WRITTEN PROJECT II, due Monday June 21

Do problems 4.12, 6.13 and 6.15 in separate pages. Clearly write your name on each page. The proofs should be written clearly, following the methods seen in class. Each problem is worth 10 points.

4.12 Prove that if $\alpha : S \to T$, $\beta : T \to U$, and $\gamma : U \to V$ are any mappings, then $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.

Solution. We need to show that the two mappings $\gamma \circ (\beta \circ \alpha)$ and $(\gamma \circ \beta) \circ \alpha$ are equal, so we need to show that they their domains are equal, their codomains are equal and that $\gamma \circ (\beta \circ \alpha)(x) = (\gamma \circ \beta) \circ \alpha(x)$ for each x in their common domain.

Domains and codomains are equal: Since $\alpha : S \to T$, $\beta : T \to U$, and $\gamma : U \to V$, we have,
$\beta \circ \alpha : S \to U$ and $\gamma \circ (\beta \circ \alpha) : S \to V$, thus the domain of $\gamma \circ (\beta \circ \alpha)$ is S and its codomain is V.
Likewise, we have $\gamma \circ \beta : T \to V$ and $(\gamma \circ \beta) \circ \alpha : S \to V$, so the domain of $(\gamma \circ \beta) \circ \alpha$ is S and its codomain is V as well.

For each $x \in S$ we have on one hand:

$$\begin{aligned} \gamma \circ (\beta \circ \alpha)(x) &= \gamma(\beta \circ \alpha(x)) \quad \text{(By def. of composition of mappings)} \\ &= \gamma(\beta(\alpha(x))) \quad \text{(By def. of composition of mappings)} \end{aligned}$$

on the other hand,

$$\begin{aligned} (\gamma \circ \beta) \circ \alpha(x) &= (\gamma \circ \beta)(\alpha(x)) \quad \text{(By def. of composition of mappings)} \\ &= \gamma(\beta(\alpha(x))) \quad \text{(By def. of composition of mappings)} \end{aligned}$$

comparing both equations we see that,
$\gamma \circ (\beta \circ \alpha)(x) = (\gamma \circ \beta) \circ \alpha(x)$ for all $x \in S$.

6.13 Prove that the group in example 5.8 is non-Abelian.

Example 5.8. Let A denote the set of all mappings $\alpha_{a,b}$, with $a, b \in \mathbb{R}$, $a \neq 0$, defined by $\alpha_{a,b}(x) = ax + b$ for all $x \in \mathbb{R}$. With composition of mappings as the operation this yields a group. Indeed, A is closed under composition since if $\alpha_{a,b}, \alpha_{c,d} \in A$, then both $a \neq 0$ and $c \neq 0$, so $ac \neq 0$ and $\alpha_{a,b} \circ \alpha_{c,d} = \alpha_{ac,ad+b} \in A$. The identity is the element $\alpha_{1,0}$ and the inverse of $\alpha_{a,b}$ is $\alpha_{a^{-1},-a^{-1}b}$.

You don't need to write Example 5.8, I put it for your reference.

Solution: We need to provide a counter example, namely, exhibit two such functions which do not commute. For example consider $\alpha_{2,5}$

and $\alpha_{3,7}$, we have $\alpha_{2,5} \circ \alpha_{3,7}(x) = \alpha_{6,19}(x) = 6x + 19$ which is different from $\alpha_{3,7} \circ \alpha_{2,5}(x) = \alpha_{6,22}(x) = 6x + 22$, therefore $\alpha_{2,5}$ and $\alpha_{3,7}$ do not commute, which makes the group nonabelian.

6.15 Assume that α and β are disjoint cycles representing elements of S_n, say $\alpha = (a_1 a_2 \cdots a_s)$ and $\beta = (b_1 b_2 \cdots b_t)$ with $a_i \neq b_j$ for all i and j.

(a) Compute $(\alpha \circ \beta)(a_k)$ and $(\beta \circ \alpha)(a_k)$ for $1 \leq k \leq s$. [Here, $(\alpha \circ \beta)(a_k)$ denotes the image of a_k under the mapping $(\alpha \circ \beta)$; that is, (a_k) is not a 1-cycle.]

Solution: $(\alpha \circ \beta)(a_k) = a_{k+1}$ if $k < s$ and $(\alpha \circ \beta)(a_s) = a_1$, since $\beta = (b_1 b_2 \cdots b_t)$ fixes each a_k as non of them occurs in β. Likewise, $(\beta \circ \alpha)(a_k) = a_{k+1}$ if $k < s$ and $(\beta \circ \alpha)(a_s) = a_1$.

(b) Compute $(\alpha \circ \beta)(b_k)$ and $(\beta \circ \alpha)(b_k)$ for $1 \leq k \leq t$.

Solution: $(\alpha \circ \beta)(b_k) = b_{k+1}$ if $k < t$ and $(\alpha \circ \beta)(b_t) = b_1$, since $\alpha = (a_1 a_2 \cdots a_s)$ fixes each b_k as non of them occurs in α. Likewise, $(\beta \circ \alpha)(b_k) = b_{k+1}$ if $k < t$ and $(\beta \circ \alpha)(b_t) = b_1$.

(c) Compute $(\alpha \circ \beta)(m)$ and $(\beta \circ \alpha)(m)$ for $1 \leq m \leq n$ with $m \neq a_i$ and $m \neq b_j$ for all i and j.

Solution: Since m does not occur in neither α nor β, they both fix it, so we have $(\alpha \circ \beta)(m) = (\beta \circ \alpha)(m) = m$ for $1 \leq m \leq n$ with $m \neq a_i$ and $m \neq b_j$ for all i and j.

(d) What do parts (a), (b), and (c), taken together prove about the relationship between $\alpha \circ \beta$ and $\beta \circ \alpha$.

Solution: By (a), (b), and (c) one has $(\alpha \circ \beta)(\ell) = (\beta \circ \alpha)(\ell)$ for $1 \leq \ell \leq n$, that is, α and β commute.