Mysterious Duality

Sasha Voronov[1] on joint work with Hisham Sati[2]

[1]University of Minnesota and [2]NYU Abu Dhabi

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Setting up the problem

Once upon a time at Harvard...

in 2001, A. Iqbal, A. Neitzke, and C. Vafa discovered a "mysterious duality"

$$\begin{array}{c}\text{Math: Algebraic geometry of del Pezzo surfaces}\\ \downarrow \\ \text{Exceptional series } E_k \text{ of root systems} \qquad 0 \le k \le 8 \\ \uparrow \\ \text{Physics: M-theory, 11d supergravity}\end{array}$$

Classical (XIX century) Algebraic Geometry: Cf. 27 lines on a cubic surface E.g., $x^3 + y^3 + z^3 + w^3 = 0$ in $\mathbb{CP}^3$
(Fermat surface)

The del Pezzo surfaces

A *del Pezzo* (*dP*) *surface* is a complex compact smooth surface whose anticanonical class $-K$ is ample (sufficiently positive). Every dP surface is isomorphic to one on the following list:

$$\mathbb{B}_0 = \mathbb{P}^2, \mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_8$$

and

$$\mathbb{P}^1 \times \mathbb{P}^1.$$

Here

$$\mathbb{B}_k = \text{blowup of } \mathbb{P}^2 \text{ at } k \text{ points in general position}$$
$$= \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$$

Blowup; picture credit: R. Hartshorne

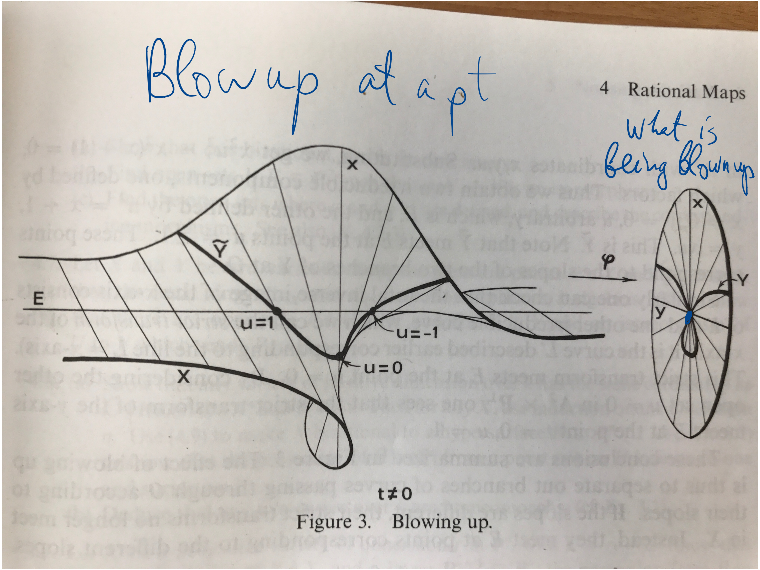
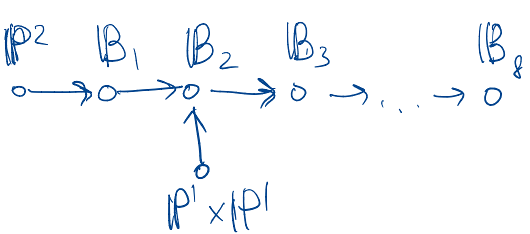


Figure 3. Blowing up.

Sequence of blowups and $E_{\cancel{8}\,10}$ Dynkin diagram

Del Pezzos:

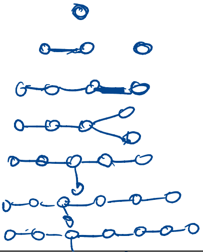


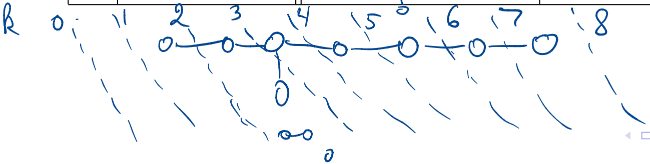
Dynkin diagram "E_{10}"

Physics: Dimensional reduction of supergravity on a k-torus $T^k = (S^1)^k$ a.k.a. toroidal compactifications of M-theory

