(p)$, where $\phi_p(\cdot)$ is the Legendre symbol modulo p. As we will see in Section 5, combining aspects of Kilbourn's proof of (2.1) (specifically (4.2)) with properties of ${}_nG_n[\cdots]$ (Lemma 4.2) we know that, for all odd primes p,

$$
{}_4G_4\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{array}\,\middle|\,1\right]_p - p = a(p).
$$

Consequently, the proof of Theorem 2.3, and hence Theorem 2.2, can be reduced to establishing the following identity.

Theorem 2.4. *Let p be an odd prime and define $s(p) := \Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)\Gamma_p\left(\frac{1}{2}\right)^2 = (-1)^{\lfloor\frac{p-1}{4}\rfloor + \lfloor\frac{p-1}{2}\rfloor}$. Then*

$$
{}_4G_4\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1 & 1 \end{array}\,\middle|\,1\right]_p - s(p) \cdot p = \omega^{\frac{p-1}{2}}(-1)\left[{}_4G_4\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{array}\,\middle|\,1\right]_p - p\right],
$$

where ω is the Teichmüller character.

We will need the following transformations to establish Theorem 2.4. Let $\delta(x)$ equal 1 if $x = 0$, and equal zero otherwise.

Theorem 2.5. *Let p be an odd prime and define $s(p) := \Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)\Gamma_p\left(\frac{1}{2}\right)^2 = (-1)^{\lfloor\frac{p-1}{4}\rfloor + \lfloor\frac{p-1}{2}\rfloor}$. For $1 \neq x \in \mathbb{F}_p^*$,*

$$
{}_3G_3\left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{array}\,\middle|\,x^{-1}\right]_p
$$
$$
= \delta(1+x) \cdot p \cdot \omega^{\frac{p-1}{2}}(-1) + s(p) \cdot \omega^{\frac{p-1}{2}}(2(1-x)) \cdot {}_3G_3\left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1 \end{array}\,\middle|\,-\frac{(1-x)^2}{4x}\right]_p,
$$

where ω is the Teichmüller character.

Theorem 2.6. *Let p be an odd prime and let $s \in \mathbb{F}_p$. For $a_i \in \mathbb{Q} \cap \mathbb{Z}_p$, $1 \leq i \leq n$,*

$$_{n+1}G_{n+1}\left[\begin{array}{ccccc} a_1 & a_2 & \cdots & a_n & \frac{1}{2} \\ 1 & 1 & \cdots & 1 & 1 \end{array}\Bigg| s\right]_p = -\sum_{t=2}^{p-1} {}_nG_n\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ 1 & 1 & \cdots & 1 \end{array}\Bigg| st\right]_p \cdot \omega^{\frac{p-1}{2}}(1-t),$$

where ω is the Teichmüller character.

Theorem 2.7. *Let $2 \leq n \in \mathbb{Z}$. For p an odd prime and $s \in \mathbb{F}_p^*$,*

$$_nG_n\left[\begin{array}{cccccc} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{array}\Bigg| s\right]_p = p \cdot \omega^{\frac{p-1}{2}}((-1)^{n+1}s) \cdot {}_nG_n\left[\begin{array}{cccccc} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & \cdots & 1 & \frac{1}{4} & \frac{3}{4} \end{array}\Bigg| s^{-1}\right]_p,$$

where ω is the Teichmüller character.

Theorem 2.5 is an analogue of the classical hypergeometric quadratic transformation due to Whipple [2, (3.1.15) page 130]

$$_3F_2\left[\begin{array}{ccc} a & b & c \\ & 1+a-b & 1+a-c \end{array}\Bigg| z\right]$$

$$= (1-z)^{-a} \, {}_3F_2\left[\begin{array}{ccc} \frac{a}{2} & \frac{a+1}{2} & 1+a-b-c \\ & 1+a-b & 1+a-c \end{array}\Bigg| -\frac{4z}{(1-z)^2}\right],$$

when $a = b = c = \frac{1}{2}$.

Many transformations exist for finite field hypergeometric functions which are analogues of classical results [11, 23]. These transformations can readily be converted to expressions involving $_nG_n[\cdots]$ using Lemma 4.2. However, the resulting transformations in $_nG_n[\cdots]$ will only be valid for primes p where the original characters existed over $\mathbb{F}_p$, i.e., for primes in certain congruence classes. It is a non-trivial exercise to then extend these results to almost all primes. There are very few identities and transformations for $_nG_n[\cdots]$ in full generality. Barman and Saikia [4, 5] provide transformations for $_2G_2[\cdots]$, by counting points on various families of elliptic curves. To the extent of our knowledge, Theorems 2.4 to 2.7 are the only other full $_nG_n[\cdots]$ identities.

Theorems 2.5 to 2.7 are extended versions of special cases of the finite field hypergeometric function results described in [11], in Corollary 4.30, Theorem 3.13 and Theorem 4.2(b) respectively.

3. Preliminaries

Let $\mathbb{Z}_p$ denote the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}_p}$ the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}_p}$.

3.1. Multiplicative Characters and Gauss Sums.
Let $\widehat{\mathbb{F}_p^*}$ denote the group of multiplicative characters of $\mathbb{F}_p^*$. We extend the domain of $\chi \in \widehat{\mathbb{F}_p^*}$ to $\mathbb{F}_p$, by defining $\chi(0) := 0$ (including the trivial character ε) and denote $\overline{\chi}$ as the inverse of χ. When p is odd we denote the character of order two of $\mathbb{F}_p^*$ by ϕ_p. We will often drop the subscript p if it is clear from the context.

