BINOMIAL COEFFICIENT–HARMONIC SUM IDENTITIES ASSOCIATED TO SUPERCONGRUENCES

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Abstract. We establish two binomial coefficient–generalized harmonic sum identities using the partial fraction decomposition method. These identities are a key ingredient in the proofs of numerous supercongruences. In particular, in other works of the author, they are used to establish modulo p^k $(k>1)$ congruences between truncated generalized hypergeometric series, and a function which extends Greene's hypergeometric function over finite fields to the p-adic setting. A specialization of one of these congruences is used to prove an outstanding conjecture of Rodriguez-Villegas which relates a truncated generalized hypergeometric series to the p-th Fourier coefficient of a particular modular form.

1. Introduction and Statement of Results

For non-negative integers i and n, we define the generalized harmonic sum, $H_n^{(i)}$, by

$$H_n^{(i)} := \sum_{j=1}^{n} \frac{1}{j^i}$$

and $H_0^{(i)} := 0$. In [3] Chu proves the following binomial coefficient-generalized harmonic sum identity using the partial fraction decomposition method. If n is a positive integer, then

$$\sum_{k=1}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \left[1 + 2kH_{n+k}^{(1)} + 2kH_{n-k}^{(1)} - 4kH_k^{(1)}\right] = 0. \tag{1.1}$$

This identity had previously been established using the WZ method [1] and was used by Ahlgren and Ono in proving the Apéry number supercongruence [2].

In [4], [5] the author establishes various supercongruences between truncated generalized hypergeometric series, and a function which extends Greene's hypergeometric function over finite fields to the p-adic setting. Specifically, let p be an odd prime and let $n \in \mathbb{Z}^+$. For $1 \leq i \leq n+1$, let $\frac{m_i}{d_i} \in \mathbb{Q} \cap \mathbb{Z}_p$ such that $0 < \frac{m_i}{d_i} < 1$. Let $\Gamma_p(\cdot)$ denote Morita's p-adic gamma function, $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ denote the fractional part of x, i.e. $x - \lfloor x \rfloor$. Then define

$${}_{n+1}G\left(\tfrac{m_1}{d_1}, \tfrac{m_2}{d_2}, \ldots, \tfrac{m_{n+1}}{d_{n+1}}\right)_p$$
$$:= \frac{-1}{p-1} \sum_{j=0}^{p-2} \left((-1)^j \Gamma_p\left(\tfrac{j}{p-1}\right)\right)^{n+1} \prod_{i=1}^{n+1} \frac{\Gamma_p\left(\langle \frac{m_i}{d_i} - \frac{j}{p-1} \rangle\right)}{\Gamma_p\left(\frac{m_i}{d_i}\right)} (-p)^{-\lfloor \frac{m_i}{d_i} - \frac{j}{p-1} \rfloor}.$$

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Note that when $p \equiv 1 \pmod{d_i}$ this function recovers Greene's hypergeometric function over finite fields. For a complex number a and a non-negative integer n let $(a)_n$ denote the rising factorial defined by

$$(a)_0 := 1 \quad \text{and} \quad (a)_n := a(a+1)(a+2)\cdots(a+n-1) \text{ for } n > 0.$$

Then, for complex numbers a_i, b_j and z, with none of the b_j being negative integers or zero, we define the truncated generalized hypergeometric series

$${}_rF_s\left[\begin{matrix} a_1, & a_2, & a_3, & \dots, & a_r \\ & b_1, & b_2, & \dots, & b_s \end{matrix} \;\middle|\; z\right]_m := \sum_{n=0}^{m} \frac{(a_1)_n (a_2)_n (a_3)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n} \, \frac{z^n}{n!}.$$

An example of one the supercongruence results from [5] is the following theorem.

Theorem 1.1 ([5] Thm. 2.7)**.** *Let $r, d \in \mathbb{Z}$ such that $2 \leq r \leq d-2$ and $\gcd(r,d) = 1$. Let p be an odd prime such that $p \equiv \pm 1 \pmod{d}$ or $p \equiv \pm r \pmod{d}$ with $r^2 \equiv \pm 1 \pmod{d}$. If $s(p) := \Gamma_p\big(\frac{1}{d}\big)\Gamma_p\big(\frac{r}{d}\big)\Gamma_p\big(\frac{d-r}{d}\big)\Gamma_p\big(\frac{d-1}{d}\big)$, then*

$${}_4G\Big(\tfrac{1}{d}, \tfrac{r}{d}, 1-\tfrac{r}{d}, 1-\tfrac{1}{d}\Big)_p \equiv {}_4F_3\left[\begin{matrix} \frac{1}{d}, & \frac{r}{d}, & 1-\frac{r}{d}, & 1-\frac{1}{d} \\ & 1, & 1, & 1 \end{matrix} \;\middle|\; 1\right]_{p-1} + s(p)p \pmod{p^3}.$$

A specialization of this congruence is used to prove an outstanding supercongruence conjecture of Rodriguez-Villegas, which relates a truncated generalized hypergeometric series to the p-th Fourier coefficient of a particular modular form [4],[6]. Similar results to Theorem 1.1 exist for ${}_4G$ with other parameters, and also ${}_2G$ and ${}_3G$.