Much more to the E_k series

The E_k lattice arises as the orthogonal complement to $-K$ (or $c_1(-K)$) in the cohomology group $H^2(\mathbb{B}_k, \mathbb{Z})$ with its intersection form $H^2(\mathbb{B}_k, \mathbb{Z}) \otimes H^2(\mathbb{B}_k, \mathbb{Z}) \to \mathbb{Z}$. root system of type E_8

k	del Pezzo	Dynkin Diagram	Type of E_k	Lie Algebra
0	$\mathbb{P}^2$	$\emptyset$	A_{-1}	$\mathfrak{sl}_0 = \emptyset$
1	$\mathbb{B}_1$, $\mathbb{P}^1 \times \mathbb{P}^1$	$\emptyset$	A_0	$\mathfrak{sl}_1 = 0$
2	$\mathbb{B}_2$		A_1	$\mathfrak{sl}_2$
3	$\mathbb{B}_3$		$A_2 \times A_1$	$\mathfrak{sl}_3 \oplus \mathfrak{sl}_2$
4	$\mathbb{B}_4$		A_4	$\mathfrak{sl}_5$
5	$\mathbb{B}_5$		D_5	$\mathfrak{so}_{10}$
6	$\mathbb{B}_6$		E_6	$\mathfrak{e}_6$
7	$\mathbb{B}_7$		E_7	$\mathfrak{e}_7$
8	$\mathbb{B}_8$		E_8	$\mathfrak{e}_8$



Branes in toroidal compactifications of M-theory

Same story but the E_k lattice shows up in the yoga of branes, such as this table for type IIA string theory = M-theory$/S^1$:

dP surface brane branes in 10d space-time

co homology class	tension	type IIA meaning
E	$R^{-1} = l_s^{-1} g_s^{-1}$	D0-brane
$H - E$	$(2\pi)^2 R l_p^{-3} = (2\pi)^{-1} l_s^{-2}$	F-string
H	$(2\pi)\, l_p^{-3} = (2\pi)^{-2}\, l_s^{-3} g_s^{-1}$	D2-brane
$2H - E$	$(2\pi)^2 R l_p^{-6} = (2\pi)^{-4}\, l_s^{-5} g_s^{-1}$	D4-brane
$2H$	$(2\pi)\, l_p^{-6} = (2\pi)^{-5}\, l_s^{-6} g_s^{-2}$	NS5-brane
$3H - 2E$	$(2\pi)^3 R^2 l_p^{-9} = (2\pi)^{-6}\, l_s^{-7} g_s^{-1}$	D6-brane
$4H - 3E$	$(2\pi)^4 R^3 l_p^{-12} = (2\pi)^{-8}\, l_s^{-9} g_s^{-1}$	D8-brane

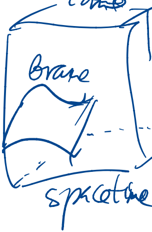


Table credit: Iqbal, Neitzke, and Vafa (2001)

Mystery: Physics and AG give rise to the E_k series, but no explicit connection between physics and del Pezzo surfaces.

Our take on Mysterious Duality

We uncover the mystery of Mysterious Duality in a broad sense as a duality between physics and mathematics by completing one side of the following triangle:

$$
\begin{tikzcd}
\text{Algebraic geometry} \arrow[rr, leftrightarrow, dotted] \arrow[dr, leftrightarrow, dotted, "\text{2001 INV's Myst. Duality}"'] & & \text{Algebraic Topology} \\
& \text{Physics} \arrow[ur, leftrightarrow, "\text{Sati – V, 2021}"'] &
\end{tikzcd}
$$

Algebraic Topology: the rational homotopy theory of iterated cyclic loop spaces $\mathcal{L}_C^k S^4$ of the four-sphere S^4

1. **Math Physics**: is explicitly related to the M-theory story;
2. **Math**: has internal E_k symmetry hidden in it.

Math part: Cyclic loop spaces $\mathcal{L}_c^k S^4$

The *free loop space* of a topological space Z:

$$\mathcal{L}Z = \mathrm{Map}(S^1, Z).$$

It admits a natural action of the group S^1 by rotating loops, and we define the *cyclic loop space* $\mathcal{L}_c Z$ to be the *homotopy quotient*

$$\mathcal{L}_c Z := \mathcal{L}Z /\!\!/ S^1 = \mathcal{L}Z \times_{S^1} ES^1,$$

the *Borel construction*. For $k \geq 0$, the *iterated cyclic loop space* $\mathcal{L}_c^k Z$ is the k-fold iteration of the cyclic loop space construction:

$$\begin{aligned} \mathcal{L}_c^0 Z &:= Z, \\ \mathcal{L}_c^k Z &:= \mathcal{L}_c(\mathcal{L}_c^{k-1} Z) \qquad \text{for } k \geq 1. \end{aligned}$$

We will be interested mostly in the iterated cyclic loop spaces $\mathcal{L}_c^k S^4$ of the 4-sphere S^4 for $0 \leq k \leq 8$.

Math part: the Sullivan minimal algebra of $\mathcal{L}_c^k S^4$

Rational homotopy theory (RHT): $X \sim Y$ iff $X \to Y$ rational homotopy equivalence, a conts map inducing isomorphisms $\pi_n(X) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_n(Y) \otimes \mathbb{Q}$ on rational homotopy groups.