We now introduce some properties of Gauss sums. For further details see [3], noting that we have adjusted results to take into account $\varepsilon(0) = 0$. Let ζ_p be a fixed primitive

p-th root of unity in $\overline{\mathbb{Q}_p}$. We define the additive character $\theta : \mathbb{F}_p \to \mathbb{Q}_p(\zeta_p)$ by $\theta(x) := \zeta_p^x$. We note that $\mathbb{Q}_p$ contains all $(p-1)$-st roots of unity and in fact they are all in $\mathbb{Z}_p^*$. Thus we can consider multiplicative characters of $\mathbb{F}_p^*$ to be maps $\chi : \mathbb{F}_p^* \to \mathbb{Z}_p^*$. Recall then that for $\chi \in \widehat{\mathbb{F}_p^*}$, the Gauss sum $g(\chi)$ is defined by $g(\chi) := \sum_{x \in \mathbb{F}_p} \chi(x)\theta(x)$.

The following useful result gives a simple expression for the product of two Gauss sums. For $\chi \in \widehat{\mathbb{F}_p^*}$ we have

$$g(\chi)g(\overline{\chi}) = \begin{cases} \chi(-1)p & \text{if } \chi \neq \varepsilon, \\ 1 & \text{if } \chi = \varepsilon. \end{cases} \tag{3.1}$$

Another important product formula for Gauss sums is the Hasse-Davenport formula.

Theorem 3.1 (Hasse, Davenport [3] Thm 11.3.5). *Let ψ be a character of order m of $\mathbb{F}_p^*$ for some positive integer m. For a character χ of $\mathbb{F}_p^*$ we have*

$$\prod_{i=0}^{m-1} g(\psi^i \chi) = g(\chi^m)\chi^{-m}(m) \prod_{i=1}^{m-1} g(\psi^i).$$

Recall also, that for $\chi, \psi \in \widehat{\mathbb{F}_p^*}$, we define the Jacobi sum by $J(\chi, \psi) := \sum_{t \in \mathbb{F}_p} \chi(t)\psi(1-t)$. We can relate Jacobi sums to Gauss sums, as follows. For $\chi, \psi \in \widehat{\mathbb{F}_q^*}$ not both trivial,

$$J(\chi, \psi) = \begin{cases} \dfrac{g(\chi)g(\psi)}{g(\chi\psi)} & \text{if } \chi\psi \neq \varepsilon, \\[4mm] -\dfrac{g(\chi)g(\psi)}{p} & \text{if } \chi\psi = \varepsilon. \end{cases} \tag{3.2}$$

3.2. p-adic Preliminaries. We define the Teichmüller character to be the primitive character $\omega : \mathbb{F}_p \to \mathbb{Z}_p^*$ satisfying $\omega(x) \equiv x \pmod{p}$ for all $x \in \{0, 1, \ldots, p-1\}$. We now recall the p-adic gamma function. For further details, see [15]. Let p be an odd prime. For $n \in \mathbb{Z}^+$ we define the p-adic gamma function as

$$\Gamma_p(n) := (-1)^n \prod_{\substack{0 < j < n \\ p \nmid j}} j$$

and extend it to all $x \in \mathbb{Z}_p$ by setting $\Gamma_p(0) := 1$ and

$$\Gamma_p(x) := \lim_{n \to x} \Gamma_p(n)$$

for $x \neq 0$, where n runs through any sequence of positive integers p-adically approaching x. This limit exists, is independent of how n approaches x, and determines a continuous function on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p^*$. We now state a product formula for the p-adic gamma function. If $m \in \mathbb{Z}^+$, $p \nmid m$ and $x = \frac{r}{p-1}$ with $0 \leq r \leq p-1$, then

$$\prod_{h=0}^{m-1} \Gamma_p\left(\tfrac{x+h}{m}\right) = \omega\left(m^{(1-x)(1-p)}\right) \Gamma_p(x) \prod_{h=1}^{m-1} \Gamma_p\left(\tfrac{h}{m}\right). \tag{3.3}$$

We note also that

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{x_0}, \tag{3.4}$$

where $x_0 \in \{1, 2, \ldots, p\}$ satisfies $x_0 \equiv x \pmod{p}$.

The Gross-Koblitz formula [12] allows us to relate Gauss sums and the p-adic gamma function. Let $\pi \in \mathbb{C}_p$ be the fixed root of $x^{p-1} + p = 0$ that satisfies $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. Then we have the following result.

Theorem 3.2 (Gross, Koblitz [12]). *For $j \in \mathbb{Z}$,*

$$g(\overline{\omega}^j) = -\pi^{(p-1)\langle \frac{j}{p-1} \rangle} \, \Gamma_p\Big(\langle \tfrac{j}{p-1} \rangle\Big).$$

We will need the following lemmas in the proofs of Theorems 2.4 and 2.6 respectively.

Lemma 3.3. *For p an odd prime and $j \in \mathbb{Z}$, with $0 < j < p - 1$,*

$$\frac{\Gamma_p\big(\langle \frac{1}{2} - \frac{j}{p-1} \rangle\big)\Gamma_p\big(\frac{j}{p-1}\big)}{\Gamma_p\big(\frac{1}{2}\big)}(-p)^{-\lfloor \frac{1}{2} - \frac{j}{p-1} \rfloor} = -\sum_{t=2}^{p-1} \overline{\omega}^j\left(\frac{4(1-t)}{t^2}\right).$$

Proof. Let $\chi \in \widehat{\mathbb{F}_p^*}$. By the Hasse-Davenport formula (Theorem 3.1) with $m = 2$, and hence $\psi = \phi$, we see that

$$g(\phi\chi)\, g(\chi) = g(\chi^2)\, g(\phi)\, \overline{\chi}(4). \tag{3.5}$$

Applying (3.5) and (3.2), we find that, for $\chi \neq \varepsilon$,

$$\frac{g(\phi\chi)g(\overline{\chi})}{g(\phi)} = \frac{g(\chi^2)g(\overline{\chi})}{g(\chi)}\overline{\chi}(4) = J(\chi^2, \overline{\chi})\,\overline{\chi}(4).$$

Now consider the Jacobi sum

$$J(\chi^2, \overline{\chi}) = \sum_{t=2}^{p-1} \chi^2(t)\,\overline{\chi}(1 - t) = \sum_{t=2}^{p-1} \chi\left(\frac{t^2}{1-t}\right).$$

