Foundations of Algebra Name (please print) ______________________

Exam I, June 14, 2004

I. (15) If A, B and C are sets, prove or disprove (by giving a counterexample) each of the following statements:

a)(5) $A \cap B \subseteq A$

Proof. Let $x \in A \cap B$ then, by the definition of intersection of sets, $x \in A$ and $x \in B$. Thus, if $x \in A \cap B$ then $x \in A$. By the definition of subset, we have then that $A \cap B \subseteq A$. □

b)(5) $A \subseteq B$ implies $A \cap B = A$

Proof. We need to show that if $A \subseteq B$ then $A \cap B = A$. The latter is true if and only if both $A \cap B \subseteq A$ and $A \subseteq A \cap B$ are true. The part $A \cap B \subseteq A$ was proven in a), so we only need to prove that $A \subseteq A \cap B$. Let $x \in A$, then since by hypothesis $A \subseteq B$, we have that $x \in B$, that is $x \in A$ and $x \in B$. By definition of intersection, this means that $x \in A \cap B$, that is $A \subseteq A \cap B$.

□

c)(5) If $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$ then $B \cap C \neq \emptyset$

This is not true, a counterexample can be given as follows: let $A = \{a, b, c\}$, $B = \{b\}$ and $C = \{c\}$, with $a \neq b$, $a \neq c$, $b \neq c$. Then $A \cap B = \{b\} \neq \emptyset$, $A \cap C = \{c\} \neq \emptyset$, however $B \cap C = \emptyset$ since $b \neq c$.

II. (5) Let $S(n) = 1 + 2 + \cdots + n$. Using the principle of mathematical induction, prove that the following statement $P(n)$ is true for each positive integer n (no credit will be given if you do not use the principle of mathematical induction):

$$P(n): \qquad S(n) = \frac{n(n+1)}{2}$$

Proof. $P(1)$ is true since $S(1) = 1 = \frac{1(1+1)}{2}$

We then need to show that if $P(k)$ is true then $P(k+1)$ is true as well, that is, if $S(k) = \frac{k(k+1)}{2}$ then $S(k+1) = \frac{(k+1)(k+2)}{2}$. So we assume that $S(k) = \frac{k(k+1)}{2}$. We have, $S(k+1) = 1+2+\cdots+k+(k+1) = S(k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$, by hypothesis. So, $S(k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$.

□

III. (5) Write the negation of the following

Statement: " For each x, $e^x > 0$"

Negation: "There is one x such that $e^x \leq 0$

IV. (5) Write the converse of the following

Statement:" If n is a nonnegative integer then n is a rational number."

Converse:" If n is a rational number then n isa nonnegative integer.'

Is the converse true? Prove it if true or give a counterexample otherwise.

Not true, for example $n = \frac{1}{2}$ is a rational number which is not an integer.

V. (5) Write the contrapositive of the following

Statement: "If A is a subset of B then $A \cap B = A$".

Contrapositive:"If $A \cap B \neq A$ then A is not a subset of B."

VI. (5) Decide which of the following statements is logically equivalent to

" If y is an integer and y is a square then y is a positive integer"

Circle your choice, you do not need to justify.

a) If y is not an integer and y is not a square then y is not a positive integer.

b) If y is a positive integer then y is an integer and y is a square.

c) If y is not a positive integer then y is not an integer or y is not a square.

Ans. c) is the correct one.

VII. (5) Prove that if $\beta : S \to T$, $\gamma : S \to T$, $\alpha : T \to U$, α is one-to-one, and $\alpha \circ \beta = \alpha \circ \gamma$ then $\beta = \gamma$.

Proof. We need to show that, under the hypothesis, $\beta = \gamma$. By the definition of when two mappings are equal, this is equivalent to showing that $\beta(x) = \gamma(x)$ for all $x \in S$, since we already know that they have same domain and codomain.

So, let $x \in S$. Since $\alpha \circ \beta = \alpha \circ \gamma$, we have $(\alpha \circ \beta)(x) = (\alpha \circ \gamma)(x)$. That is, $\alpha(\beta(x)) = \alpha(\gamma(x))$, by the definition of composition of functions. Now, since α is one-to-one and $\alpha(\beta(x)) = \alpha(\gamma(x))$, for $x \in S$ it has to be $\beta(x) = \gamma(x)$, for $x \in S$. This concludes the proof.

$\square$

a)(3) Prove that the set of even positive intergers

$$A = \{n \in \mathbb{N} : \ n = 2m, \ \text{for some } m \in \mathbb{N}\}$$

is closed under $+$.

Proof. Let $n_1, n_2 \in A$ then there are $m_1, m_2 \in \mathbb{N}$ such that $n_1 = 2m_1$ and $n_2 = 2m_2$. Therefore, $n_1 + n_2 = 2m_1 + 2m_2 = 2(m_1 + m_2)$, so $n_1 + n_2 \in A$, being an even positive integer itself. $\square$

b)(2) Prove that the set of odd positive intergers

$$B = \{n \in \mathbb{N} : \ n = 2k + 1, \ \text{for some } k \in \mathbb{N}\}$$

is not closed under $+$. You can prove it either by exhibiting a specific counterexample or in general.

Either, for instance, $5 + 7 = 12 \neq 2k + 1$ for all $k \in \mathbb{N}$ or, if $n_1 = 2k_1 + 1, n_2 = 2k_2 + 1$, with $k_1, k_2 \in \mathbb{N}$ then $n_1 + n_2 = 2(k_1 + k_2) + 2 = 2(k_1 + k_2 + 1)$ so $n_1 + n_2 \notin A$ as it is not an odd positive integer.