The main results of the current paper, Theorems 1.2 and 1.3 below, are two binomial coefficient–generalized harmonic sum identities which factor heavily into the proofs of all the ${}_4G$ congruences. Taking particular values for n, m, l, c_1 and c_2 in these identities allows the vanishing of certain terms in the proofs. Note that letting $m = n$ in Theorem 1.2 recovers (1.1).

Theorem 1.2. *Let m, n be positive integers with $m \geq n$. Then*

$$\sum_{k=0}^{n} \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k}\left[1 + k\left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)}\right)\right]$$
$$+ \sum_{k=n+1}^{m} (-1)^{k-n} \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k} \Big/ \binom{k-1}{n} = (-1)^{m+n}.$$

Theorem 1.3. *Let l, m, n be positive integers with $l > m \geq n \geq \frac{l}{2}$ and $c_1, c_2 \in \mathbb{Q}$ some constants. Then*

$$\sum_{k=0}^{n} \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k}\Bigg\{\Big[1 + k\Big(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)}\Big)\Big]$$
$$\cdot \Big[c_1\left(H_{k+n}^{(1)} - H_{k+l-n-1}^{(1)}\right) + c_2\Big(H_{k+m}^{(1)} - H_{k+l-m-1}^{(1)}\Big)\Big] - k\Big[c_1\left(H_{k+n}^{(2)} - H_{k+l-n-1}^{(2)}\right)$$
$$+ c_2\left(H_{k+m}^{(2)} - H_{k+l-m-1}^{(2)}\right)\Big]\Bigg\} + \sum_{k=n+1}^{m} (-1)^{k-n}\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\Big/\binom{k-1}{n}$$
$$\cdot \Big[c_1\left(H_{k+n}^{(1)} - H_{k+l-n-1}^{(1)}\right) + c_2\left(H_{k+m}^{(1)} - H_{k+l-m-1}^{(1)}\right)\Big] = 0.$$

The remainder of this paper is spent proving Theorems 1.2 and 1.3.

2. Proofs

We first develop two algebraic identities of which the binomial coefficient–harmonic sum identities are limiting cases.

Theorem 2.1. *Let x be an indeterminate and let m, n positive integers with $m \geq n$. Then*

$$\sum_{k=0}^{n} \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k}$$
$$\cdot\left\{\frac{-k}{(x+k)^2} + \frac{1 + k\left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)}\right)}{x+k}\right\}$$
$$+ \sum_{k=n+1}^{m} \frac{(-1)^{k-n}}{x+k}\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\Big/\binom{k-1}{n} = \frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_{m+1}}. \tag{2.1}$$

Proof. Using partial fraction decomposition we can write

$$f(x) := \frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_{m+1}} = \frac{A}{x} + \sum_{k=1}^{n}\left\{\frac{B_k}{(x+k)^2} + \frac{C_k}{x+k}\right\} + \sum_{k=n+1}^{m} \frac{D_k}{x+k}$$

for some A, B_k, C_k and $D_k \in \mathbb{Q}$. We now isolate these coefficients by taking various limits of $f(x)$ as follows.

$$A = \lim_{x\to 0} xf(x) = \lim_{x\to 0} \frac{(1-x)_n(1-x)_m}{(1+x)_n(1+x)_m} = 1.$$

