WRITTEN PROJECT I, due Friday June 4th

Do problems 1.28 and 1.29 in separate pages. Clearly write your name on each page. The proofs should be written clearly, following the methods seen in class. Each problem is worth 10 points. You will have an opportunity to correct your mistakes if you don't get it right the first time.

1.28.

a) Prove that if $\alpha : S \to T$ and A and B are subsets of S, then $\alpha(A \cap B) \subseteq \alpha(A) \cap \alpha(B)$.
b) Give an example (specific S, T, A, B and α) to show equality need not hold in part a). (For the simplest example S will have two elements.)

SOLUTION

a) *Proof.* We need to show that $\alpha(A \cap B) \subseteq \alpha(A) \cap \alpha(B)$ which is equivalent to showing that if $y \in \alpha(A \cap B)$ then $y \in \alpha(A) \cap \alpha(B)$.

   Let $y \in \alpha(A \cap B)$, then by the definition of $\alpha(A \cap B)$ (that is, by the definition of image of a subset), there is $x \in A \cap B$ such that $y = \alpha(x)$ (or, equivalently, $y = \alpha(x)$ for some $x \in A \cap B$). Since $x \in A \cap B$ we know that $x \in A$ and $x \in B$, by definition of intersection of sets. Therefore $y = \alpha(x) \in \alpha(A)$ and $y = \alpha(x) \in \alpha(B)$. Thus $y = \alpha(x) \in \alpha(A) \cap \alpha(B)$, which is what we wanted to prove.

   $\square$

b) Consider the sets $S = \{x_1, x_2\}$, $T = \{y\}$, $A = \{x_1\}$, $B = \{x_2\}$, where $x_1 \neq x_2$ and let $\alpha : S \to T$ be given by $\alpha(x_1) = \alpha(x_2) = y$. We have $A \subseteq S$, $B \subseteq S$ and $\alpha(A) = \alpha(B) = \{y\}$. Thus $\alpha(A) \cap \alpha(B) = \{y\}$. However, $A \cap B = \emptyset$, so $\alpha(A) \cap \alpha(B) \nsubseteq \alpha(A \cap B) = \emptyset$.

1.29. Prove that a mapping $\alpha : S \to T$ is one-to-one iff $\alpha(A \cap B) = \alpha(A) \cap \alpha(B)$ for every pair of subsets A and B of S. (Compare Problem 1.28.)

SOLUTION

Proof. We have a biconditional statement " P iff Q", thus we need to prove both the necessity (that is, "if P then Q") and the sufficiency (that is, "if Q then P").

Necessity: We need to show that if a mapping $\alpha : S \to T$ is one-to-one then $\alpha(A \cap B) = \alpha(A) \cap \alpha(B)$ for every pair of subsets A and B of S. So, we assume that the mapping $\alpha : S \to T$ is one-to-one. In order to prove the equality $\alpha(A \cap B) = \alpha(A) \cap \alpha(B)$ we need to show that both inclusions $\alpha(A \cap B) \subseteq \alpha(A) \cap \alpha(B)$ and $\alpha(A) \cap \alpha(B) \subseteq \alpha(A \cap B)$ hold. The first inclusion, $\alpha(A \cap B) \subseteq \alpha(A) \cap \alpha(B)$, was proved to be true for every mapping α (that is, α does not need to be one-to-one for the inclusion to be true) in problem 1.28. So, we only need to show $\alpha(A) \cap \alpha(B) \subseteq \alpha(A \cap B)$ provided that α is one-to-one. Let $y \in \alpha(A) \cap \alpha(B)$, then, by the definition of intersection of two sets, $y \in \alpha(A)$ and $y \in \alpha(B)$. Since $y \in \alpha(A)$, we have $y = \alpha(x_1)$ for some $x_1 \in A$ (by definition of image of a subset); likewise, since $y \in \alpha(B)$, we have $y = \alpha(x_2)$ for some $x_2 \in A$. Since α is one-to-one and $y = \alpha(x_1) = \alpha(x_2)$ it has to be $x_1 = x_2$; thus $x_1 \in A \cap B$, that is $y = \alpha(x_1)$ where $x_1 \in A \cap B$, namely, $y \in \alpha(A \cap B)$. This proves the necessity.

Sufficiency: Now we need to show that if a mapping $\alpha : S \to T$ is such that

$$\alpha(A \cap B) = \alpha(A) \cap \alpha(B) \qquad (*)$$

for every pair of subsets A and B of S then α is one-to-one. So, we assume we have a mapping $\alpha : S \to T$ with the property $(*)$ and let $x_1, x_2 \in S$ be any pair of elements in S such that $x_1 \neq x_2$. We need to show that $\alpha(x_1) \neq \alpha(x_2)$, that is α is one-to-one. Since the property $(*)$ is true for any pair of subsets A and B of S, in particular it will be true if we let $A = \{x_1\}$ and $B = \{x_2\}$. Now, we have that $\alpha(A) = \{y \in T : y = \alpha(x), x \in A\} = \{\alpha(x_1)\}$; likewise, $\alpha(B) = \{\alpha(x_2)\}$. We also have, $A \cap B = \emptyset$ since $x_1 \neq x_2$, thus $\alpha(A \cap B) = \emptyset$. Since $\alpha(A \cap B) = \alpha(A) \cap \alpha(B)$, it has to be then $\alpha(A) \cap \alpha(B) = \emptyset$ and therefore $\alpha(x_1) \neq \alpha(x_2)$. This proves the sufficiency and concludes the proof.

$\square$