Therefore, for $\chi \neq \varepsilon$,

$$\frac{g(\phi\chi)g(\overline{\chi})}{g(\phi)} = \sum_{t=2}^{p-1} \chi\left(\frac{t^2}{4(1-t)}\right).$$

Letting $\chi = \omega^j$, and using the Gross-Koblitz formula (Theorem 3.2) to convert the left-hand side to an expression in p-adic gamma functions, completes the proof. $\square$

Lemma 3.4. *For p an odd prime and $j \in \mathbb{Z}$, with $0 < j < p - 1$,*

$$\frac{\Gamma_p\big(\langle \frac{1}{2} - \frac{j}{p-1} \rangle\big)\Gamma_p\big(\frac{j}{p-1}\big)}{\Gamma_p\big(\frac{1}{2}\big)}(-p)^{-\lfloor \frac{1}{2} - \frac{j}{p-1} \rfloor} = -\sum_{t=2}^{p-1} \omega^j(-t)\,\omega^{\frac{p-1}{2}}(t(t-1)).$$

Proof. Let $\chi \in \widehat{\mathbb{F}_p^*}$. Using (3.1) and (3.2), we find that for $\chi \neq \varepsilon$,

$$\frac{g(\phi\chi)g(\overline{\chi})}{g(\phi)} = \frac{g(\phi\chi)g(\phi)\,\chi(-1)}{g(\chi)\,\phi(-1)} = \phi\chi(-1)J(\phi\chi, \phi) = \sum_{t=2}^{p-1} \chi(-t)\,\phi(t(t-1)).$$

Again using (3.1) and (3.2) we see that

$$\phi(-1)J(\phi, \phi) = -\frac{\phi(-1)g(\phi)g(\phi)}{p} = -1 = \frac{g(\phi)g(\varepsilon)}{g(\phi)}.$$

Therefore, for all $\chi \in \widehat{\mathbb{F}_p^*}$,

$$\frac{g(\phi\chi)g(\overline{\chi})}{g(\phi)} = \phi\chi(-1)J(\phi\chi, \phi) = \sum_{t=2}^{p-1} \chi(-t)\,\phi(t(t-1)).$$

Letting $\chi = \omega^j$, and using the Gross-Koblitz formula (Theorem 3.2) to convert the left-hand side to an expression in p-adic gamma functions, completes the proof. $\square$

4. Finite Field Hypergeometric Functions

Hypergeometric functions over finite fields were originally defined by Greene [11], who first established these functions as analogues of classical hypergeometric functions. Functions of this type were also introduced by Katz [13] about the same time. In the present article we use a normalized version of these functions defined by the second author in [23], which is more suitable for our purposes. The reader is directed to [23, §2] for the precise connections among these three classes of functions.

Definition 4.1. [23, Definition 1.4] For $A_0, A_1, \ldots, A_n, B_1, \ldots, B_n \in \widehat{\mathbb{F}_p^*}$ and $x \in \mathbb{F}_p$ define

$$_{n+1}F_n\left(\begin{array}{cccc} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{array} \Big| x \right)_p$$

$$:= \frac{1}{p-1} \sum_{\chi \in \widehat{\mathbb{F}_p^*}} \prod_{i=0}^{n} \frac{g(A_i\chi)}{g(A_i)} \prod_{j=1}^{n} \frac{g(\overline{B_j\chi})}{g(\overline{B_j})} g(\overline{\chi})\chi(-1)^{n+1}\chi(x).$$

Many of the results concerning hypergeometric functions over finite fields that we quote from other articles were originally stated using Greene's function. If this is the case, note then that we have reformulated them in terms of $_{n+1}F_n(\cdots)$ as defined above.

Kilbourn [14] proved that, if p is an odd prime, then

$$_4F_3\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 \end{array} \Big| 1 \right]_{p-1} \equiv {_4F_3}\left(\begin{array}{cccc} \phi & \phi & \phi & \phi \\ & \varepsilon & \varepsilon & \varepsilon \end{array} \Big| 1 \right)_p - p \pmod{p^3}. \qquad (4.1)$$

To establish (2.1), he combined (4.1) with the following result of Ahlgren and Ono [1] :

$$_4F_3\left(\begin{array}{cccc} \phi & \phi & \phi & \phi \\ & \varepsilon & \varepsilon & \varepsilon \end{array} \Big| 1 \right)_p - p = a(p). \qquad (4.2)$$

The main method of relating finite field hypergeometric functions and Fourier coefficients of modular forms has been via the Eichler-Selberg trace formula [1, 7, 8, 9, 10, 17, 19, 25]. However, apart from a number of special cases, including Ahlgren and Ono's relation above, most of these results are either restricted to primes in certain congruence classes to facilitate the existence of characters of certain orders, which appear as arguments in the finite field hypergeometric functions, or require much more complex relations than (4.2). This makes finite field hypergeometric functions unsuitable for establishing the remaining Rodriguez-Villegas supercongruence conjectures in full generality.

For example, let

$$g(z) := g_1(z) + 5g_2(z) + 20g_3(z) + 25g_4(z) + 25g_5(z) = \sum_{n=1}^{\infty} b(n)q^n$$

where $g_i(z) := \eta^{5-i}(z)\,\eta^4(5z)\,\eta^{i-1}(25z)$, $\eta(z) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function and $q := e^{2\pi i z}$. Then g is a newform of weight four on the congruence subgroup $\Gamma_0(25)$, and when $p \equiv 1 \pmod 5$, we see via [21, 22] that

$$_4F_3\left[\begin{array}{cccc} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ & 1 & 1 & 1 \end{array}\Big|\ 1\right]_{p-1} \overset{(\mathrm{mod}\ p^3)}{\equiv} {}_4F_3\left(\begin{array}{cccc} \psi & \psi^2 & \psi^3 & \psi^4 \\ & \varepsilon & \varepsilon & \varepsilon \end{array}\Big|\ 1\right)_p - p = b(p) \qquad (4.3)$$

where $\psi \in \widehat{\mathbb{F}_p^*}$ is a character of order 5. Because the finite field hypergeometric function above is based on characters of order 5, we know that the result cannot be extended beyond $p \equiv 1 \pmod 5$ using this function. Extending (4.3) to all primes, and proving another supercongruence conjecture of Rodriguez-Villegas, was the motivation of the second author to develop ${}_nG_n[\cdots]$. We have the following relationship between ${}_nG_n[\cdots]$ and ${}_{n+1}F_n(\cdots)$.