For $1 \leq k \leq n$,

$$B_k = \lim_{x \to -k} (x+k)^2 f(x) = \lim_{x \to -k} \frac{x(1-x)_n(1-x)_m}{(x)_k^2 (x+k+1)_{n-k}(x+k+1)_{m-k}}$$

$$= \frac{-k(k+1)_n(k+1)_m}{(-k)_k^2 (1)_{n-k}(1)_{m-k}}$$

$$= \frac{-k(k+1)_n(k+1)_m}{(-1)^{2k} k!^2 (n-k)!(m-k)!}$$

$$= -k \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k},$$

and, using L'Hôspital's rule,

$$C_k = \lim_{x \to -k} \frac{(x+k)^2 f(x) - B_k}{x+k}$$

$$= \lim_{x \to -k} \frac{d}{dx}\left[(x+k)^2 f(x)\right]$$

$$= \lim_{x \to -k} \frac{d}{dx}\left[\frac{x(1-x)_n(1-x)_m}{(x)_k^2 (x+k+1)_{n-k}(x+k+1)_{m-k}}\right]$$

$$= \lim_{x \to -k} \left\{ \left[\frac{(1-x)_n(1-x)_m}{(x)_k^2 (x+k+1)_{n-k}(x+k+1)_{m-k}}\right] \left[1 - x\left(\sum_{s=1}^{n}(-x+s)^{-1} \right.\right.\right.$$
$$\left.\left.\left. + \sum_{s=1}^{m}(-x+s)^{-1} + \sum_{s=1}^{n-k}(x+k+s)^{-1} + \sum_{s=1}^{m-k}(x+k+s)^{-1} + 2\sum_{s=0}^{k-1}(x+s)^{-1}\right)\right]\right\}$$

$$= \left[\frac{(1+k)_n(1+k)_m}{(-k)_k^2 (1)_{n-k}(1)_{m-k}}\right]\left[1 + k\left(\sum_{s=1}^{n}(k+s)^{-1} + \sum_{s=1}^{m}(k+s)^{-1} + \sum_{s=1}^{n-k}(s)^{-1}\right.\right.$$
$$\left.\left. + \sum_{s=1}^{m-k}(s)^{-1} + 2\sum_{s=0}^{k-1}(-k+s)^{-1}\right)\right]$$

$$= \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k}\left[1 + k\left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)}\right)\right].$$

Similarly, for $n+1 \leq k \leq m$,

$$D_k = \lim_{x \to -k} (x+k) f(x) = \lim_{x \to -k} \frac{x(1-x)_n (1-x)_m}{(x)_{n+1} (x)_k (x+k+1)_{m-k}}$$

$$= \frac{-k(k+1)_n (k+1)_m}{(-k)_{n+1} (-k)_k (1)_{m-k}}$$

$$= (-1)^{k-n} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \Big/ \binom{k-1}{n}.$$

□

Theorem 2.2. *Let x be an indeterminate and let l, m, n be positive integers with $l > m \geq n \geq \frac{l}{2}$ and $c_1, c_2 \in \mathbb{Q}$ some constants. Then*

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{1}{x+k} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \binom{n}{k} \Bigg\{ \Bigg[c_1 \left(H_{k+n}^{(1)} - H_{k+l-n-1}^{(1)} \right) \\
&\quad + c_2 \left(H_{k+m}^{(1)} - H_{k+l-m-1}^{(1)} \right) \Bigg] \cdot \left[\frac{x}{x+k} + k \left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)} \right) \right] \\
&\quad - k \left[c_1 \left(H_{k+n}^{(2)} - H_{k+l-n-1}^{(2)} \right) + c_2 \left(H_{k+m}^{(2)} - H_{k+l-m-1}^{(2)} \right) \right] \Bigg\} \\
&+ \sum_{k=n+1}^{m} \frac{(-1)^{k-n}}{x+k} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \Big/ \binom{k-1}{n} \\
&\quad \cdot \left[c_1 \left(H_{k+n}^{(1)} - H_{k+l-n-1}^{(1)} \right) + c_2 \left(H_{k+m}^{(1)} - H_{k+l-m-1}^{(1)} \right) \right] \\
&\qquad = \frac{x(1-x)_n (1-x)_m}{(x)_{n+1} (x)_{m+1}} \left[c_1 \sum_{s=l-n}^{n} (-x+s)^{-1} + c_2 \sum_{s=l-m}^{m} (-x+s)^{-1} \right]. \qquad (2.2)
\end{aligned}$$

Proof. Using partial fraction decomposition we can write

$$f(x) := \frac{x(1-x)_n (1-x)_m}{(x)_{n+1} (x)_{m+1}} \left[c_1 \sum_{s=l-n}^{n} (-x+s)^{-1} + c_2 \sum_{s=l-m}^{m} (-x+s)^{-1} \right]$$

$$= \frac{A}{x} + \sum_{k=1}^{n} \left\{ \frac{B_k}{(x+k)^2} + \frac{C_k}{x+k} \right\} + \sum_{k=n+1}^{m} \frac{D_k}{x+k}$$

for some A, B_k, C_k and $D_k \in \mathbb{Q}$. As in the proof of Theorem 2.1, we isolate the coefficients A, B_k, C_k and D_k by taking various limits of $f(x)$. For brevity, we first let

$$T_a^{(r)} := c_1 \sum_{s=l-n}^{n} (a+s)^{-r} + c_2 \sum_{s=l-m}^{m} (a+s)^{-r}$$

and

$$U^{(r)} := c_1 \left(H_{k+n}^{(r)} - H_{k+l-n-1}^{(r)} \right) + c_2 \left(H_{k+m}^{(r)} - H_{k+l-m-1}^{(r)} \right).$$

