Topologically nontrivial counterexamples to Sard's theorem

Pure Mathematics Colloquium

Texas Tech University

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February 15, 2021

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P. Goldstein, P. Hajłasz, P. Pankka, Topologically Nontrivial Counterexamples to Sard's Theorem Int. Math. Res. Not. IMRN. 2018.

P. Goldstein, P Hajłasz, C^1 mappings in $\mathbb{R}^5$ with derivative of rank at most 3 cannot be uniformly approximated by C^2 mappings with derivative of rank at most 3. J. Math. Anal. Appl. 468 (2018), 1108–1114.

The Sard theorem

- If $f : \mathbb{R}^m \to \mathbb{R}^n$, $m \geq n$, is of class C^1, then the critical points are

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- Critical values are **bad**, regular values are **nice**. We want the set of critical values to be small.

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- **A new idea is needed.**

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One needs to work with forms that are weakly closed and weakly exact and use the L^p Hodge decomposition and Sobolev spaces.

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- Theorem (Goldstein, Hajłasz, Pankka). For $n \geq 4$ there is $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ with $\operatorname{rank} df < n$, that is not homotopic to a constant map.

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- $n = 2$, $f \in C^1(\mathbb{S}^3, \mathbb{S}^2)$, $\mathrm{rank}\, df < 2$, by Hopf invariant. Well known.

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The Sard theorem: counterexample & homotopy

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- Larry Guth (2013): We don't know any homotopically non-trivial C^1-mappings from $\mathbb{S}^m$ to $\mathbb{S}^n$ with rank $< n$. Does one exist?

The Sard theorem: counterexample & homotopy

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We find it somewhat surprising that the situation changes at the dimension $n = 4$: $\pi_4(\mathbb{S}^3) = \pi_5(\mathbb{S}^4) = \mathbb{Z}_2$, but the claim of the theorem is different in these dimensions.

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- S. Wenger, R. Young, Lipschitz homotopy groups of the Heisenberg groups. Geom. Funct. Anal. 24 (2014), 387–402.

The Sard theorem: counterexample & homotopy

Theorem (Goldstein-Hajłasz-Pankka). For each $n \geq 4$, there is a map $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ that is not homotopic to a constant map and such that $\operatorname{rank} df < n$ everywhere.

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Sketch of the construction.

Construction of a mapping: a graphic novel

- $\pi_m(\mathbb{S}^k) \neq 0$ so $\pi_{m-1}(\mathbb{S}^{k-1}) \neq 0$, because the suspension map $\sigma : \pi_{m-1}(\mathbb{S}^{k-1}) \to \pi_m(\mathbb{S}^k)$ is a surjection

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- The suspension maps each $(m-1)$-dimensional sphere parallel to the equator to the corresponding $(k-1)$-dimensional sphere parallel to the equator as a scaled and translated copy of h.

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- It suffices to extend it to $F : \mathbb{B}^{m+1} \to \mathbb{B}^{k+1}$ in such a way that $\operatorname{rank} dF \leq k$.

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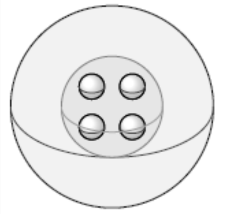
- It suffices to extend it to $F : \mathbb{B}^{m+1} \to \mathbb{B}^{k+1}$ in such a way that $\operatorname{rank} dF \leq k$.
- Indeed, gluing together two copies of such mappings along equators we will obtain a map from $\mathbb{S}^{m+1}$ to $\mathbb{S}^{k+1}$ that is not homotopic to a constant map (by Freduenthal's theorem).

Construction of a mapping: a graphic novel

Now we will describe how to extend $H : \mathbb{S}^m \to \mathbb{S}^k$ to $F : \mathbb{B}^{m+1} \to \mathbb{B}^{k+1}$ in such a way that $\operatorname{rank} dF \leq k$.

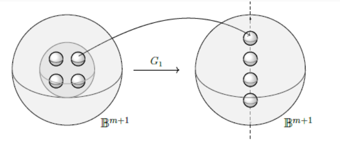
Construction of a mapping: a graphic novel

In the ball $\mathbb{B}^{m+1}$ we select many smaller balls:



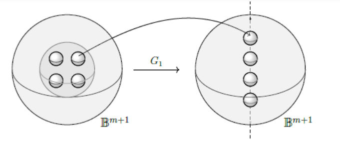
Construction of a mapping: a graphic novel

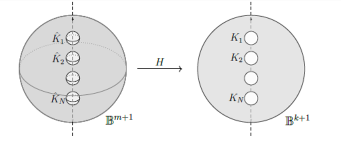
We apply a diffeomorphism that arranges balls vertically:



Construction of a mapping: a graphic novel

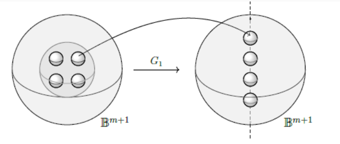
On each sphere $\partial\hat{K}_i$ and on $\partial\mathbb{B}^{m+1}$, we have a copy of the suspension map $H : \mathbb{S}^m \to \mathbb{S}^k$ and we extend it to $\mathbb{B}^{m+1} \setminus \bigcup_i \hat{K}_i$ using cylindrical coordinates.