Lemma 4.2. [24, Lemma 3.3] *For a fixed odd prime p, let $A_i, B_k \in \widehat{\mathbb{F}_p^*}$ be given by $\overline{\omega}^{a_i(p-1)}$ and $\overline{\omega}^{b_k(p-1)}$ respectively, where ω is the Teichmüller character, and let $s \in \mathbb{F}_p$. Then*

$$_{n+1}F_n\left(\begin{array}{cccc} A_0 & A_1 & \cdots & A_n \\ & B_1 & \cdots & B_n \end{array}\Big|\ s\right)_p = {}_{n+1}G_{n+1}\left[\begin{array}{cccc} a_0 & a_1 & \cdots & a_n \\ 1 & b_1 & \cdots & b_n \end{array}\Big|\ s^{-1}\right]_p.$$

This relation opens the possibility of extending results involving finite field hypergeometric functions to all but finitely many primes by using ${}_nG_n[\cdots]$ instead. This has been possible in many cases [21, 22, 24], including for (4.3). When $p \neq 5$ and $h(p) := \Gamma_p(\frac{1}{5})\Gamma_p(\frac{2}{5})\Gamma_p(\frac{3}{5})\Gamma_p(\frac{4}{5})$, we see by [21, 22] that

$$_4F_3\left[\begin{array}{cccc} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ & 1 & 1 & 1 \end{array}\Big|\ 1\right]_{p-1} \overset{(\mathrm{mod}\ p^3)}{\equiv} {}_4G_4\left[\begin{array}{cccc} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ 1 & 1 & 1 & 1 \end{array}\Big|\ 1\right]_p - h(p)\cdot p = b(p), \qquad (4.4)$$

which was the second of Rodriguez-Villegas' conjectures to be proven.

5. Proofs

We begin by proving three transformations of ${}_nG_n[\cdots]_p$.

Proof of Theorem 2.5. Let p be an odd prime. Taking $A = B = C = \phi_p$ in [11, Theorem 4.28] and reformulating in terms of Gauss sums and ${}_{n+1}F_n(\cdots)$, as defined in Definition 4.1, we get that, for $x \neq 1$,

$$_3F_2\left(\begin{array}{ccc} \phi & \phi & \phi \\ & \varepsilon & \varepsilon \end{array}\Big|\ x\right)_p$$

$$= \delta(1+x)\cdot p\cdot\phi(-1) + \frac{\phi(1-x)}{p-1}\sum_{\chi\in\widehat{\mathbb{F}_p^*}} \frac{g(\phi\chi^2)g(\phi\chi)g(\overline{\chi})^3}{g(\phi)^2}\,\chi\left(\frac{x}{(1-x)^2}\right). \qquad (5.1)$$

By definition of the Teichmüller character we know that $\{\omega^j \mid 0 \leq j \leq p-2\} = \widehat{\mathbb{F}_p^*}$. Using Lemma 4.2 and the Gross-Koblitz formula, Theorem 3.2, to re-write equation (5.1) in the p-adic setting, we now have that

$$
{}_3G_3\left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{array} \middle| x^{-1}\right]_p = \delta(1+x) \cdot p \cdot \omega^{\frac{p-1}{2}}(-1) - \frac{\omega^{\frac{p-1}{2}}(1-x)}{p-1}
$$
$$
\times \sum_{j=0}^{p-2} (-p)^{-\lfloor \frac{1}{2} - \frac{2j}{p-1}\rfloor - \lfloor \frac{1}{2} - \frac{j}{p-1}\rfloor} \frac{\Gamma_p\left(\langle \frac{1}{2} - \frac{2j}{p-1}\rangle\right) \Gamma_p\left(\langle \frac{1}{2} - \frac{j}{p-1}\rangle\right) \Gamma_p\left(\frac{j}{p-1}\right)^3}{\Gamma_p\left(\frac{1}{2}\right)^2} \, \omega^j\left(\frac{x}{(1-x)^2}\right).
$$
(5.2)

Applying (3.3) with $x = \langle \frac{1}{2} - \frac{2j}{p-1}\rangle$ and $m = 2$ yields

$$
\Gamma_p\left(\left\langle \frac{1}{2} - \frac{2j}{p-1}\right\rangle\right) = \frac{\Gamma_p\left(\frac{1}{2}\langle\frac{1}{2} - \frac{2j}{p-1}\rangle\right) \Gamma_p\left(\frac{1}{2}\langle\frac{1}{2} - \frac{2j}{p-1}\rangle + \frac{1}{2}\right)}{\Gamma_p\left(\frac{1}{2}\right) \omega\left(2^{(1 - \langle \frac{1}{2} - \frac{2j}{p-1}\rangle)(1-p)}\right)}.
$$

By considering j in the intervals $[0, \lfloor \frac{p-1}{4}\rfloor]$, $(\lfloor \frac{p-1}{4}\rfloor, \lfloor \frac{3(p-1)}{4}\rfloor]$ and $(\lfloor \frac{3(p-1)}{4}\rfloor, p-2]$ it straightforward to verify that

$$
\Gamma_p\left(\frac{1}{2}\left\langle \frac{1}{2} - \frac{2j}{p-1}\right\rangle\right) \Gamma_p\left(\frac{1}{2}\left\langle \frac{1}{2} - \frac{2j}{p-1}\right\rangle + \frac{1}{2}\right) = \Gamma_p\left(\left\langle \frac{1}{4} - \frac{j}{p-1}\right\rangle\right) \Gamma_p\left(\left\langle \frac{3}{4} - \frac{j}{p-1}\right\rangle\right)
$$

and

$$
\omega\left(2^{(1 - \langle \frac{1}{2} - \frac{2j}{p-1}\rangle)(1-p)}\right) = \omega^{\frac{p-1}{2}}(2) \, \overline{\omega}^j(4).
$$

