Foundations of Algebra, Practice problems for Exam I

1. Prove that the statements $P(n)$ below are true for each positive integer n, using the principle of mathematical induction

   a) $P(n)$: 2 is a factor of $n^2 + n$

   b) $P(n) : S(n) = x^n + x^{n-1} + \cdots + x + 1 = \dfrac{x^{n+1} - 1}{x - 1}$

2. If A, B and C are sets prove or disprove (by giving a counter example) each of the following statements:
   a) $A \subseteq A \cup B$
   b) $A \cap B \subseteq A$
   c) $A \subseteq B$ implies $A \cup B = B$
   d) If $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$ then $B \cap C \neq \emptyset$
   e) $A \subseteq B$ implies $A \cap B = A$
   f) $A \subseteq B$ implies $B \subseteq A$

3. Write the negation of the following statements:

   a) " For each x in $[2, 3]$, $\dfrac{1}{x}$ is less than 1"

   Negation:

   b) "There is a positive integer n such that $10^n = 5$"

   Negation:

4. Write the converse of the following statement:
   " If a mapping $f : \mathbb{R} \to \mathbb{R}$ is one-to-one then $f(0) \neq f(1)$"

   Converse:

   Is the converse true? Prove it if true or give a counterexample otherwise.

5. Write the contrapositive of the following statement:
   If A is a subset of B then $A \cup B$ is equal to B.

   Contrapositive:

6. Decide which of the following statements is logically equivalent to
   " If x an integer and y is a rational number then xy is a rational number"

   a) If xy is a rational number then x is an integer and y is a rational number.
   b) If xy is not a rational number then x is not an integer or y is not a rational number.
   c) if xy is a rational number then x is not an integer and y is a rational number.

7. Prove that if $\alpha : S \to T$, $\beta : T \to U$, $\gamma : T \to U$, α is onto, and $\beta \circ \alpha = \gamma \circ \alpha$, then $\beta = \gamma$.

8. Prove that if $\beta : S \to T$, $\gamma : S \to T$, $\alpha : T \to U$, α is one-to-one, and $\alpha \circ \beta = \alpha \circ \gamma$ then $\beta = \gamma$.
9. Prove that if $*$ is an associative operation onf S and $a \in S$ then the set

$$C(a) = \{x : x * a = a * x\}$$

is closed under $*$.

10. Prove that multiplication is not commutative as an operation on the set of 2×2 matrices with real numbers as entries.