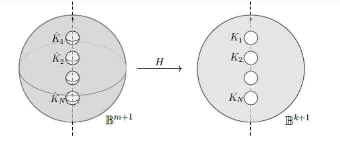


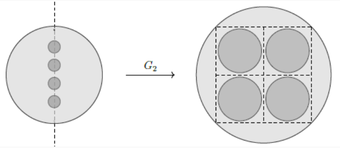


Construction of a mapping: a graphic novel

We rearrange the balls in $\mathbb{B}^{k+1}$ by a diffeomorphism so they are inside a cubical grid.

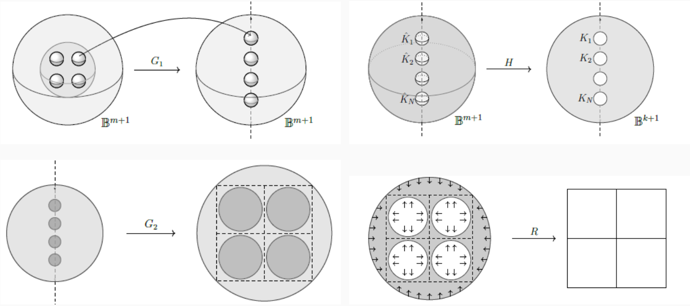






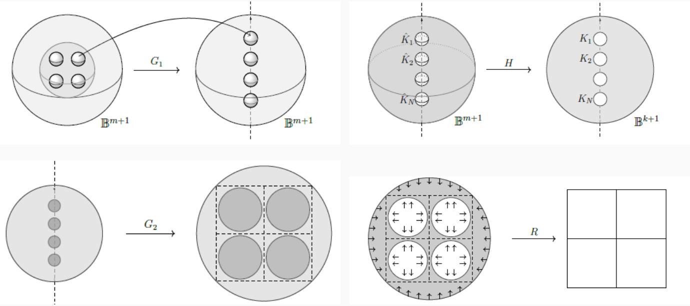
Construction of a mapping: a graphic novel

We compose the map with the (C^∞ smooth!) projection R on the cubical grid.

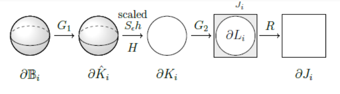


Construction of a mapping: a graphic novel

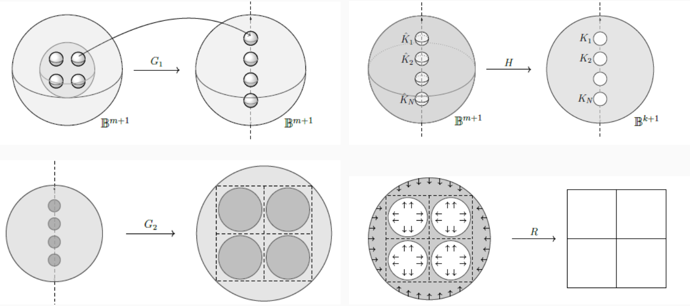
We have a map from $\mathbb{B}^{m+1} \setminus \bigcup_i \mathbb{B}_i$ onto the boundary of the cubical grid. Boundary $\partial\mathbb{B}^{m+1}$ is mapped onto the boundary of the big cube.

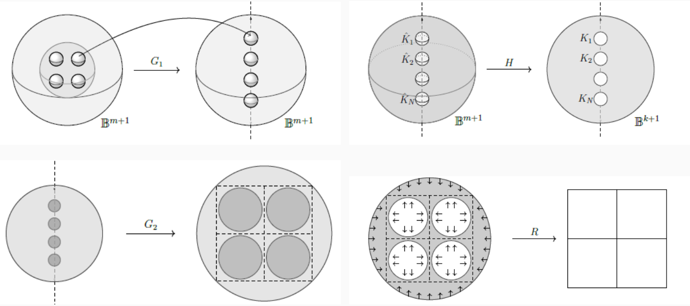


Boundary of each small ball is mapped onto the boundary of a small cube.