Therefore

$$
\Gamma_p\left(\left\langle \frac{1}{2} - \frac{2j}{p-1}\right\rangle\right) = \frac{\Gamma_p\left(\langle\frac{1}{4} - \frac{j}{p-1}\rangle\right) \Gamma_p\left(\langle\frac{3}{4} - \frac{j}{p-1}\rangle\right)}{\Gamma_p\left(\frac{1}{2}\right) \omega^{\frac{p-1}{2}}(2) \, \overline{\omega}^j(4)}.
$$
(5.3)

Note also that

$$
\left\lfloor \frac{1}{2} - \frac{2j}{p-1}\right\rfloor = \left\lfloor \frac{1}{4} - \frac{j}{p-1}\right\rfloor + \left\lfloor \frac{3}{4} - \frac{j}{p-1}\right\rfloor.
$$
(5.4)

Accounting for (5.3) and (5.4) in (5.2) yields

$$
{}_3G_3\left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{array} \middle| x^{-1}\right]_p = \delta(1+x) \cdot p \cdot \omega^{\frac{p-1}{2}}(-1) - \frac{\omega^{\frac{p-1}{2}}(2(1-x))}{p-1}
$$
$$
\times \sum_{j=0}^{p-2} \overline{\omega}^j\left(\frac{(1-x)^2}{4x}\right) \frac{\Gamma_p\left(\langle\frac{1}{2} - \frac{j}{p-1}\rangle\right) \Gamma_p\left(\langle\frac{1}{4} - \frac{j}{p-1}\rangle\right) \Gamma_p\left(\langle\frac{3}{4} - \frac{j}{p-1}\rangle\right) \Gamma_p\left(\frac{j}{p-1}\right)^3}{\Gamma_p\left(\frac{1}{2}\right)^3}
$$
$$
\cdot (-p)^{-\lfloor \frac{1}{2} - \frac{j}{p-1}\rfloor - \lfloor \frac{1}{4} - \frac{j}{p-1}\rfloor - \lfloor \frac{3}{4} - \frac{j}{p-1}\rfloor}.
$$

Noting that $\overline{\omega}^j(-1) = (-1)^j$ and $\Gamma_p\left(\frac{1}{2}\right)^4 = 1$, by (3.4), yields the result. $\qquad \square$

Proof of Theorem 2.6. Let p be an odd prime. By definition

$$
{}_{n+1}G_{n+1}\left[\begin{array}{ccccc|c} a_1 & a_2 & \dots & a_n & \frac{1}{2} \\ 1 & 1 & \dots & 1 & 1 \end{array}\;\middle|\; s\right]_p = \frac{-1}{p-1}\sum_{j=0}^{p-2}(-1)^{j(n+1)}\,\overline{\omega}^j(s)
$$

$$
\times\prod_{i=1}^{n}\frac{\Gamma_p\big(\langle a_i - \frac{j}{p-1}\rangle\big)\,\Gamma_p\big(\frac{j}{p-1}\big)}{\Gamma_p(\langle a_i\rangle)}(-p)^{-\lfloor\langle a_i\rangle - \frac{j}{p-1}\rfloor}
$$

$$
\times\frac{\Gamma_p\big(\langle\frac{1}{2} - \frac{j}{p-1}\rangle\big)\Gamma_p\big(\frac{j}{p-1}\big)}{\Gamma_p\big(\frac{1}{2}\big)}(-p)^{-\lfloor\frac{1}{2} - \frac{j}{p-1}\rfloor}.
$$

Applying Lemma 3.4 to the expression on the last line above and tidying up yields

$$
{}_{n+1}G_{n+1}\left[\begin{array}{ccccc|c} a_1 & a_2 & \dots & a_n & \frac{1}{2} \\ 1 & 1 & \dots & 1 & 1 \end{array}\;\middle|\; s\right]_p
$$

$$
= -\sum_{t=2}^{p-1} {}_nG_n\left[\begin{array}{cccc|c} a_1 & a_2 & \dots & a_n \\ 1 & 1 & \dots & 1 \end{array}\;\middle|\; st^{-1}\right]_p \cdot \omega^{\frac{p-1}{2}}(t(t-1))
$$

Letting $t \to t^{-1}$ and noting that $\omega^{\frac{p-1}{2}}(t^2) = 1$ completes the proof. $\qquad\square$

Proof of Theorem 2.7. Let p be an odd prime and let $2 \le n \in \mathbb{Z}$. By definition

$$
{}_nG_n\left[\begin{array}{cccccc|c} \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & \dots & 1 & 1 & 1 \end{array}\;\middle|\; s\right]_p
$$

$$
= \frac{-1}{p-1}\sum_{j=0}^{p-2}(-1)^{jn}\,\overline{\omega}^j(s)\frac{\Gamma_p\big(\langle\frac{1}{2} - \frac{j}{p-1}\rangle\big)^{n-2}\Gamma_p\big(\langle\frac{1}{4} - \frac{j}{p-1}\rangle\big)\Gamma_p\big(\langle\frac{3}{4} - \frac{j}{p-1}\rangle\big)\Gamma_p\big(\langle\frac{j}{p-1}\rangle\big)^n}{\Gamma_p\big(\frac{1}{2}\big)^{n-2}\Gamma_p\big(\frac{1}{4}\big)\Gamma_p\big(\frac{3}{4}\big)}
$$

$$
\times (-p)^{-(n-2)\lfloor\frac{1}{2} - \frac{j}{p-1}\rfloor - \lfloor\frac{1}{4} - \frac{j}{p-1}\rfloor - \lfloor\frac{3}{4} - \frac{j}{p-1}\rfloor - n\lfloor\frac{j}{p-1}\rfloor}.
$$