Rational homotopy category: topological spaces with inverses of rational h. equivalences formally added.

Fact (Quillen, Sullivan, '60–70s): the rational homotopy category (of good enough spaces) is equivalent to a category of DGCAs of a certain type:

(model)

$$X \mapsto M(X), \quad \text{the } \textit{Sullivan minimal algebra} \text{ of } X.$$

Examples and Math Physics part

Fiorenza, Schreiber, Sati

$$M(S^4) = (\mathbb{Q}[g_4, g_7], d),$$

$$dg_4 = 0, \qquad dg_7 = -\tfrac{1}{2} g_4^2,$$

$$|g_4| = 4, \qquad |g_7| = 7.$$

Equations of motion in M-theory

$$M(\mathcal{L}_c S^4) = (\mathbb{R}[g_4, g_7, sg_4, sg_7, w], d),$$

$$dg_4 = sg_4 \cdot w, \qquad dg_7 = -\tfrac{1}{2} g_4^2 + sg_7 \cdot w,$$

$$dsg_4 = 0, \quad dsg_7 = sg_4 \cdot g_4, \quad dw = 0.$$

Equations of motion in type IIA string theory

This pattern continues for higher k's

Math part: E_k from $\mathcal{L}_c^k S^4$

Toroidal symmetries of $M(\mathcal{L}_c^k S^4)$ (and of the rational homotopy type of $\mathcal{L}_c^k S^4$):

Theorem (Sati-V)

For each k, $0 \leq k \leq 8$, the automorphism group of the Sullivan minimal model $M = M(\mathcal{L}_c^{k} S^4) \otimes_{\mathbb{Q}} \mathbb{R}$ is a real algebraic group~~s~~ which contains a canonically defined maximal $\mathbb{R}$-split torus $T \cong (\mathbb{R}^\times)^{k+1} \subseteq \operatorname{Aut} M$. $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$

The Sullivan minimal model $M = M(\mathcal{L}_c S^4)$ splits into a weight decomposition

$$M = \bigoplus_{\alpha \in X(T)} M_\alpha$$

indexed by the *character group* $X(T) = \mathrm{Mor}(T, \mathbb{G}_m)_{\mathbb{R}}$ of real algebraic group morphisms from T to the multiplicative group $\mathbb{G}_m$, so that T acts on each *weight space* M_α by the character α:

$$M_\alpha = \{m \in M \mid t \cdot m = \alpha(t) m \quad \text{for all } t \in T\}.$$

Math part: E_k from $\mathcal{L}_c^k S^4$

Theorem (Sati-V)

The abelian Lie algebra $\mathfrak{h}_k = \mathrm{Lie}(T)$ of the torus T (of $M(\mathcal{L}_c^k S^4)$) has a natural basis, giving a lattice $\mathfrak{h}_k^{\mathbb{Z}} \subseteq \mathfrak{h}_k$, an integral inner product, and a distinguished element $\omega_k \in \mathfrak{h}_k^{\mathbb{Z}}$. The triple $(\mathfrak{h}_k^{\mathbb{Z}}, (-,-), \omega_k)$ associated to the cyclic loop spaces $\mathcal{L}_c^k S^4$ and their Sullivan minimal models $M(\mathcal{L}_c^k S^4)$ consists of

1. *a free abelian group $\mathfrak{h}_k^{\mathbb{Z}}$ with a basis $h_0, h_1, \ldots, h_k$;*
2. *a symmetric bilinear form $\mathfrak{h}_k^{\mathbb{Z}} \otimes \mathfrak{h}_k^{\mathbb{Z}} \to \mathbb{Z}$ given by*

$$(h_0, h_0) = 1, \qquad (h_i, h_j) = -\delta_{ij}, \qquad i > 0, j \geq 0;$$

3. *an element $\omega_k = -3h_0 + h_1 + \cdots + h_k$.*

acts on $M(\mathcal{L}_c^k S^4)$ by degree

In the same way as for del Pezzo surfaces, this algebraic structure produces the root system E_k and the Weyl group $W(E_k)$, now in the context of cyclic loop spaces of S^4.

Conjecture: duality between dP surfaces and loop spaces of S^4

Algebraic Geometry $\overset{?}{\longleftrightarrow}$ Algebraic Topology

Conjecture

There must be an explicit relation between the series of del Pezzo surfaces $\mathbb{B}_k$, $0 \leq k \leq 8$, and the series of iterated cyclic loop spaces $\mathcal{L}_c^k S^4$, $0 \leq k \leq 8$. This relation should match the E_k symmetry patterns occurring in both series, as well as relate other geometric data, such as the volumes of curves on del Pezzo surfaces, with some topological data on the iterated loop spaces.

Recommended Reading :

Sheldon Katz, Enumerative geometry and String theory, 2006, 206 pages, AMS