Practice Problems for Exam I

The test will cover Sections 1.1-2.9 On each proof clearly state the definitions involved.

The exam will consist of four of these problems.

1. Prove that if A and B are subsets of S then $(A \cap B)' = A' \cup B'$, where the apostrophe (') indicates the complement relative to S.
2. Prove that if $f : A \to B$ is a function and $X \subset B$, $Y \subset B$, then $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.
3. Prove that the set $\mathbb{R}$ is uncountable. To do it follow these steps:
   a) Assume that $\mathbb{R}$ is countable, thus $\mathbb{R} = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$
   b) For each n consider the interval $I_n = (x_i - \frac{1}{2^{n+1}}, x_i + \frac{1}{2^{n+1}})$, then the length of I_n is 2^{-n} and the sum of the lengths of all the I_n is $2^{-1} + 2^{-2} + \cdots + 2^{-n} + ... = 1$ (justify this step with a geometric series argument)
   c) Reach a contradiction since each $x_i \in \mathbb{R}$ is in the interval I_i and therefore $\mathbb{R} \subset \cup_{n=1}^{\infty} I_n$. But then the length of $\mathbb{R}$ (the real line) is infinite while the length of $\cup_{n=1}^{\infty} I_n$ is finite.
   This is Corollary 1.6B's "another proof".
4. Prove that if the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent to L, then any subsequence of $\{S_n\}_{n=1}^{\infty}$ is also convergent to L.
5. Prove that if the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ diverges to infinity, then so does any subsequence of $\{S_n\}_{n=1}^{\infty}$.
6. Prove that if $\{S_n\}_{n=1}^{\infty}$ diverges to infinity then $\{S_n\}_{n=1}^{\infty}$ is not convergent.
7. Prove that if $\{S_n\}_{n=1}^{\infty}$ is a nonincreasing sequence which is bounded below then it is convergent.
8. Prove that if $\{S_n\}_{n=1}^{\infty}$ is a nonincreasing sequence which is not bounded below then it diverges to minus infinity.
9. Prove that if $\{S_n\}_{n=1}^{\infty}$ is bounded above and has a subsequence that is bounded below by A then $\limsup_{n\to\infty} S_n \geq A$.
10. Prove that if $\{S_n\}_{n=1}^{\infty}$ has no subsequence that is bounded below then $\limsup_{n\to\infty} S_n = -\infty$.
11. Prove that if $\{S_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers then $\liminf_{n\to\infty} S_n = \lim_{n\to\infty} S_n$.
12. Prove that if $\{S_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if $\liminf_{n\to\infty} S_n = \limsup_{n\to\infty} S_n = -\infty$, then $\{S_n\}_{n=1}^{\infty}$ diverges to $-\infty$.
13. Homework problems 1.1 – 2.8 without *.