Letting $j = \frac{p-1}{2} - k$ this becomes

$$
{}_nG_n\left[\begin{array}{cccccc|c} \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & \dots & 1 & 1 & 1 \end{array}\;\middle|\; s\right]_p
$$

$$
= \frac{-1}{p-1}\sum_{k=1-\frac{p-1}{2}}^{\frac{p-1}{2}}(-1)^{n\left(\frac{p-1}{2} - k\right)}\,\overline{\omega}^{\frac{p-1}{2} - k}(s)
$$

$$
\times \frac{\Gamma_p\big(\langle\frac{k}{p-1}\rangle\big)^{n-2}\Gamma_p\big(\langle-\frac{1}{4} + \frac{k}{p-1}\rangle\big)\Gamma_p\big(\langle\frac{1}{4} + \frac{k}{p-1}\rangle\big)\Gamma_p\big(\langle\frac{1}{2} - \frac{k}{p-1}\rangle\big)^n}{\Gamma_p\big(\frac{1}{2}\big)^{n-2}\Gamma_p\big(\frac{1}{4}\big)\Gamma_p\big(\frac{3}{4}\big)}
$$

$$
\times (-p)^{-(n-2)\lfloor\frac{k}{p-1}\rfloor - \lfloor-\frac{1}{4} + \frac{k}{p-1}\rfloor - \lfloor\frac{1}{4} + \frac{k}{p-1}\rfloor - n\lfloor\frac{1}{2} - \frac{k}{p-1}\rfloor}.
$$

Note the summand is invariant under $k \to k+(p-1)$ so we can take the limits of summation to be from $k = 0$ to $k = p - 2$. Therefore,

$$
{}_nG_n\left[\begin{array}{ccccccc} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{array} \;\middle|\; s\right]_p
$$

$$
= \frac{-1}{p-1}\sum_{k=0}^{p-2}(-1)^{n\left(\frac{p-1}{2}-k\right)}\,\overline{\omega}^{\frac{p-1}{2}-k}(s)
$$

$$
\times \frac{\Gamma_p\!\left(\langle\frac{k}{p-1}\rangle\right)^{n-2}\Gamma_p\!\left(\langle-\frac14+\frac{k}{p-1}\rangle\right)\Gamma_p\!\left(\langle\frac14+\frac{k}{p-1}\rangle\right)\Gamma_p\!\left(\langle\frac12-\frac{k}{p-1}\rangle\right)^n}{\Gamma_p\!\left(\frac12\right)^{n-2}\Gamma_p\!\left(\frac14\right)\Gamma_p\!\left(\frac34\right)}
$$

$$
\times (-p)^{-(n-2)\lfloor\frac{k}{p-1}\rfloor-\lfloor-\frac14+\frac{k}{p-1}\rfloor-\lfloor\frac14+\frac{k}{p-1}\rfloor-n\lfloor\frac12-\frac{k}{p-1}\rfloor}
$$

$$
= \frac{-p\,\overline{\omega}^{\frac{p-1}{2}}((-1)^{n+1}s)}{p-1}\sum_{k=0}^{p-2}(-1)^{nk}\,\overline{\omega}^k(s^{-1})
$$

$$
\times \frac{\Gamma_p\!\left(\langle\frac12-\frac{k}{p-1}\rangle\right)^n\Gamma_p\!\left(\langle\frac{k}{p-1}\rangle\right)^{n-2}\Gamma_p\!\left(\langle\frac14+\frac{k}{p-1}\rangle\right)\Gamma_p\!\left(\langle\frac34+\frac{k}{p-1}\rangle\right)}{\Gamma_p\!\left(\frac12\right)^n\Gamma_p\!\left(\frac14\right)\Gamma_p\!\left(\frac34\right)}
$$

$$
\times (-p)^{-n\lfloor\frac12-\frac{k}{p-1}\rfloor-(n-2)\lfloor\frac{k}{p-1}\rfloor-\lfloor\frac14+\frac{k}{p-1}\rfloor-\lfloor\frac34+\frac{k}{p-1}\rfloor}
$$

$$
= p\cdot\omega^{\frac{p-1}{2}}((-1)^{n+1}s)\cdot {}_nG_n\left[\begin{array}{ccccccc} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & \cdots & 1 & \frac{1}{4} & \frac{3}{4} \end{array} \;\middle|\; s^{-1}\right]_p,
$$

using the facts that $\Gamma_p\!\left(\frac12\right)^2 = -\omega^{\frac{p-1}{2}}(-1)$ and $(-1)^{\frac{p-1}{2}} = \phi(-1) = \omega^{\frac{p-1}{2}}(-1)$. □

We now prove Theorem 2.4 by using the transformations in Theorems 2.5-2.7, together with Lemma 3.3.

Proof of Theorem 2.4. Let p be an odd prime. By letting $t \to t^{-1}$ in Theorem 2.6, we see that

$$
{}_4G_4\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{array} \;\middle|\; 1\right]_p = -\sum_{t=2}^{p-1}{}_3G_3\left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{array} \;\middle|\; t^{-1}\right]_p\cdot\omega^{\frac{p-1}{2}}(t(t-1)).
$$

Applying Theorem 2.5 to the right-hand side gives us

$$
{}_4G_4\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{array} \;\middle|\; 1\right]_p
$$

$$
= -p\cdot\omega^{\frac{p-1}{2}}(-2) - s(p)\cdot\omega^{\frac{p-1}{2}}(-2)\cdot\sum_{t=2}^{p-1}{}_3G_3\left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1 \end{array} \;\middle|\; -\frac{(1-t)^2}{4t}\right]_p\cdot\omega^{\frac{p-1}{2}}(t).
$$