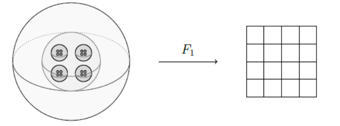


Construction of a mapping: a graphic novel





Inside each small ball we include a scaled copy of the above map defined above:



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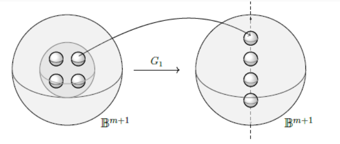

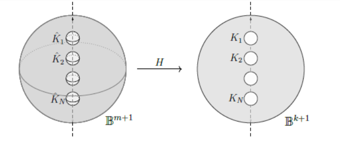

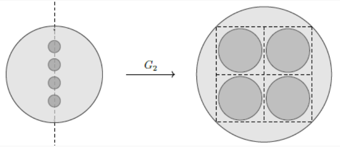

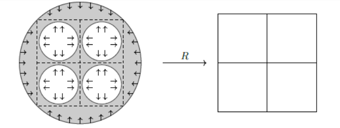

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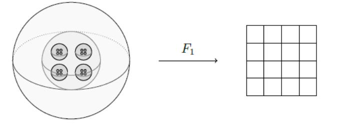

We iterate the procedure infinitely many times.

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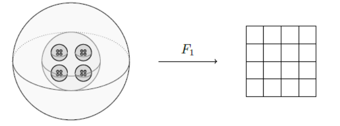

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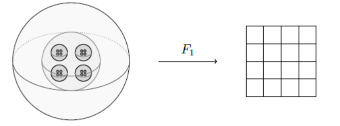

We iterate the procedure infinitely many times.

- The resulting map will be a C^1 map $F_1 : \mathbb{B}^{m+1} \to Q^{k+1}$.

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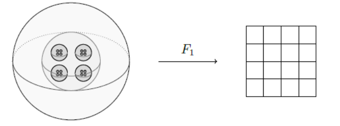

We iterate the procedure infinitely many times.

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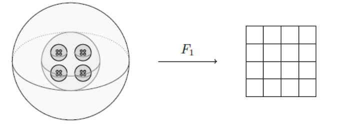

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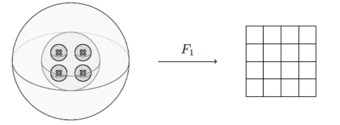

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- **Theorem (Goldstein-Hajłasz-Pankka).** For each $n \geq 4$, there is a map $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ that is not homotopic to a constant map and such that $\mathrm{rank}\, df < n$ everywhere.

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- **Theorem (Guth).** If $n \geq 2$ and $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ satisfies $\mathrm{rank}\, df < \left[\frac{n+2}{2}\right]$, then f is homotopic to a constant map.

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Our result answers Conjecture 2 in the positive when $n = 4$. Other cases $n \geq 5$ are open.

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- **Conjecture (Gałęski).** If $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ satisfies $\operatorname{rank} df \leq k$, $k < m$, then there is a sequence $f_i \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$, $\operatorname{rank} df_i \leq k$ that converges to f uniformly.

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- The following result is easy to prove:
- **Theorem.** Let $1 \leq k < m$ be integers. If $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ satisfies $\operatorname{rank} df \leq k$ everywhere in $\mathbb{R}^m$, then there is an open and dense set $\Omega \subset \mathbb{R}^m$ such that for every point $x \in \Omega$ there is a neighborhood $\mathbb{B}^m(x, \varepsilon) \subset \Omega$ and a sequence $f_i \in C^\infty(\mathbb{B}^m(x, \varepsilon), \mathbb{R}^m)$ such that $\operatorname{rank} df_i \leq k$ and f_i converge to f in $C^1(\mathbb{B}^m(x, \varepsilon), \mathbb{R}^m)$ (i.e. both f_i and their first derivatives converge uniformly).

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- **Example.** There is $f \in C^1(\mathbb{R}^5, \mathbb{R}^5)$ with $\operatorname{rank} df \leq 3$ that cannot be approximated in the supremum norm by mappings $g \in C^2(\mathbb{R}^5, \mathbb{R}^5)$ satisfying $\operatorname{rank} dg \leq 3$.

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- **Theorem.** Suppose that $k + 1 \leq m < 2k - 1$, $\ell \geq m + 1$, $r \geq k + 1$, and the homotopy group $\pi_m(\mathbb{S}^k)$ is non-trivial. Then there is a map $f \in C^1(\mathbb{R}^\ell, \mathbb{R}^r)$ with $\operatorname{rank} df \leq k$ in $\mathbb{R}^\ell$ that cannot be approximated by maps of class C^{m-k+1}.

Questions?

Thank you!