Then we have

$$\begin{aligned} A = \lim_{x\to 0} x f(x) &= c_1 \lim_{x\to 0} \sum_{s=l-n}^{n} \frac{(1-x)_n(1-x)_m}{(1+x)_n(1+x)_m(s-x)} + c_2 \lim_{x\to 0} \sum_{s=l-m}^{m} \frac{(1-x)_n(1-x)_m}{(1+x)_n(1+x)_m(s-x)} \\ &= c_1 \sum_{s=l-n}^{n} s^{-1} + c_2 \sum_{s=l-m}^{m} s^{-1} \\ &= c_1 \left(H_n^{(1)} - H_{l-n-1}^{(1)} \right) + c_2 \left(H_m^{(1)} - H_{l-m-1}^{(1)} \right). \end{aligned}$$

For $1 \leq k \leq n$,

$$\begin{aligned} B_k = \lim_{x\to -k} (x+k)^2 f(x) &= \lim_{x\to -k} \frac{x(1-x)_n(1-x)_m}{(x)_k^2 (x+k+1)_{n-k}(x+k+1)_{m-k}} \, T_{-x}^{(1)} \\ &= \frac{-k(k+1)_n(k+1)_m}{(-k)_k^2(1)_{n-k}(1)_{m-k}} \, T_k^{(1)} \\ &= -k \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k} U^{(1)} \end{aligned}$$

and

$$\begin{aligned} C_k &= \lim_{x\to -k} \frac{d}{dx} \left[(x+k)^2 f(x) \right] \\ &= \lim_{x\to -k} \frac{d}{dx} \left[\frac{x(1-x)_n(1-x)_m}{(x)_k^2(x+k+1)_{n-k}(x+k+1)_{m-k}} \, T_{-x}^{(1)} \right] \\ &= \lim_{x\to -k} \left\{ \left[\frac{(1-x)_n(1-x)_m}{(x)_k^2(x+k+1)_{n-k}(x+k+1)_{m-k}} \right] \left[x\, T_{-x}^{(2)} + \; T_{-x}^{(1)} - x\, T_{-x}^{(1)} \right. \right. \\ &\quad \cdot \left(\sum_{s=1}^{n} (-x+s)^{-1} + \sum_{s=1}^{m} (-x+s)^{-1} + \sum_{s=1}^{n-k} (x+k+s)^{-1} \right. \\ &\qquad \left. \left. \left. + \sum_{s=1}^{m-k} (x+k+s)^{-1} + 2 \sum_{s=0}^{k-1} (x+s)^{-1} \right) \right] \right\} \end{aligned}$$

$$= \left[\frac{(1+k)_n(1+k)_m}{(-k)_k^2(1)_{n-k}(1)_{m-k}}\right]\left[-kT_k^{(2)} + T_k^{(1)}\left(1 + k\left(\sum_{s=1}^{n}(k+s)^{-1} + \sum_{s=1}^{m}(k+s)^{-1} + \sum_{s=1}^{n-k}(s)^{-1} + \sum_{s=1}^{m-k}(s)^{-1} + 2\sum_{s=0}^{k-1}(-k+s)^{-1}\right)\right)\right]$$

$$= \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k} \cdot \left[-k\,U^{(2)} + \left(1 + k\Big(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)}\Big)\right)U^{(1)}\right].$$

For $n+1 \le k \le m$,

$$D_k = \lim_{x\to -k}(x+k)f(x) = \lim_{x\to -k}\frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_k(x+k+1)_{m-k}}\,T_{-x}^{(1)}$$

$$= \frac{-k(k+1)_n(k+1)_m}{(-k)_{n+1}(-k)_k(1)_{m-k}}\,T_k^{(1)}$$

$$= (-1)^{k-n}\,U^{(1)}\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\Big/\binom{k-1}{n}.$$

□

Proofs of Theorems 1.2 and 1.3. Multiply both sides of (2.1) and (2.2) respectively by x and take the limit as $x \to \infty$. □

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