Now using Theorem 2.7 and Lemma 3.3 we see that

$$_4G_4\left[\begin{array}{cccc}\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1 & 1\end{array}\ \Big|\ 1\right]_p$$

$$= p\cdot\omega^{\frac{p-1}{2}}(-1)\cdot {}_4G_4\left[\begin{array}{cccc}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & \frac{1}{4} & \frac{3}{4}\end{array}\ \Big|\ 1\right]_p$$

$$= \frac{-p\cdot\omega^{\frac{p-1}{2}}(-1)}{p-1}\sum_{j=0}^{p-2}\frac{\Gamma_p\left(\langle\frac{1}{2}-\frac{j}{p-1}\rangle\right)^4\Gamma_p\left(\frac{j}{p-1}\right)^2\Gamma_p\left(\langle\frac{1}{4}+\frac{j}{p-1}\rangle\right)\Gamma_p\left(\langle\frac{3}{4}+\frac{j}{p-1}\rangle\right)}{\Gamma_p\left(\frac{1}{2}\right)^4\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)}$$

$$\times (-p)^{-4\lfloor\frac{1}{2}-\frac{j}{p-1}\rfloor-\lfloor\frac{3}{4}+\frac{j}{p-1}\rfloor-\lfloor\frac{1}{4}+\frac{j}{p-1}\rfloor}$$

$$= \frac{p\cdot\omega^{\frac{p-1}{2}}(-1)}{p-1}\left[\sum_{j=1}^{p-2}\frac{\Gamma_p\left(\langle\frac{1}{2}-\frac{j}{p-1}\rangle\right)^3\Gamma_p\left(\frac{j}{p-1}\right)\Gamma_p\left(\langle\frac{1}{4}+\frac{j}{p-1}\rangle\right)\Gamma_p\left(\langle\frac{3}{4}+\frac{j}{p-1}\rangle\right)}{\Gamma_p\left(\frac{1}{2}\right)^3\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)}\right.$$

$$\left.\times(-p)^{-3\lfloor\frac{1}{2}-\frac{j}{p-1}\rfloor-\lfloor\frac{3}{4}+\frac{j}{p-1}\rfloor-\lfloor\frac{1}{4}+\frac{j}{p-1}\rfloor}\sum_{t=2}^{p-1}\overline{\omega}^j\left(\frac{4(1-t)}{t^2}\right)-1\right]$$

$$= -p\cdot\omega^{\frac{p-1}{2}}(-1)\cdot\sum_{t=2}^{p-1}{}_3G_3\left[\begin{array}{ccc}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{4} & \frac{3}{4}\end{array}\ \Big|\ -\frac{4(1-t)}{t^2}\right]_p - p\cdot\omega^{\frac{p-1}{2}}(-1)$$

$$= -p\cdot\omega^{\frac{p-1}{2}}(-1)\cdot\sum_{t=2}^{p-1}{}_3G_3\left[\begin{array}{ccc}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{4} & \frac{3}{4}\end{array}\ \Big|\ -\frac{4t}{(1-t)^2}\right]_p - p\cdot\omega^{\frac{p-1}{2}}(-1)$$

$$= -\sum_{t=2}^{p-1}{}_3G_3\left[\begin{array}{ccc}\frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1\end{array}\ \Big|\ -\frac{(1-t)^2}{4t}\right]_p\omega^{\frac{p-1}{2}}(t) - p\cdot\omega^{\frac{p-1}{2}}(-1),$$

where, in the second last step, we let $t\to 1-t$. We note that $\omega^{\frac{p-1}{2}}(2)=s(p)$. Therefore

$$_4G_4\left[\begin{array}{cccc}\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1 & 1\end{array}\ \Big|\ 1\right]_p - s(p)\cdot p$$

$$= -\sum_{t=2}^{p-1}{}_3G_3\left[\begin{array}{ccc}\frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1\end{array}\ \Big|\ -\frac{(1-t)^2}{4t}\right]_p\omega^{\frac{p-1}{2}}(t) - p\cdot\left(s(p)+\omega^{\frac{p-1}{2}}(-1)\right)$$

$$= \omega^{\frac{p-1}{2}}(-1)\left[{}_4G_4\left[\begin{array}{cccc}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & \cdots & 1\end{array}\ \Big|\ 1\right]_p - p\right].$$

$\square$

We conclude this section by using Theorem 2.4 to verify Theorem 2.3. Then, we apply Theorems 2.1 and 2.3 to prove the supercongruence in Theorem 2.2.

Proof of Theorem 2.3. Combining (4.2) and Lemma 4.2 we know that, for all odd primes p,

$$_4G_4\left[\begin{array}{cccc}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1\end{array}\ \Big|\ 1\right]_p - p = a(p).$$

Now, f_2 is a quadratic twist of f_1 with $c(p) = \phi_p(-1)a(p)$. To see this, we consider the quadratic twist $f_{1,\psi} = \sum_{n=1}^{\infty} \psi(n)a(n)q^n$ of f_1 by the character $\psi(\cdot) = (\frac{-4}{\cdot})$, the Kronecker symbol. A priori, $f_{1,\psi} \in S_4(\Gamma_0(32))$ by [16, §3 Prop. 17(b)]; however, the proof of [16, §3 Prop. 17(b)] reveals in fact that $f_{1,\psi} \in S_4(\Gamma_0(\text{lcm}(8, 4^2))) = S_4(\Gamma_0(16))$. Using Sage Mathematics Software [27], one easily computes the dimensions $\text{Dim}(S_4(\Gamma_0(8))) = 1$ and $\text{Dim}(S_4(\Gamma_0(16))) = 3$. Therefore, $f_1(z)$ and $f_1(2z)$ form a basis for $S_4^{\text{old}}(\Gamma_0(16))$ and so $f_{1,\psi} \in S_4^{\text{new}}(\Gamma_0(16))$. Therefore $f_{1,\psi}$ is a multiple of f_2, the unique newform in $S_4(\Gamma_0(16))$, and by comparing coefficients we see that $f_{1,\psi} = f_2$. As a result, $c(p) = \psi(p)a(p) = \phi_p(-1)a(p)$.

Noting that $\phi_p(-1) = \omega^{(p-1)/2}(-1)$, it therefore suffices to prove

$$
{}_4G_4\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1 & 1 \end{array} \,\middle|\, 1 \right]_p - s(p) \cdot p = \omega^{(p-1)/2}(-1)\left[{}_4G_4\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{array} \,\middle|\, 1 \right]_p - p \right].
$$

This has been proved in Theorem 2.4. $\qquad\square$

Proof of Theorem 2.2. Appealing to Theorem 2.1 with $d_1 = 2$, $d_2 = 4$, it suffices to prove

$$
{}_4G_4\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 1 & 1 & 1 & 1 \end{array} \,\middle|\, 1 \right]_p - s(p) \cdot p = c(p),
$$

where $s(p) = \Gamma_p(\frac{1}{4})\Gamma_p(\frac{3}{4})\Gamma_p(\frac{1}{2})^2$. This has been proved in Theorem 2.3. $\qquad\square$

6. Concluding Remarks

The proof of the supercongruence in this paper relies on the fact that the modular form in question is a twist of another modular form that appears in one of the other known supercongruences. This allows us to link the proof of the two supercongruences via an identity for ${}_nG_n[\cdots]$. Unfortunately, these are the only two modular forms in Rodriguez-Villegas' conjectures which are linked in this way. Thus, it does not appear that discovering other identities for ${}_nG_n[\cdots]$ that are similar to Theorem 2.4 will reduce the burden of proof for establishing the remaining 11 conjectures.

References

[1] S. Ahlgren, K. Ono, *A Gaussian hypergeometric series evaluation and Apéry number congruences*, J. Reine Angew. Math. **518** (2000), 187–212.

[2] G. Andrews, R. Askey, R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, **71**, Cambridge University Press, Cambridge, 1999.

[3] B. Berndt, R. Evans, K. Williams, *Gauss and Jacobi Sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1998.

[4] R. Barman, N Saikia, *p-Adic gamma function and the trace of Frobenius of elliptic curves*, J Number Theory, **140** (2014), no. 7, 181–195.

[5] R. Barman, N Saikia, *Certain transformations for hypergeometric series in the p-adic setting*, arXiv:1403.3607.

[6] R. Evans, *Hypergeometric $_3F_2(1/4)$ evaluations over finite fields and Hecke eigenforms*, Proc. Amer. Math. Soc. **138** (2010), 517–531.

[7] J. Fuselier, *Hypergeometric functions over finite fields and relations to modular forms and elliptic curves*, Ph.D. thesis, Texas A&M University, 2007.

[8] J. Fuselier, *Hypergeometric functions over $\mathbb{F}_p$ and relations to elliptic curves and modular forms*, Proc. Amer. Math. Soc. **138** (2010), no.1, 109–123.

[9] J. Fuselier, *Traces of Hecke operators in level 1 and Gaussian hypergeometric functions.*, Proc. Amer. Math. Soc. **141** (2013), no.6, 1871–1881.

[10] S. Frechette, K. Ono, and M. Papanikolas, *Gaussian hypergeometric functions and traces of Hecke operators*, Int. Math. Res. Not. **2004**, no. 60, 3233–3262.

[11] J. Greene, *Hypergeometric functions over finite fields*, Trans. Amer. Math. Soc. **301** (1987), no. 1, 77–101.

[12] B. Gross, N. Koblitz, *Gauss sums and the p-adic Γ-function*, Ann. Math. **109** (1979), no. 3, 569–581.

[13] N. M. Katz, *Exponential Sums and Differential Equations*, Princeton Univ. Press, Princeton, 1990.

[14] T. Kilbourn, *An extension of the Apéry number supercongruence*, Acta Arith. **123** (2006), no. 4, 335–348.

[15] N. Koblitz, *p-adic analysis: a short course on recent work*, London Math. Soc. Lecture Note Series, **46**. Cambridge University Press, Cambridge-New York, 1980.

[16] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, 2nd ed., Springer-Verlag, New York, 1993.

[17] C. Lennon, *Arithmetic and analytic properties of finite field hypergeometric functions*, Ph. D. thesis, Massachusetts Institute of Technology, 2011.

[18] C. Lennon, *Gaussian hypergeometric evaluations of traces of Frobenius for elliptic curves*, Proc. Amer. Math. Soc. **139** (2011), no. 6, 1931–1938.

[19] C. Lennon, *Trace formulas for Hecke operators, Gaussian hypergeometric functions, and the modularity of a threefold.*, J. Number Theory **131** (2011), no. 12, 2320–2351.

[20] E. Mortenson, *Supercongruences for truncated $_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms*, Proc. Amer. Math. Soc. **133** (2005), no. 2, 321–330.

[21] D. McCarthy, *Extending Gaussian hypergeometric series to the p-adic setting*, Int. J. Number Theory **8** (2012), no. 7, 1581–1612.

[22] D. McCarthy, *On a supercongruence conjecture of Rodriguez-Villegas*, Proc. Amer. Math. Soc. **140** (2012), 2241–2254.

[23] D. McCarthy, *Transformations of well-poised hypergeometric functions over finite fields*, Finite Fields and Their Applications, **18** (2012), no. 6, 1133–1147.

[24] D. McCarthy, *The trace of Frobenius of elliptic curves and the p-adic gamma function*, Pacific Journal of Mathematics, **261** (2013), no. 1, 219–236.

[25] D. McCarthy, M. Papanikolas, *A finite field hypergeometric function associated to eigenvalues of a Siegel eigenform*, arXiv:1205.1006.

[26] F. Rodriguez-Villegas, *Hypergeometric families of Calabi-Yau manifolds*, Calabi-Yau varieties and mirror symmetry (Toronto, Ontario, 2001), Fields Inst. Commun. **38**, Amer. Math. Soc., 2003, 223–231.

[27] W. A. Stein et al., *Sage Mathematics Software (Version 4.8)*, The Sage Development Team, 2012, http://www.sagemath.org.

[28] V. Vega, *Relations between hypergeometric functions over finite fields and algebraic curves*, Int. J. Number Theory **7** (2011), no. 8, 2171–2195.

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